Research Article

Moments for Generating Functions of Al-Salam-Carlitz Polynomials

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We employ the moment representations for Al-Salam-Carlitz polynomials and show how to deduce bilinear, trilinear, and multilinear generating functions for Al-Salam-Carlitz polynomials. Moreover, we obtain two terminating generating functions for Al-Salam-Carlitz polynomials by the method of moments.

Dedicated to Professor H. M. Srivastava

1. Introduction

The Al-Salam-Carlitz polynomials [1, equations (4.1) and (4.2)]

$$\sum_{n=0}^{\infty} U_n^{(a)}(x) \frac{t^n}{(q; q)_n} = \frac{(t, at; q)_{\infty}}{(xt; q)_{\infty}}, \quad \sum_{n=0}^{\infty} V_n^{(a)}(x) \frac{t^n}{(q; q)_n} = \frac{(xt; q)_{\infty}}{(t, at; q)_{\infty}} \quad (1.1)$$

are important q -orthogonal polynomials whose applications varies in many fields such as the q -harmonic oscillator, theta function, quantum groups, and coding theory [2–4]. The generalized Al-Salam-Carlitz polynomials $U_n(x, y, a | q)$ [5, equation (1.2)] and $V_n(x, y, a | q)$ are defined by the following

$$\sum_{n=0}^{\infty} U_n(x, y, a | q) \frac{t^n}{(q; q)_n} = \frac{(at, yt; q)_{\infty}}{(xt; q)_{\infty}}, \quad \sum_{n=0}^{\infty} V_n(x, y, a | q) \frac{t^n}{(q; q)_n} = \frac{(xt; q)_{\infty}}{(at, yt; q)_{\infty}}, \quad (1.2)$$

where $U_n(x, 1, a | q) = U_n^{(a)}(x)$, $V_n(x, 1, a | q) = V_n^{(a)}(x)$ and

$$\begin{aligned} U_n(x, y, a | q) &= (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \frac{y}{x} \\ 0 \end{matrix} ; q, \frac{qx}{a} \right], \\ V_n(x, y, a | q) &= a^n {}_2\phi_0 \left[\begin{matrix} q^{-n}, \frac{x}{y} \\ - \end{matrix} ; q, \frac{yq^n}{a} \right]. \end{aligned} \quad (1.3)$$

There are close relationships between Al-Salam-Carlitz polynomials and other polynomials, such as q -Bessel polynomials [6], Stieltjes-Wigert polynomials [7], and Rogers-Szegő polynomials [5]. For more information, please refer to [5–7].

Al-Salam and Carlitz [1] defined moments of two discrete distribution $d\alpha^{(a)}(x)$ and $d\beta^{(a)}(x)$ by Rogers-Szegő polynomials as follows

$$\begin{aligned} \int_{-\infty}^{\infty} x^n d\alpha^{(a)}(x) &= h_n(a | q) \triangleq V_n(0, 1, a | q), \\ \int_{-\infty}^{\infty} x^n d\beta^{(a)}(x) &= g_n(a | q) \triangleq (-1)^n q^{\binom{n}{2}} U_n(0, 1, a | q), \end{aligned} \quad (1.4)$$

where $\alpha^{(a)}(x)$ is a step function whose jumps occur at the points q^k and aq^k for $k \in \mathbb{N}$, while the jumps of $\beta^{(a)}(x)$ occur at the points q^{-k} for $k \in \mathbb{N}$. These jumps are given by the following

$$\begin{aligned} d\alpha^{(a)}(q^k) &= \frac{q^k}{(a; q)_{\infty} (q, q/a; q)_k}, & d\alpha^{(a)}(aq^k) &= \frac{q^k}{(1/a; q)_{\infty} (q, aq; q)_k}, \\ d\beta^{(a)}(q^{-k}) &= \frac{a^k q^{k^2} (aq^{k+1}; q)_{\infty}}{(q; q)_k}. \end{aligned} \quad (1.5)$$

In view of the fact that the discrete probability measure $\alpha^{(a)}$ of moments in (1.4) on $[a, 1]$ may be given by the sum of two terms [8, equation (3.3)],

$$\alpha^{(a)} = \sum_{k=0}^{\infty} \left[\frac{q^k}{(a; q)_{\infty} (q, q/a; q)_k} \varepsilon_{q^k} + \frac{q^k}{(1/a; q)_{\infty} (q, aq; q)_k} \varepsilon_{aq^k} \right]. \quad (1.6)$$

However, $\beta^{(a)}$ is given by [8, equation (3.15)],

$$\beta^{(a)} = (aq; q)_{\infty} \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(aq, q; q)_k} \varepsilon_{q^{-k}}, \quad (1.7)$$

where $k \in \mathbb{N}$ and ε_y denotes a unit mass supported at y , see also [9, 10].

Liu gained the following expression of bivariate Rogers-Szegö polynomials by the technique of partial fraction [11, equation (4.20)],

$$h_n(a, b | q) = \frac{a^n}{(b/a; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(n+1)k}}{(q, aq/b; q)_k} + \frac{b^n}{(a/b; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(n+1)k}}{(q, qb/a; q)_k}. \quad (1.8)$$

