Research Article

Multiple-Set Split Feasibility Problems for Asymptotically Strict Pseudocontractions

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Received 5 October 2011; Revised 5 December 2011; Accepted 7 December 2011

Abstract

In this paper, we introduce an iterative method for solving the multiple-set split feasibility problems for asymptotically strict pseudocontractions in infinite-dimensional Hilbert spaces, and, by using the proposed iterative method, we improve and extend some recent results given by some authors.

1. Introduction

The split feasibility problem (SFP) in finite dimensional spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [3–5].

The split feasibility problem in an infinite dimensional Hilbert space can be found in [2, 4, 6–8].

Throughout this paper, we always assume that $H_1$, $H_2$ are real Hilbert spaces, “→”, “⇀” are denoted by strong and weak convergence, respectively.
The purpose of this paper is to introduce and study the following multiple-set split feasibility problem for asymptotically strict pseudocontraction (MSSFP) in the framework of infinite-dimensional Hilbert spaces. Find \( x^* \in C \) such that

\[
Ax^* \in Q,
\]

where \( A : H_1 \to H_2 \) is a bounded linear operator, \( \{S_i\} \) and \( \{T_i\} \), \( i = 1, 2, \ldots, M \), are the families of mappings \( S_i : H_1 \to H_1 \) and \( T_i : H_2 \to H_2 \), respectively, \( C := \bigcap_{i=1}^M F(S_i) \) and \( Q := \bigcap_{i=1}^M F(T_i) \), where \( F(S_i) = \{x_i \in H_1 : S_ix_i = x_i\} \) and \( F(T_i) = \{y_i \in H_2 : T_iy_i = y_i\} \) denote the sets of fixed points of \( S_i \) and \( T_i \), respectively. In the sequel, we use \( \Gamma \) to denote the set of solutions of the problem (MSSFP), that is,

\[
\Gamma = \{x \in C : Ax \in Q\}.
\]

2. Preliminaries

We first recall some definitions, notations, and conclusions which will be needed in proving our main results.

Let \( E \) be a Banach space. A mapping \( T : E \to E \) is said to be demiclosed at origin if, for any sequence \( \{x_n\} \subset E \) with \( x_n \to x^* \) and \( \| (I-T)x_n \| \to 0 \), we have \( x^* = Tx^* \). A Banach space \( E \) is said to have Opial’s property if, for any sequence \( \{x_n\} \) with \( x_n \to x^* \), we have

\[
\liminf_{n \to \infty} \|x_n - x^*\| < \liminf_{n \to \infty} \|x_n - y\|
\]

for all \( y \in E \) with \( y \neq x^* \).

Remark 2.1. It is well known that each Hilbert space possesses Opial’s property.

Definition 2.2. Let \( H \) be a real Hilbert space.

1. A mapping \( G : H \to H \) is called a \((\gamma, \{k_n\})\)-asymptotically strict pseudocontraction if there exists a constant \( \gamma \in [0,1) \) and a sequence \( \{k_n\} \subset [1, \infty) \) with \( k_n \to 1 \) such that

\[
\|G^n x - G^n y\| \leq k_n \|x - y\| + \gamma \|(I - G^n)x - (I - G^n)y\|^2, \quad \forall x, y \in H.
\]

Especially, if \( k_n = 1 \) for each \( n \geq 1 \) in (2.2) and there exists \( \gamma \in [0,1) \) such that

\[
\|Gx - Gy\|^2 \leq \|x - y\|^2 + \gamma \|(I - G)x - (I - G)y\|^2, \quad \forall x, y \in H,
\]

then \( G : H \to H \) is called a \( \gamma \)-strict pseudocontraction.

2. A mapping \( G : H \to H \) is said to be uniformly \( L \)-Lipschitzian if there exists a constant \( L > 0 \) such that

\[
\|G^n x - G^n y\| \leq L\|x - y\|, \quad \forall x, y \in H, \quad n \geq 1.
\]
(3) A mapping $G : H \to H$ is said to be semicompact if, for any bounded sequence $\{x_n\} \subset H$ with $\lim_{n \to \infty} \|x_n - Gx_n\| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i}$ converges strongly to a point $x^* \in H$.

Now, we give one example of the $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction mapping.

**Example 2.3.** Let $B$ be the unit ball in a Hilbert space $l_2^3$, and define a mapping $T : B \to B$ by

$$T = (x_1, x_2, \ldots) = (0, x_1^2, a_2x_2, a_3x_3, \ldots),$$

where $\{a_i\}$ is a sequence in $(0, 1)$ such that $\prod_{i=2}^n a_i = 1/2$. It is proved in Goebel and Kirk [9] that

(a) $\|Tx = Ty\| \leq 2\|x - y\|$ for all $x, y \in B$,

(b) $\|T^n x - T^n y\| \leq 2\sum_{j=2}^n a_j$ for all $n \geq 2$ and $x, y \in B$.

Denote by $k_1^{1/2} = 2, k_n^{1/2} = 2\prod_{j=2}^n a_j (n \geq 2)$ and $\gamma \in [0, 1)$. Then, we have

$$\lim_{n \to \infty} k_n = \lim_{n \to \infty} \left(2\prod_{j=2}^n a_j\right)^{1/2} = 1,$$

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + 2\gamma\|x - y - (T^n x - T^n y)\|^2, \quad \forall n \geq 1, x, y \in B,$$

and so the mapping $T$ is a $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction.

**Remark 2.4.** (1) If we put $\gamma = 0$ in (2.2), then the mapping $G : H \to H$ is asymptotically nonexpansive.

(2) If we put $\gamma = 0$ in (2.3), then the mapping $G : H \to H$ is nonexpansive.

(3) Each $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction and each $\gamma$-strictly pseudocontraction both are demiclosed at origin [10].

**Proposition 2.5.** Let $G : H \to H$ be a $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction. If $F(G) \neq \emptyset$, then, for any $q \in F(G)$ and $x \in H$, the following inequalities hold and they are equivalent:

$$\|G^n x - q\|^2 \leq k_n\|x - q\|^2 + 2\gamma\|x - G^n x\|^2, \quad \gamma \in [0, 1),$$

$$\langle x - G^n x, x - q \rangle \geq \frac{1 - \gamma}{2}\|x - G^n x\|^2 - \frac{k_n - 1}{2}\|x - q\|^2, \quad \gamma \in [0, 1),$$

$$\langle x - G^n x, q - G^n x \rangle \leq \frac{1 + \gamma}{2}\|x - G^n x\|^2 + \frac{k_n - 1}{2}\|x - q\|^2, \quad \gamma \in [0, 1).$$

**Lemma 2.6 (see [11]).** Let $\{a_n\}, \{b_n\},$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then the limit $\lim_{n \to \infty} a_n$ exists.
3. Multiple-Set Split Feasibility Problem

For solving the multiple-set split feasibility problem (1.1), let us assume that the following conditions are satisfied:

(C1) $H_1$ and $H_2$ are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator;

(C2) $S_i : H_1 \to H_1$, $i = 1, 2, \ldots, M$, is a uniformly $L_i$-Lipschitzian and $(\beta_i, \{k_{i,n}\})$-asymptotically strict pseudocontraction, and $T_i : H_2 \to H_2$, $i = 1, 2, \ldots, M$, is a uniformly $L_i$-Lipschitzian and $(\mu_i, \{\bar{k}_{i,n}\})$-asymptotically strict pseudocontraction satisfying the following conditions:

\begin{enumerate}
  \item $C := \bigcap_{i=1}^{M} F(S_i) \neq \emptyset$ and $Q := \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$,
  \item $\beta = \max_{1 \leq i \leq M} \beta_i < 1$ and $\mu = \max_{1 \leq i \leq M} \mu_i < 1$,
  \item $L := \max_{1 \leq i \leq M} L_i < \infty$ and $\bar{L} := \max_{1 \leq i \leq M} \bar{L}_i < \infty$,
  \item $k_n = \max_{1 \leq i \leq M} \{k_{i,n}, \bar{k}_{i,n}\}$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$.
\end{enumerate}

We are now in a position to give the following result.

Theorem 3.1. Let $H_1$, $H_2$, $A$, $\{S_i\}$, $\{T_i\}$, $C$, $Q$, $\beta$, $\mu$, $L$, $\bar{L}$, and $\{k_n\}$ be the same as above. Let $\{x_n\}$ be the sequence generated by

\begin{align}
  x_1 & \in H_1 \text{ chosen arbitrarily,} \\
  x_{n+1} & = (1 - \alpha_n) u_n + \alpha_n S_n^n (u_n), \\
  u_n & = x_n + \gamma A^* (T_n^n - I) Ax_n, \quad \forall n \geq 1,
\end{align}

where $S_n^n = S_n^n (\mod M)$, $T_n^n = T_n^n (\mod M)$ for all $n \geq 1$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\gamma > 0$ is a constant satisfying the following conditions:

\begin{enumerate}
  \item $\alpha_n \in (\beta, 1 - \beta)$ for all $n \geq 1$ and $\gamma \in (0, (1 - \mu)/\|A\|^2)$, where $\delta \in (0, 1 - \beta)$ is a positive constant.
\end{enumerate}

(1) If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.