So it's natural to define the generalized discrete probability measure $\alpha^{(a,b)}$ and $\beta^{(a,b)}$ by the following

$$\alpha^{(a,b)} = \sum_{k=0}^{\infty} \left[\frac{q^k}{(a/b; q)_\infty (q, qb/a; q)_k} \varepsilon_{bq^k} + \frac{q^k}{(b/a; q)_\infty (q, aq/b; q)_k} \varepsilon_{aq^k} \right], \quad (1.9)$$

$$\beta^{(a,b)} = \left(\frac{aq}{b}; q \right)_\infty \sum_{k=0}^{\infty} \frac{a^k b^{-k} q^{k^2}}{(aq/b, q; q)_k} \varepsilon_{bq^{-k}}, \quad (1.10)$$

where the bivariate Rogers-Szegö polynomials expressed by the following

$$\begin{aligned} h_n(a, b | q) &= \int_{-\infty}^{+\infty} x^n d\alpha^{(a,b)}(x) \triangleq V_n(0, a, b | q), \\ g_n(a, b | q) &= \int_{-\infty}^{+\infty} x^n d\beta^{(a,b)}(x) \triangleq U_n(0, a, b | q), \end{aligned} \quad (1.11)$$

and their generating functions given by [11, equations (2.3) and (2.13)],

$$\sum_{n=0}^{\infty} h_n(a, b | q) \frac{t^n}{(q; q)_n} = \frac{1}{(at, bt; q)_\infty}, \quad \sum_{n=0}^{\infty} g_n(a, b | q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = (at, bt; q)_\infty. \quad (1.12)$$

Al-Salam and Ismail [9] built relations between multilinear generating functions for Rogers-Szegö polynomials and ${}_8W_7$ by the method of moments. Berg and Ismail [8] gained orthogonality relations of generating functions for Rogers-Szegö polynomials by moments. Ismail and Masson [12] obtained several q -beta integrals and Bailey's ${}_6\psi_6$ from q -Hermite polynomials by moments. Ismail and Stanton [13, 14] derived generating functions for Al-Salam-Chihara and continuous q -Hermite polynomials by moments. For more information, please refer to [8, 9, 12–18].

There are many ways to explore generating functions for Al-Salam-Carlitz polynomials. Carlitz [19] used combinatorial method. Srivastava and Jain [20] utilized the technique of transformation theory. Kim [21] obtained combinatorial interpretation by moments. Wang [22] employed the q -integral representation. Chen et al. [5] applied q -exponential operator. For more information, please refer to [2, 5, 19–22].

One may ask naturally a question: can Al-Salam-Carlitz polynomials be expressed by moments and some related problems deduced by moments? In this paper, we would like to represent Al-Salam-Carlitz polynomials in terms of moments (see Theorem 2.3 below) and

show how to deduce generating functions for Al-Salam-Carlitz polynomials by the method of moments.

This paper is organized as follows. In Section 3, two bilinear generating functions for Al-Salam-Carlitz polynomials are obtained by the method of moments. In Section 4, a trilinear generating function for Al-Salam-Carlitz polynomials is gained and a corollary is achieved. In Section 5, a multilinear generating function for Al-Salam-Carlitz polynomials is supplied naturally. In Section 6, two miscellaneous generating functions for Al-Salam-Carlitz polynomials are given.

2. Notations and Moments for Al-Salam-Carlitz Polynomials

In this paper, we follow the notations and terminology in [23] and suppose that $0 < q < 1$. The q -series and its compact factorials are defined, respectively, by the following

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k) \quad (2.1)$$

and $(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n$, where m is a positive integer and n is a nonnegative integer or ∞ . In the context, convergence of basic hypergeometric series is no issue at all because they are the terminating q -series.

The q -Chu-Vandermonde formula reads that [23, equations (II.6) and (II.7)],

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, a \\ c \end{matrix} ; q, q \right] = \frac{(c/a; q)_n}{(c; q)_n} a^n, \quad {}_2\phi_1 \left[\begin{matrix} q^{-n}, a \\ c \end{matrix} ; q, \frac{cq^n}{a} \right] = \frac{(c/a; q)_n}{(c; q)_n}. \quad (2.2)$$

The q -difference operators D_a and θ_a defined by [23, 24],

$$D_a \{f(a)\} = \frac{f(a) - f(aq)}{a}, \quad \theta_a \{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \quad (2.3)$$

Before the proof of main results, we need the following lemmas.

Lemma 2.1 (q -Leibniz formula [25, 26]). *For $n \in \mathbb{N}$, one has*

$$\theta_a^n \{f(a)g(a)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \theta_a^k \{f(a)\} \theta_a^{n-k} \{g(aq^{-k})\}, \quad (2.4)$$

$$D_a^n \{f(a)g(a)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_a^k \{f(a)\} D_a^{n-k} \{g(aq^k)\}. \quad (2.5)$$

Lemma 2.2 (equations (III.12) and (III.13)). *For $n \in \mathbb{N}$, one has*

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix} ; q, q \right] = \frac{(e/c; q)_n c^n}{(e; q)_n} {}_3\phi_2 \left[\begin{matrix} q^{-n}, c, \frac{d}{b} \\ d, \frac{cq^{1-n}}{e} \end{matrix} ; q, \frac{bq}{e} \right], \quad (2.6)$$

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix} ; q, \frac{deq^n}{bc} \right] = \frac{(e/c; q)_n}{(e; q)_n} {}_3\phi_2 \left[\begin{matrix} q^{-n}, c, \frac{d}{b} \\ d, \frac{cq^{1-n}}{e} \end{matrix} ; q, q \right], \quad (2.7)$$

for $d \rightarrow 0$ in (2.6) and $d \rightarrow \infty$ in (2.7), one has

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, b, c \\ 0, e \end{matrix} ; q, q \right] = \frac{(e/c; q)_n c^n}{(e; q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, c \\ \frac{cq^{1-n}}{e} \end{matrix} ; q, \frac{bq}{e} \right], \quad (2.8)$$

$${}_3\phi_1 \left[\begin{matrix} q^{-n}, b, c \\ e \end{matrix} ; q, \frac{eq^n}{bc} \right] = \frac{(e/c; q)_n}{(e; q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, c \\ \frac{cq^{1-n}}{e} \end{matrix} ; q, \frac{q}{b} \right]. \quad (2.9)$$