(2) In addition, if there exists a positive integer $j$ such that $S_j$ is semicompact, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to a point $x^* \in \Gamma$.

Proof. (1) The proof is divided into 5 steps as follows.

Step 1. We first prove that, for any $p \in \Gamma$, the limit

\begin{equation}
  \lim_{n \to \infty} \|x_n - p\| \quad (3.2)
\end{equation}
exists. In fact, since \( p \in \Gamma, p \in C := \bigcap_{i=1}^{M} F(S_i) \), and \( Ap \in Q := \bigcap_{i=1}^{M} F(T_i) \). From (3.1) and (2.8), it follows that

\[
\| x_{n+1} - p \|^2 = \| u_n - p - \alpha_n(u_n - S_n u_n) \|^2 \\
= \| u_n - p \|^2 - 2\alpha_n \langle u_n - p, u_n - S_n u_n \rangle + \alpha_n^2 \| u_n - S_n u_n \|^2 \\
\leq \| u_n - p \|^2 - \alpha_n \left\{ (1 - \beta) \| u_n - S_n u_n \|^2 - (k_n - 1) \| u_n - p \|^2 \right\} + \alpha_n^2 \| u_n - S_n u_n \|^2 \\
= (1 + \alpha_n (k_n - 1)) \| u_n - p \|^2 - \alpha_n (1 - \beta - \alpha_n) \| u_n - S_n u_n \|^2.
\]

On the other hand, since

\[
\| u_n - p \|^2 = \| x_n - p + \gamma A^*(T^n_n - I)Ax_n \|^2 \\
= \| x_n - p \|^2 + \gamma^2 \| A^*(T^n_n - I)Ax_n \|^2 + 2\gamma \langle x_n - p, A^*(T^n_n - I)Ax_n \rangle,
\]

\[
\gamma^2 \| A^*(T^n_n - I)Ax_n \|^2 = \gamma^2 \langle A^*(T^n_n - I)Ax_n, A^*(T^n_n - I)Ax_n \rangle \\
= \gamma^2 \langle AA^*(T^n_n - I)Ax_n, (T^n_n - I)Ax_n \rangle \\
\leq \gamma^2 \| A \|^2 \| T^n_n Ax_n - Ax_n \|^2,
\]

\[
2\gamma \langle x_n - p, A^*(T^n_n - I)Ax_n \rangle = 2\gamma \langle Ax_n - Ap, (T^n_n - I)Ax_n \rangle \\
= 2\gamma \langle (Ax_n - Ap) + (T^n_n - I)Ax_n - (T^n_n - I)Ax_n, (T^n_n - I)Ax_n \rangle \\
= 2\gamma \left\{ (T^n_n Ax_n - Ap, T^n_n Ax_n - Ax_n) - \| (T^n_n - I)Ax_n \|^2 \right\}.
\]

Further, letting \( x = Ax_n, \gamma^2 = T^n_n, q = Ap, \gamma = \mu \) in (2.9) and noting \( Ap \in F(T_n) \), it follows that

\[
(T^n_n Ax_n - Ap, T^n_n Ax_n - Ax_n) \leq \frac{1 + \mu}{2} \| (T^n_n - I)Ax_n \|^2 + \frac{k_n - 1}{2} \| Ax_n - Ap \|^2 \\
\leq \frac{1 + \mu}{2} \| (T^n_n - I)Ax_n \|^2 + \frac{(k_n - 1)\| A \|^2}{2} \| x_n - p \|^2.
\]

Substituting (3.7) into (3.6) and simplifying it, we have

\[
2\gamma \langle x_n - p, A^*(T^n_n - I)Ax_n \rangle \leq \gamma (\mu - 1) \| (T^n_n - I)Ax_n \|^2 + (k_n - 1)\gamma \| A \|^2 \| x_n - p \|^2.
\]