Theorem 2.3. *For $n \in \mathbb{N}$, one has*

$$U_n(x, y, a | q) = \frac{(-1)^n q^{\binom{n}{2}}}{(qx/y, qx/a; q)_\infty} \int_{-\infty}^{\infty} \mu^n \left(\frac{qx\mu}{ay}; q \right) d\beta^{(a,y)}(\mu), \quad (2.10)$$

$$V_n(x, y, a | q) = \left(\frac{x}{y}, \frac{x}{a}; q \right)_\infty \int_{-\infty}^{\infty} \frac{\mu^n}{(x\mu/(ay); q)_\infty} d\alpha^{(a,y)}(\mu), \quad (2.11)$$

where $\alpha^{(y,a)}(\mu)$ and $\beta^{(y,a)}(\mu)$ are defined by (1.6) and (1.7).

Theorem 2.4. *For $f \in \mathbb{N}$ and $s/r = q^{-f}$, if $\max\{|ar|, |at|, |bt|, |aw|, |bw|\} < 1$, one has*

$$\int_{-\infty}^{\infty} \frac{(s\mu; q)_\infty}{(r\mu, t\mu, w\mu; q)_\infty} d\alpha^{(a,b)}(\mu) = \frac{(as, abtw; q)_\infty}{(ar, at, bt, aw, bw; q)_\infty} {}_3\phi_2 \left[\begin{matrix} \frac{s}{r}, at, aw \\ as, abtw \end{matrix} ; q, br \right]. \quad (2.12)$$

For $r/s = q^{-f}$, if $\max\{|as|, |abtw/q|\} < 1$, one has

$$\int_{-\infty}^{\infty} \frac{(r\mu, t\mu, w\mu; q)_\infty}{(s\mu; q)_\infty} d\beta^{(a,b)}(\mu) = \frac{(ar, at, bt, aw, bw; q)_\infty}{(as, abtw/q; q)_\infty} {}_3\phi_2 \left[\begin{matrix} \frac{r}{s}, \frac{q}{(at)}, \frac{q}{(aw)} \\ \frac{q}{as}, \frac{q^2}{abtw} \end{matrix} ; q, q \right]. \quad (2.13)$$

Proof of Theorem 2.3. By (1.11) and (1.12), we have

$$\int_{-\infty}^{\infty} \frac{1}{(t\mu; q)_{\infty}} d\alpha^{(a,b)}(\mu) = \frac{1}{(at, bt; q)_{\infty}}. \quad (2.14)$$

Differentiating by D_t^n and using q -Leibniz formula (2.5) for D_t on both sides of (2.14), we have

$$\int_{-\infty}^{\infty} \frac{\mu^n}{(t\mu; q)_{\infty}} d\alpha^{(a,b)}(\mu) = \frac{a^n}{(at, bt; q)_{\infty}} {}_2\phi_0 \left[\begin{matrix} q^{-n}, at \\ - \end{matrix} ; q, \frac{bq^n}{a} \right]. \quad (2.15)$$

Comparing (1.2), (1.3), and (2.15), we have (2.11). Similarly, by q -Leibniz formula (2.4) for θ_t , we obtain

$$\int_{-\infty}^{\infty} \mu^n (t\mu; q)_{\infty} d\beta^{(a,b)}(\mu) = a^n (at, bt; q)_{\infty} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \frac{q}{(at)} \\ 0 \end{matrix} ; q, bt \right], \quad (2.16)$$

so we achieve (2.10) by (1.3). The proof is complete. $\square$

Proof of Theorem 2.4. Multiplying $w^n / (q; q)_n$ and summing n over $0 \leq n \leq \infty$ on both sides of (2.15) yields

$$\int_{-\infty}^{\infty} \frac{1}{(t\mu, w\mu; q)_{\infty}} d\alpha^{(a,b)}(\mu) = \frac{(abt w; q)_{\infty}}{(at, bt, aw, bw; q)_{\infty}}. \quad (2.17)$$

Differentiating by D_t^n and using (2.5) on both sides of (2.17), we have

$$\begin{aligned} D_t^n \left\{ \frac{(abt w; q)_{\infty}}{(at, bt; q)_{\infty}} \right\} &= \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_t^k \left\{ \frac{(abt w; q)_{\infty}}{(bt; q)_{\infty}} \right\} D_t^{n-k} \left\{ \frac{1}{(at q^k; q)_{\infty}} \right\} \\ &= \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{b^k (aw; q)_k (abt w q^k; q)_{\infty}}{(bt; q)_{\infty}} \frac{(aq^k)^{n-k}}{(at q^k; q)_{\infty}}, \end{aligned} \quad (2.18)$$

that is

$$\int_{-\infty}^{\infty} \frac{\mu^n}{(t\mu, w\mu; q)_{\infty}} d\alpha^{(a,b)}(\mu) = \frac{a^n (abt w; q)_{\infty}}{(at, bt, aw, bw; q)_{\infty}} {}_3\phi_1 \left[\begin{matrix} q^{-n}, aw, at \\ abt w \end{matrix} ; q, \frac{bq^n}{a} \right]. \quad (2.19)$$