Substituting (3.5) and (3.8) into (3.4) and simplifying it, we have

\[
\| u_n - p \|^2 \leq \| x_n - p \|^2 + \gamma^2 \| A \|^2 \| T^n_n Ax_n - Ax_n \|^2 \\
+ \gamma (\mu - 1) \| (T^n_n - I)Ax_n \|^2 + (k_n - 1)\gamma \| A \|^2 \| x_n - p \|^2.
\]
\[
= \|x_n - p\|^2 - \gamma \left(1 - \mu - \gamma \|A\|^2\right)\|T_n^p Ax_n - Ax_n\|^2 \\
+ (k_n - 1)\gamma \|A\|^2 \|x_n - p\|^2.
\]

Again, substituting (3.9) into (3.3) and simplifying it, we have

\[
\|x_{n+1} - p\|^2 \leq (1 + \alpha_n(k_n - 1)) \times \left\{\|x_n - p\|^2 - \gamma \left(1 - \mu - \gamma \|A\|^2\right)\|T_n^p Ax_n - Ax_n\|^2 + (k_n - 1)\gamma \|A\|^2 \|x_n - p\|^2\right\}
\]

\[
- \alpha_n(1 - \beta - \alpha_n)\|u_n - S_n^a u_n\|^2
\]

\[
\leq (1 + \alpha_n(k_n - 1))\|x_n - p\|^2 - \gamma \left(1 - \mu - \gamma \|A\|^2\right)\|T_n^p Ax_n - Ax_n\|^2
\]

\[
+ (1 + \alpha_n(k_n - 1))(k_n - 1)\gamma \|A\|^2 \|x_n - p\|^2 - \alpha_n(1 - \beta - \alpha_n)\|u_n - S_n^a u_n\|^2.
\]

By the condition (e), we have

\[
\|x_{n+1} - p\|^2 \leq (1 + \alpha_n(k_n - 1))\|x_n - p\|^2 + (1 + \alpha_n(k_n - 1))(k_n - 1)\gamma \|A\|^2 \|x_n - p\|^2
\]

\[
\leq (1 + K(k_n - 1))\|x_n - p\|^2,
\]

where

\[
K = \sup_{n \geq 1} \left(\alpha_n + (1 + \alpha_n(k_n - 1))\gamma \|A\|^2\right) < \infty.
\]

By the condition (d), \(\sum_{n=1}^\infty (k_n - 1) < \infty\); hence, from Lemma 2.6, we know that the following limit exists:

\[
\lim_{n \to \infty} \|x_n - p\|.\]

**Step 2.** We will now prove that, for each \(p \in \Gamma\), the limit

\[
\lim_{n \to \infty} \|u_n - p\|
\]

exists. In fact, from (3.10) and (3.13), it follows that

\[
\gamma \left(1 - \mu - \gamma \|A\|^2\right)\|T_n^p Ax_n\|^2 + \alpha_n(1 - \beta - \alpha_n)\|u_n - S_n^a u_n\|^2
\]

\[
\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + K(k_n - 1)\|x_n - p\|^2 \to 0 \quad (n \to \infty).
\]

This together with the condition (e) implies that

$$\lim_{n \to \infty} \|u_n - S_n^p u_n\| = 0, \quad \text{(3.16)}$$

$$\lim_{n \to \infty} \|(T_n^n - I)Ax_n\| = 0. \quad \text{(3.17)}$$

Therefore, it follows from (3.4), (3.13), and (3.17) that the limit $\lim_{n \to \infty} \|u_n - p\|$ exists.

**Step 3.** Now, we prove that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0, \quad \lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. \quad \text{(3.18)}$$

In fact, it follows from (3.1) that

$$\|x_{n+1} - x_n\| = \|(1 - \alpha_n)u_n + \alpha_nS_n^p(u_n) - x_n\|
= \|(1 - \alpha_n)(x_n + \gamma A^*(T_n^n - I)Ax_n) + \alpha_nS_n^p(u_n) - x_n\|
= \|(1 - \alpha_n)\gamma A^*(T_n^n - I)Ax_n + \alpha_n(S_n^p(u_n) - u_n) + \alpha_n(u_n - x_n)\|
= \|(1 - \alpha_n)\gamma A^*(T_n^n - I)Ax_n + \alpha_n(S_n^p(u_n) - u_n) + \alpha_n\gamma A^*(T_n^n - I)Ax_n\|
= \|\gamma A^*(T_n^n - I)Ax_n + \alpha_n(S_n^p(u_n) - u_n)\|.$$  

(3.19)

In view of (3.16) and (3.17), we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \quad \text{(3.20)}$$

Similarly, it follows from (3.1), (3.17), and (3.20) that

$$\|u_{n+1} - u_n\| = \|x_{n+1} + \gamma A^*(T_{n+1}^{n+1} - I)Ax_{n+1} - (x_n + \gamma A^*(T_n^n - I)Ax_n)\|
\leq \|x_{n+1} - x_n\| + \gamma \|A^*(T_{n+1}^{n+1} - I)Ax_{n+1}\|
+ \gamma \|A^*(T_n^n - I)Ax_n\| \to 0 \quad (n \to \infty). \quad \text{(3.21)}$$

The conclusion (3.18) is proved.

**Step 4.** Next, we prove that, for each $j = 1, 2, \ldots, M$,

$$\|u_{M+j} - S_j u_{M+j}\| \to 0, \quad \|Ax_{M+j} - T_j Ax_{M+j}\| \to 0 \quad (i \to \infty). \quad \text{(3.22)}$$

In fact, from (3.16), it follows that

$$\hat{u}_{M+j} := \|u_{M+j} - S_j^{M+j} u_{M+j}\| \to 0 \quad (i \to \infty). \quad \text{(3.23)}$$
Since $S_j$ is uniformly $L_j$-Lipschitzian continuous, it follows from (3.18) and (3.23) that

\[
\|u_{iM+j} - S_j u_{iM+j}\| \\
\leq \|u_{iM+j} - S_j u_{iM+j}\| + \|S_j u_{iM+j} - S_j u_{iM+j}\| \\
\leq \xi_{iM+j} + L_j \|S_j^{iM+j-1} u_{iM+j} - u_{iM+j}\| \\
\leq \xi_{iM+j} + L_j \left\{ \|S_j^{iM+j-1} u_{iM+j} - S_j^{iM+j-1} u_{iM+j-1}\| + \|S_j^{iM+j-1} u_{iM+j-1} - u_{iM+j}\| \right\} \\
\leq \xi_{iM+j} + L_j(1 + L_j) \|u_{iM+j} - u_{iM+j-1}\| + L_j \xi_{iM+j-1} \rightarrow 0 \quad (i \to \infty). \tag{3.24}
\]

Similarly, for each $j = 1, 2, \ldots, M$, it follows from (3.17) that

\[
\xi_{iM+j} := \|A x_{iM+j} - T_j^M A x_{iM+j}\| \rightarrow 0 \quad (i \to \infty). \tag{3.25}
\]

Since $T_j$ is uniformly $\tilde{L}_j$-Lipschitzian continuous, by the same way as above, from (3.18) and (3.25), we can also prove that

\[
\|A x_{iM+j} - T_j A x_{iM+j}\| \rightarrow 0 \quad (i \to \infty). \tag{3.26}
\]

Step 5. Finally, we prove that $x_n \rightarrow x^*$ and $u_n \rightarrow x^*$, which is a solution of the problem (MSSFP). In fact, since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightarrow x^* \in H_1$. Hence, for any positive integer $j = 1, 2, \ldots, M$, there exists a subsequence $\{n_i(j)\} \subset \{n_i\}$ with $n_i(j) \mod M = j$ such that $u_{n_i(j)} \rightarrow x^*$. Again, from (3.22), it follows that

\[
\|u_{n_i(j)} - S_j u_{n_i(j)}\| \rightarrow 0 \quad (n_i(j) \to \infty). \tag{3.27}
\]

Since $S_j$ is demiclosed at zero (see Remark 2.4), it follows that $x^* \in F(S_j)$. By the arbitrariness of $j = 1, 2, \ldots, M$, we have $x^* \in C := \bigcap_{j=1}^M F(S_j)$.