Multiplying $(s/r; q)_n r^n / (q; q)_n$ and summing on both sides of (2.19), we obtain (2.12) after simplification. Similarly, we have

$$\int_{-\infty}^{\infty} (t\mu, w\mu; q)_{\infty} d\beta^{(a,b)}(\mu) = \frac{(at, bt, aw, bw; q)_{\infty}}{(abt w / q; q)_{\infty}}, \quad (2.20)$$

by q -Leibniz formula for θ_t , we gain

$$\theta_t^n \left\{ \frac{(at, bt; q)_\infty}{(abt w / q; q)_\infty} \right\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{-\binom{k+1}{2}} (abw)^k (q/(aw); q)_k (bt; q)_\infty (-aq^{-k})^{n-k} (atq^{-k}; q)_\infty}{(abt w q^{-1-k}; q)_\infty}, \quad (2.21)$$

so we have

$$\int_{-\infty}^{\infty} \mu^n(t\mu, w\mu; q)_\infty d\beta^{(a,b)}(x) = \frac{a^n (aw, bw, at, bt; q)_\infty}{(abt w / q; q)_\infty} {}_3\phi_2 \left[\begin{matrix} q^{-n}, \frac{q}{(aw)}, \frac{q}{(at)} \\ \frac{q^2}{(abt w)}, 0 \end{matrix}; q, q \right], \quad (2.22)$$

which is (2.13) after multiplying $(r/s; q)_n s^n / (q; q)_n$ and summing on both sides of (2.22). The proof is complete. $\square$

3. Bilinear Generating Function for Al-Salam-Carlitz Polynomials

Chen et al. [5] gave the following bilinear generating functions for Al-Salam-Carlitz polynomials.

Theorem 3.1 (see [5, Theorem 5.3]). *Assume that $\max\{|xq/a|, |uq/b|\} < 1$. One has*

$$\begin{aligned} & \sum_{k=0}^{\infty} (-1)^{m+n+k} q^{-\binom{n+k}{2} - \binom{m}{2} - mk} U_{n+k}(x, y, a | q) U_{m+k}(u, v, b | q) \frac{z^k}{(q; q)_k} \\ &= \frac{(yq/a, vq/b, abz; q)_\infty a^n b^m}{(xq/a, uq/b; q)_\infty} \sum_{k=0}^{\infty} \frac{(y/x, q/(abz); q)_k}{(q, yq/a; q)_k} \left(\frac{xbz}{q^n} \right)^k \\ & \quad \times {}_2\phi_1 \left[\begin{matrix} \frac{v}{u}, \frac{q^{1+k}}{(abz)} \\ \frac{vq}{b} \end{matrix}; q, \frac{auz}{q^{m+k}} \right], \end{aligned} \quad (3.1)$$

where $y/x = q^{-f}$ and $v/u = q^{-g}$ for nonnegative integers f, g and $|auzq^{-(m+k)}| < 1$.

In this section, we gain (3.1) and the following dual ones by the method of moments.

Theorem 3.2. Assume that $\max\{|yq/a|, |vq/b|\} < 1$. One has

$$\begin{aligned} & \sum_{k=0}^{\infty} V_{n+k}(x, y, a | q) V_{m+k}(u, v, b | q) \frac{(zq)^k}{(q; q)_k} \\ &= \frac{(x/a, u/b; q)_{\infty} a^n b^m}{(y/a, abz, v/b; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(x/y, abz; q)_k q^{(1+n)k}}{(q, aq/y; q)_k} \\ & \quad \times {}_2\phi_1 \left[\begin{matrix} \frac{u}{v}, abzq^k \\ \frac{bq}{v} \end{matrix} ; q, q^{1+m} \right], \end{aligned} \quad (3.2)$$

where $x/y = q^{-f}$ and $u/v = q^{-g}$ for nonnegative integers f, g .

Corollary 3.3 (see [5, Theorem 4.5]). For $f \in \mathbb{N}$, one has

$$\begin{aligned} & \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k}{2}} U_k(x, y, a | q) U_k(u, v, b | q) \frac{(zq)^k}{(q; q)_k} \\ &= \frac{(abzq, avzq, yq/a; q)_{\infty}}{(auzq, xq/a; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} \frac{y}{x}, \frac{1}{(abz)', (avz)'} \\ \frac{1}{(auz)', \frac{yq}{a}} \end{matrix} ; q, \frac{xbvzq}{u} \right], \end{aligned} \quad (3.3)$$

provided that $y/x = q^{-f}$ or $v/u = q^{-f}$ and $\max\{|auzq^{-f}|, |xq/a|, |xbvz/u|\} < 1$.

Corollary 3.4. For $f \in \mathbb{N}$, one has

$$\begin{aligned} & \sum_{k=0}^{\infty} V_k(x, y, a | q) V_k(u, v, b | q) \frac{z^k}{(q; q)_k} \\ &= \frac{(auz, x/a; q)_{\infty}}{(abz, avz, y/a; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} abz, \frac{x}{y}, avz \\ auz, \frac{aq}{y} \end{matrix} ; q, q \right], \end{aligned} \quad (3.4)$$

provided that $x/y = q^{-f}$ or $u/v = q^{-f}$ and $\max\{|abz|, |avz|, |y/a|\} < 1$.

Remark 3.5. For $m = n = 0$, formula (3.1) and (3.2) reduce to (3.3) and (3.4), respectively, by q -Gauss sum [23, equation (II.8)],

$${}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, \frac{c}{ab} \right] = \frac{(c/a, c/b; q)_{\infty}}{(c, c/(ab); q)_{\infty}} \quad (3.5)$$

and q -Chu-Vandermonde formula (2.2).