Moreover, it follows from (3.1) and (3.17) that

\[
x_n = u_n - \gamma A^* (T_n^m - I) A x_n \rightarrow x^*. \tag{3.28}
\]

Since $A$ is a linear bounded operator, it follows that $A x_n \rightarrow A x^*$. For any positive integer $k = 1, 2, \ldots, M$, there exists a subsequence $\{n_i(k)\} \subset \{n_i\}$ with $n_i(k) \mod M = k$ such that $A x_{n_i(k)} \rightarrow A x^*$. In view of (3.22), we have

\[
\|A x_{n_i(k)} - T_k A x_{n_i(k)}\| \rightarrow 0 \quad (n_i(k) \to \infty). \tag{3.29}
\]

Since $T_k$ is demiclosed at zero, we have $A x^* \in F(T_k)$. By the arbitrariness of $k = 1, 2, \ldots, M$, it follows that $A x^* \in Q := \bigcap_{k=1}^M F(T_k)$. This together with $x^* \in C$ shows that $x^* \in \Gamma$, that is, $x^*$ is a solution to the problem (MSSFP).
Corollary 3.2. Let \( x_n \to x^* \) and \( u_n \to x^* \). In fact, assume that there exists another subsequence \( \{u_{n_j}\} \subset \{u_n\} \) such that \( u_{n_j} \to y^* \in \Gamma \) with \( y^* \neq x^* \). Consequently, by virtue of (3.2) and Opial’s property of Hilbert space, we have

\[
\liminf_{n_i \to \infty} \|u_{n_i} - x^*\| < \liminf_{n_j \to \infty} \|u_{n_j} - y^*\| = \lim_{n \to \infty} \|u_n - y^*\| = \lim_{n_i \to \infty} \|u_{n_i} - y^*\| < \liminf_{n_j \to \infty} \|u_{n_j} - x^*\| = \lim_{n \to \infty} \|u_n - x^*\| = \liminf_{n_i \to \infty} \|u_{n_i} - x^*\|. \tag{3.30}
\]

This is a contradiction. Therefore, \( u_n \to x^* \). By using (3.1) and (3.17), we have

\[
x_n = u_n - \gamma A^*(T_n^\beta - I)Ax_n \to x^*. \tag{3.31}
\]

Therefore, the conclusion (I) follows.

(2) Without loss of generality, we can assume that \( S_1 \) is semicompact. It follows from (3.27) that

\[
\|u_{n_i}(1) - S_1u_{n_i}(1)\| \to 0 \quad (n_i \to \infty). \tag{3.32}
\]

Therefore, there exists a subsequence of \( \{u_{n_i}(1)\} \) (for the sake of convenience, we still denote it by \( \{u_{n_i}(1)\} \)) such that \( u_{n_i}(1) \to u^* \in H \). Since \( u_{n_i}(1) \to x^*, x^* = u^* \) and so \( u_{n_i}(1) \to x^* \in \Gamma \). By virtue of (3.2), we know that

\[
\lim_{n \to \infty} \|u_n - x^*\| = 0, \quad \lim_{n \to \infty} \|x_n - x^*\| = 0, \tag{3.33}
\]

that is, \( \{u_n\} \) and \( \{x_n\} \) both converge strongly to the point \( x^* \in \Gamma \). This completes the proof. \( \Box \)

If we put \( \gamma = 0 \) in Theorem 3.1, we can get the following.

**Corollary 3.2.** Let \( H, C, L \) and \( \{k_n\} \) be the same as above and \( \{S_i\} \) a family of asymptotically nonexpansive mappings. Let \( \{x_n\} \) be the sequence generated by

\[
x_1 \in H_1 \text{ chosen arbitrarily},
\]

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S_n^n(x_n), \quad \forall n \geq 1,
\]

where \( S_n^n = S_{n \mod M}^n \) for all \( n \geq 1 \) and \( \{\alpha_n\} \) is a sequence in \([0,1]\) satisfying the following conditions.

(a) \( \alpha_n \in (\delta, 1 - \beta) \) for all \( n \geq 1 \), where \( \delta \in (0, 1 - \beta) \) is a positive constant.
(1) If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.

(2) In addition, if there exists a positive integer $j$ such that $S_j$ is semicompact, then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$.

The following theorem can be obtained from Theorem 3.1 immediately.

**Theorem 3.3.** Let $H_1$ and $H_2$ be two real Hilbert spaces, $A : H_1 \to H_2$ a bounded linear operator, $S_i : H_1 \to H_1$, $i = 1, 2, \ldots, M$, a uniformly $L_i$-Lipschitzian and $\beta_i$-strict pseudocontraction, and $T_i : H_2 \to H_2$, $i = 1, 2, \ldots, M$, a uniformly $\tilde{L}_i$-Lipschitzian and $\mu_i$-strict pseudocontraction satisfying the following conditions:

(a) $C := \bigcap_{i=1}^M F(S_i) \neq \emptyset$ and $Q := \bigcap_{i=1}^M F(T_i) \neq \emptyset$,

(b) $\beta = \max_{1 \leq i \leq M} \beta_i < 1$ and $\mu = \sup_{1 \leq i \leq M} \mu_i < 1$.