Proof of Theorems 3.1 and 3.2. The left-hand side of (3.1) is equal to

$$\begin{aligned}
& \sum_{k=0}^{\infty} (-1)^{m+n+k} q^{-\binom{n+k}{2} - \binom{m}{2} - mk} \frac{(-1)^{n+k} q^{\binom{n+k}{2}}}{(qx/y, qx/a; q)_{\infty}} \int_{-\infty}^{\infty} \mu^{n+k} \left(\frac{qx\mu}{(ay)}; q \right)_{\infty} d\beta^{(a,y)}(\mu) \\
& \times \frac{(-1)^{m+k} q^{\binom{m+k}{2}}}{(qu/v, qu/b; q)_{\infty}} \int_{-\infty}^{\infty} w^{m+k} \left(\frac{quw}{(bv)}; q \right)_{\infty} d\beta^{(b,v)}(w) \frac{z^k}{(q; q)_k} \\
& = \frac{1}{(qx/y, qx/a, qu/v, qu/b; q)_{\infty}} \iint_{-\infty}^{\infty} \mu^n w^m \left(\frac{qx\mu}{(ay)}, \frac{quw}{(bv)}, \mu wz; q \right)_{\infty} d\beta^{(a,y)}(\mu) d\beta^{(b,v)}(w) \\
& = \frac{1}{(qu/v, qu/b; q)_{\infty}} \int_{-\infty}^{\infty} \left\{ \frac{1}{(qx/y, qx/a; q)_{\infty}} \int_{-\infty}^{\infty} \mu^n \left(\frac{qx\mu}{(ay)}, \mu wz; q \right)_{\infty} d\beta^{(a,y)}(\mu) \right\} \\
& \times w^m \left(\frac{quw}{(bv)}; q \right)_{\infty} d\beta^{(b,v)}(w) \\
& = \frac{1}{(qu/v, qu/b; q)_{\infty}} \int_{-\infty}^{\infty} \left\{ \frac{a^n (awz, ywz; q)_{\infty}}{(xwz; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} q^{-n}, \frac{q}{(awz)}, \frac{y}{x} \\ \frac{q}{(xwz)}, 0 \end{matrix}; q, q \right] \right\} \text{ by (2.8)} \\
& \times w^m \left(\frac{quw}{(bv)}; q \right)_{\infty} d\beta^{(b,v)}(w) \\
& = \frac{1}{(qu/v, qu/b; q)_{\infty}} \int_{-\infty}^{\infty} \left\{ \frac{a^n (awz, ywz; q)_{\infty}}{(xwz; q)_{\infty}} \frac{(qy/a, xwz; q)_{\infty}}{(qx/a, ywz; q)_{\infty}} \right. \\
& \quad \left. \times {}_2\phi_1 \left[\begin{matrix} \frac{y}{x}, \frac{q}{(awz)} \\ \frac{yq}{a} \end{matrix}; q, \frac{xwz}{q^n} \right] \right\} \\
& \times w^m \left(\frac{quw}{(bv)}; q \right)_{\infty} d\beta^{(b,v)}(w) \\
& = \frac{a^n (qy/a; q)_{\infty}}{(qu/v, qu/b, qx/a; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(y/x; q)_k (-1)^k q^{\binom{k+1}{2}}}{(q, yq/a; q)_k} \left(\frac{x}{aq^n} \right)^k \\
& \times \int_{-\infty}^{\infty} w^m \left(\frac{quw}{(bv)}, awzq^{-k}; q \right)_{\infty} d\beta^{(b,v)}(w) \text{ by (2.22)} \\
& = \frac{a^n (qy/a; q)_{\infty}}{(qu/v, qu/b, qx/a; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(y/x; q)_k (-1)^k q^{\binom{k+1}{2}}}{(q, yq/a; q)_k} \left(\frac{x}{aq^n} \right)^k \\
& \times \frac{b^m (qu/b, qu/v, abzq^{-k}, avzq^{-k}; q)_{\infty}}{(auzq^{-k}; q)_{\infty}} \times {}_3\phi_2 \left[\begin{matrix} q^{-m}, \frac{q^{1+k}}{(abz)}, \frac{v}{u} \\ \frac{q^{1+k}}{(auz)}, 0 \end{matrix}; q, q \right] \text{ by (2.8)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a^n(qy/a; q)_\infty}{(qu/v, qu/b, qx/a; q)_\infty} \sum_{k=0}^{\infty} \frac{(y/x; q)_k (-1)^k q^{\binom{k+1}{2}}}{(q, yq/a; q)_k} \left(\frac{x}{aq^n} \right)^k \\
&\quad \times \frac{b^m(qu/b, qu/v, abzq^{-k}, avzq^{-k}; q)_\infty}{(auzq^{-k}; q)_\infty} \frac{(qv/b, auzq^{-k}; q)_\infty}{(uq/b, avzq^{-k}; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{v}{u}, \frac{q^{1+k}}{abz} \\ \frac{qv}{b} \end{matrix} ; q, \frac{auz}{q^{m+k}} \right], \\
&\hspace{25em} (3.6)
\end{aligned}$$

which is the right hand side of (3.1) after simplification. Similarly, the right hand side of (3.2) is equivalent to

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left(\frac{x}{y}, \frac{x}{a}; q \right)_\infty \int_{-\infty}^{\infty} \frac{\mu^{n+k}}{(x\mu/(ay); q)_\infty} d\alpha^{(a,y)}(\mu) \left(\frac{u}{v}, \frac{u}{b}; q \right)_\infty \\
&\quad \times \int_{-\infty}^{\infty} \frac{w^{m+k}}{(uw/(bv); q)_\infty} d\alpha^{(b,v)}(w) \frac{(zq)^k}{(q; q)_k} \\
&= \left(\frac{u}{v}, \frac{u}{b}; q \right)_\infty \int_{-\infty}^{\infty} \left\{ \left(\frac{x}{y}, \frac{x}{a}; q \right)_\infty \int_{-\infty}^{\infty} \frac{\mu^n}{(x\mu/(ay), \mu wz; q)_\infty} d\alpha^{(a,y)}(\mu) \right\} \\
&\quad \times \frac{w^m}{(uw/(bv); q)_\infty} d\alpha^{(b,v)}(w) \\
&= \left(\frac{u}{v}, \frac{u}{b}; q \right)_\infty \int_{-\infty}^{\infty} \frac{a^n(xwz; q)_\infty}{(awz, ywz; q)_\infty} {}_3\phi_1 \left[\begin{matrix} q^{-n}, awz, \frac{x}{y} \\ xwz \end{matrix} ; q, \frac{yq^n}{a} \right] \\
&\quad \times \frac{w^m}{(uw/(bv); q)_\infty} d\alpha^{(b,v)}(w) \text{ by (2.9)} \\
&= \left(\frac{u}{v}, \frac{u}{b}; q \right)_\infty \int_{-\infty}^{\infty} \frac{a^n(xwz; q)_\infty}{(awz, ywz; q)_\infty} \frac{(x/a, ywz; q)_\infty}{(xwz, y/a; q)_\infty} {}_2\phi_1 \left[\begin{matrix} \frac{x}{y}, awz \\ \frac{aq}{y} \end{matrix} ; q, q^{1+n} \right] \\
&\quad \times \frac{w^m}{(uw/(bv); q)_\infty} d\alpha^{(b,v)}(w) \\
&= \frac{a^n(u/v, u/b, x/a; q)_\infty}{(y/a; q)_\infty} \sum_{k=0}^{\infty} \frac{(x/y; q)_k q^{(1+n)k}}{(q, aq/y; q)_k} \\
&\quad \times \int_{-\infty}^{\infty} \frac{w^m}{(uw/(bv), awzq^k; q)_\infty} d\alpha^{(b,v)}(w) \text{ by (2.19)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a^n(u/v, u/b, x/a; q)_\infty}{(y/a; q)_\infty} \sum_{k=0}^{\infty} \frac{(x/y; q)_k q^{(1+n)k}}{(q, aq/y; q)_k} \frac{b^m(auzq^k; q)_\infty}{(u/v, u/b, abzq^k, avzq^k; q)_\infty} \\
&\quad \times {}_3\phi_1 \left[\begin{matrix} q^{-m}, abzq^k, \frac{u}{v} \\ auzq^k \end{matrix} ; q, \frac{vq^m}{b} \right],
\end{aligned} \tag{3.7}$$

which is the right hand side of (3.2) after using (2.9). The proof is complete. $\square$

4. Trilinear Generating Function for Al-Salam-Carlitz Polynomials

Verma and Jain deduced the following trilinear generating functions for Rogers-Szegő polynomials.

Proposition 4.1 (see [27, equation (2.14)]). *For $s \in \mathbb{N}$, one has*

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_{n+s}(x|q) h_n(y|q) h_n(z|q) \frac{w^n}{(q; q)_n} \\
&= \frac{(yzw^2; q)_\infty}{(x, w, yw, zw, yzw; q)_\infty} {}_6\phi_5 \left[\begin{matrix} w, yw, zw, yzw, 0, 0 \\ \frac{q}{x}, w\sqrt{yz}, -w\sqrt{yz}, w\sqrt{yzq}, -w\sqrt{yzq} \end{matrix} ; q, q^{1+s} \right] \\
&\quad + \frac{x^s(yzx^2w^2; q)_\infty}{(1/x, xw, xyw, xzw, xyzw; q)_\infty} \\
&\quad \times {}_6\phi_5 \left[\begin{matrix} xw, xyw, xzw, xyzw, 0, 0 \\ xq, xw\sqrt{yz}, -xw\sqrt{yz}, xw\sqrt{yzq}, -xw\sqrt{yzq} \end{matrix} ; q, q^{1+s} \right].
\end{aligned} \tag{4.1}$$

In this section, we obtain the following trilinear generating functions for Al-Salam-Carlitz polynomials.

Theorem 4.2. *For $k \in \mathbb{N}$, one has*

$$\begin{aligned}
&\sum_{n=0}^{\infty} V_n(x, y, a|q) V_n(u, v, b|q) V_{n+k}(r, s, c|q) \frac{w^n}{(q; q)_n} \\
&= \frac{(r/s, auws, bxws; q)_\infty s^k}{(c/s, abws, avws, byws; q)_\infty} \sum_{j=0}^{\infty} \frac{(r/c, abws, avws, byws; q)_j q^{(1+k)j}}{(q, qs/c, auws, bxws; q)_j} \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} \frac{x}{y}, \frac{u}{v}, abwsq^j \\ auwsq^j, bxwsq^j \end{matrix} ; q, yvwsq^j \right] + \frac{(r/c, auwc, bxwc; q)_\infty c^k}{(s/c, abwc, avwc, bywc; q)_\infty} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(r/s, abwc, avwc, bywc; q)_\infty q^{(1+k)j}}{(q, qc/s, auwc, bxwc; q)_j} {}_3\phi_2 \left[\begin{matrix} \frac{x}{y}, \frac{u}{v}, abwcq^j \\ auwcq^j, bxwcq^j \end{matrix} ; q, yvwcq^j \right],
\end{aligned} \tag{4.2}$$

provided that the right hand side of (4.2) is convergent.

Remark 4.3. Ismail and Stanton [14] developed a method for deriving integral representations of certain orthogonal polynomials as moments and obtained [14, equation (4.13)] trilinear generating functions for q -Hermite polynomials $H_n(\cos \theta \mid q)$, which is equivalent to Proposition 4.1. For more information, please refer to [14].