Let $\{x_n\}$ be the sequence generated by

$$
x_1 \in H_1 \text{ chosen arbitrarily,}
$$

$$
x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S_n(u_n),
$$

$$
\alpha_n = \gamma A^* (T_n - I)Ax_n, \quad \forall n \geq 1,
$$

(3.35)

where $S_n = S_{n(\text{mod } M)}$, $T_n = T_{n(\text{mod } M)}$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $0 < \gamma < 1$ is a constant. If $\Gamma \neq \emptyset$ and the following condition is satisfied:

(c) $\alpha_n \in (\delta, 1 - \beta)$ for all $n \geq 1$ and $\gamma \in (0, (1 - \mu)/\|A\|^2)$, where $\delta \in (0, 1 - \beta)$ is a constant,

then the sequence $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$. In addition, if there exists a positive integer $j$ such that $S_j$ is semicompact, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to the point $x^*$.

**Proof.** By the same way as given in the proof of Theorem 3.1 and using the case of strict pseudocontraction with the sequence $\{k_n = 1\}$, we can prove that, for each $p \in \Gamma$, the limits $\lim_{n \to \infty} \|x_n - p\|$ and $\lim_{n \to \infty} \|u_n - p\|$ exist,

$$
\|u_n - S_n u_n\| \to 0, \quad \|Ax_n - T_n Ax_n\| \to 0, \quad \|u_n - u_{n+1}\| \to 0, \quad \|x_n - x_{n+1}\| \to 0, \quad x_n \to x^*, \quad u_n \to x^* \in \Gamma.
$$

(3.36)

In addition, if there exists a positive integer $j$ such that $S_j$ is semicompact, we can also prove that $\{x_n\}$ and $\{u_n\}$ both converge strongly to the point $x^*$. This completes the proof. \(\square\)

If you put $S_i = T_i$ or $T_i = I$ (: the identity mapping) for each $i = 1, 2, \ldots, M$ in Theorem 3.3, then we have the following.

**Corollary 3.4.** Let $H$ be a real Hilbert space and $S_i : H \to H$, $i = 1, 2, \ldots, M$, a uniformly $L_i$-Lipschitzian and $\beta_i$-strict pseudocontraction satisfying the following conditions:

(a) $C := \bigcap_{i=1}^M F(S_i) \neq \emptyset$,

(b) $\beta = \max_{1 \leq i \leq M} \beta_i < 1$. 


Let \( \{x_n\} \) be the sequence generated by
\[
x_1 \in H_1 \text{ chosen arbitrarily},
\]
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S_n(x_n), \quad \forall n \geq 1,
\]
where \( S_n = S_n(\mod M) \) and \( \{\alpha_n\} \) is a sequence in \([0,1]\). If \( \Gamma \neq \emptyset \) and the following condition is satisfied:

(c) \( \alpha_n \in (\delta, 1 - \beta) \) for all \( n \geq 1 \), where \( \delta \in (0, 1 - \beta) \) is a constant,

then the sequence \( \{x_n\} \) converges weakly to a point \( x^* \in \Gamma \). In addition, if there exists a positive integer \( j \) such that \( S_j \) is semicompact, then the sequences \( \{x_n\} \) converges strongly to the point \( x^* \).

**Remark 3.5.** Theorems 3.1 and 3.3 improve and extend the corresponding results of Censor et al. \([1, 4, 5]\), Byrne \([2]\), Yang \([7]\), Moudafi \([12]\), Xu \([13]\), Censor and Segal \([14]\), Masad and Reich \([15]\), and others in the following aspects:

1. for the framework of spaces, we extend the space from finite dimension Hilbert space to infinite dimension Hilbert space;
2. for the mappings, we extend the mappings from nonexpansive mappings, quasi-nonexpansive mapping or demicontractive mappings to finite families of asymptotically strictly pseudocontractions;
3. for the algorithms, we propose some new hybrid iterative algorithms which are different from ones given in \([1, 2, 4, 5, 7, 14, 15]\). And, under suitable conditions, some weak and strong convergences for the algorithms are proved.

**Acknowledgments**

The authors would like to express their thanks to the referees for their helpful suggestions and comments. This work was supported by the Natural Science Foundation of Yunnan University of Economics and Finance and the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant no. 2011–0021821).

**References**

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