Proof of Theorem 4.2. The left hand side of (4.2) is equal to

$$\begin{aligned}
& \left(\frac{r}{s}, \frac{r}{c}; q \right) \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mu^{n+k}}{(r\mu/(cs); q)_{\infty}} d\alpha^{(c,s)}(\mu) V_n(x, y, a \mid q) V_n(u, v, b \mid q) \frac{w^n}{(q; q)_n} \\
&= \left(\frac{r}{s}, \frac{r}{c}; q \right) \int_{-\infty}^{\infty} \frac{\mu^k}{(r\mu/(cs); q)_{\infty}} \sum_{n=0}^{\infty} V_n(x, y, a \mid q) V_n(u, v, b \mid q) \frac{(w\mu)^n}{(q; q)_n} d\alpha^{(c,s)}(\mu) \text{ by (3.4)} \\
&= \left(\frac{r}{s}, \frac{r}{c}; q \right) \int_{-\infty}^{\infty} \frac{\mu^k (auw\mu, x/a; q)_{\infty}}{(r\mu/(cs), abw\mu, avw\mu, y/a; q)_{\infty}} \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} abw\mu, \frac{x}{y}, avw\mu \\ auw\mu, \frac{aq}{y} \end{matrix}; q, q \right] d\alpha^{(c,s)}(\mu) \\
&= \left(\frac{r}{s}, \frac{r}{c}; q \right) \int_{-\infty}^{\infty} \frac{\mu^k (auw\mu, x/a; q)_{\infty}}{(r\mu/(cs), abw\mu, avw\mu, y/a; q)_{\infty}} \\
&\quad \times \frac{(bxw\mu, y/a; q)_{\infty}}{(byw\mu, x/a; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} \frac{x}{y}, \frac{u}{v}, abw\mu \\ auw\mu, bxw\mu \end{matrix}; q, yvw\mu \right] d\alpha^{(c,s)}(\mu) \text{ by (1.9)} \\
&= \left(\frac{r}{s}, \frac{r}{c}; q \right) \sum_{j=0}^{\infty} \frac{q^j}{(c/s; q)_{\infty} (q, qs/c; q)_j} \frac{(sq^j)^k (auwsq^j, bxwsq^j; q)_{\infty}}{(rq^j/c, abwsq^j, avwsq^j, bywsq^j; q)_{\infty}} \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} \frac{x}{y}, \frac{u}{v}, abwsq^j \\ auwsq^j, bxwsq^j \end{matrix}; q, yvwsq^j \right] + \left(\frac{r}{s}, \frac{r}{c}; q \right) \sum_{j=0}^{\infty} \frac{q^j}{(s/c; q)_{\infty} (q, qc/s; q)_j} \\
&\quad \times \frac{(cq^j)^k (auwcq^j, bxwcq^j; q)_{\infty}}{(rq^j/s, abwcq^j, avwcq^j, bywcq^j; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} \frac{x}{y}, \frac{u}{v}, abwcq^j \\ auwcq^j, bxwcq^j \end{matrix}; q, yvwcq^j \right], \tag{4.3}
\end{aligned}$$

which is the right hand of (4.2) after simplification. The proof is complete. $\square$

Proof of Proposition 4.1. Noting the fact that $V_n(0, y, 1 \mid q) = h_n(y \mid q)$ and

$$\begin{aligned}
(vw^2y; q)_{2j} &= (w\sqrt{vy}, -w\sqrt{vy}, w\sqrt{vyq}, -w\sqrt{vyq}; q)_j, \\
(s^2vw^2y; q)_{2j} &= (sw\sqrt{vy}, -sw\sqrt{vy}, sw\sqrt{vyq}, -sw\sqrt{vyq}; q)_j. \tag{4.4}
\end{aligned}$$

If letting $x = u = r = 0$ and $a = b = c = 1$ in (4.2), the right hand side of (4.2) is equal to

$$\begin{aligned} & \frac{s^k (s^2 v w^2 y; q)_\infty}{(1/s, ws, vws, yws, yvws; q)_\infty} \sum_{j=0}^{\infty} \frac{(ws, vws, yws, yvws; q)_j q^{(1+k)j}}{(q, qs; q)_j (s^2 v w^2 y; q)_{2j}} \\ & + \frac{(v w^2 y; q)_\infty}{(s, w, vw, yw, yvw; q)_\infty} \sum_{j=0}^{\infty} \frac{(w, vw, yw, yvw; q)_j}{(q, q/s; q)_j (v w^2 y; q)_{2j}}, \end{aligned} \quad (4.5)$$

which reduces to the right hand side of (4.1) after setting $(s, v) = (x, z)$. The proof is complete. $\square$

5. Multilinear Generating Functions for Al-Salam-Carlitz Polynomials

Ismail and Stanton gained the following multilinear generating functions for Rogers-Szegő polynomials.

Proposition 5.1 (see [28, equation (5.14)]). *One has*

$$\begin{aligned} & \sum_{m_1, \dots, m_k=0}^{\infty} h_{m_1+\dots+m_k}(a | q) h_{m_1}\left(\frac{t_1}{t_2} | q\right) \cdots h_{m_k}\left(\frac{t_{2k-1}}{t_{2k}} | q\right) \prod_{j=1}^k \frac{t_{2j}^{m_j}}{(q; q)_{m_j}} \\ & = {}_{2k}\phi_{2k-1} \frac{\left[\begin{smallmatrix} t_1, \dots, t_{2k} \\ q/a, 0, \dots, 0 \end{smallmatrix}; q, q \right]}{\left\{ (a; q)_\infty \prod_{j=1}^{2k} (t_j; q)_\infty \right\}} + {}_{2k}\phi_{2k-1} \frac{\left[\begin{smallmatrix} at_1, \dots, at_{2k} \\ aq, 0, \dots, 0 \end{smallmatrix}; q, q \right]}{\left\{ (1/a; q)_\infty \prod_{j=1}^{2k} (at_j; q)_\infty \right\}}. \end{aligned} \quad (5.1)$$

In this section, we obtain the following multilinear generating functions for Al-Salam-Carlitz polynomials.

Theorem 5.2. *For $s \in \mathbb{N}$, one has*

$$\begin{aligned} & \sum_{m_1, \dots, m_k=0}^{\infty} V_{m_1+\dots+m_k+s}(y, x, a | q) V_{m_1}(y_1, x_1, a_1 | q) \cdots V_{m_k}(y_k, x_k, a_k | q) \frac{t_1^{m_1} \cdots t_k^{m_k}}{(q; q)_{m_1} \cdots (q; q)_{m_k}} \\ & = \frac{a^s (y/a, y_1 t_1 a, \dots, y_k t_k a; q)_\infty}{(x/a, a_1 t_1 a, x_1 t_1 a, \dots, a_k t_k a, x_k t_k a; q)_\infty} \\ & \quad \times {}_{2k+1}\phi_{2k} \left[\begin{smallmatrix} \frac{y}{x}, a_1 t_1 a, x_1 t_1 a, \dots, a_k t_k a, x_k t_k a \\ \frac{aq}{x}, y_1 t_1 a, \dots, y_k t_k a \end{smallmatrix}; q, q^{1+s} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{x^s (y/x, y_1 t_1 x, \dots, y_k t_k x; q)_\infty}{(a/x, a_1 t_1 x, x_1 t_1 x, \dots, a_k t_k x, x_k t_k x; q)_\infty} \\
& \times {}_{2k+1}\phi_{2k} \left[\begin{matrix} \frac{y}{a}, a_1 t_1 x, x_1 t_1 x, \dots, a_k t_k x, x_k t_k x \\ \frac{xq}{a}, y_1 t_1 x, \dots, y_k t_k x \end{matrix} ; q, q^{1+s} \right],
\end{aligned} \tag{5.2}$$

provided that the right hand side of (5.2) is convergent.

Corollary 5.3 (see [20, equation (1.12)]). For $s \in \mathbb{N}$, one has

$$\begin{aligned}
& \sum_{n_1, \dots, n_k=0}^{\infty} h_{n_1+\dots+n_k+s}(x|q) h_{n_1}(x_1|q) \dots h_{n_k}(x_k|q) \frac{t_1^{n_1}}{(q; q)_{n_1}} \dots \frac{t_k^{n_k}}{(q; q)_{n_k}} \\
& = \frac{1}{(x, t_1, x_1 t_1, \dots, t_k, x_k t_k; q)_\infty} {}_{2k}\phi_{2k-1} \left[\begin{matrix} t_1, x_1 t_1, \dots, t_k, x_k t_k \\ \frac{q}{x}, 0, \dots, 0 \end{matrix} ; q, q^{1+s} \right] \\
& + \frac{x^s}{(1/x, x t_1, x x_1 t_1, \dots, x t_k, x x_k t_k; q)_\infty} {}_{2k}\phi_{2k-1} \left[\begin{matrix} x t_1, x x_1 t_1, \dots, x t_k, x x_k t_k \\ q x, 0, \dots, 0 \end{matrix} ; q, q^{1+s} \right].
\end{aligned} \tag{5.3}$$

Remark 5.4. Letting $y = y_1 = \dots = y_k = s = 0$, $a = a_1 = \dots = a_k = 1$ and taking $(x, x_i, t_i) = (a, t_i/t_{i+1}, t_{2i})$ in Theorem 5.2, formula (5.2) reduces to (5.1). Setting $y = y_1 = \dots = y_k = 0$ and $a = a_1 = \dots = a_k = 1$ in Theorem 5.2, formula (5.2) reduces to (5.3).

Proof. The left hand side of (5.2) is equal to

$$\begin{aligned}
& \left(\frac{y}{x}, \frac{y}{a}; q \right)_\infty \int_{-\infty}^{\infty} \sum_{m_1, \dots, m_k=0}^{\infty} V_{m_1}(y_1, x_1, a_1|q) \dots V_{m_k}(y_k, x_k, a_k|q) \\
& \times \frac{(t_1 \mu)^{m_1} \dots (t_k \mu)^{m_k}}{(q; q)_{m_1} \dots (q; q)_{m_k}} \frac{\mu^s d\alpha^{(a,x)}(\mu)}{(y\mu/(ax); q)_\infty} \\
& = \left(\frac{y}{x}, \frac{y}{a}; q \right)_\infty \int_{-\infty}^{\infty} \frac{\mu^s (y_1 t_1 \mu, \dots, y_k t_k \mu; q)_\infty}{(a_1 t_1 \mu, x_1 t_1 \mu, \dots, a_k t_k \mu, x_k t_k \mu, y\mu/(ax); q)_\infty} d\alpha^{(a,x)}(\mu) \\
& = \left(\frac{y}{x}, \frac{y}{a}; q \right)_\infty \left[\sum_{r=0}^{\infty} \frac{q^r}{(x/a; q)_\infty (q, qa/x; q)_r} \right. \\
& \quad \times \frac{(aq^r)^s (y_1 t_1 aq^r, \dots, y_k t_k aq^r; q)_\infty}{(a_1 t_1 aq^r, x_1 t_1 aq^r, \dots, a_k t_k aq^r, x_k t_k aq^r, yq^r/x; q)_\infty} \\
& \quad + \sum_{r=0}^{\infty} \frac{q^r}{(a/x; q)_\infty (q, qx/a; q)_r} \\
& \quad \times \left. \frac{(xq^r)^s (y_1 t_1 xq^r, \dots, y_k t_k xq^r; q)_\infty}{(a_1 t_1 xq^r, x_1 t_1 xq^r, \dots, a_k t_k xq^r, x_k t_k xq^r, yq^r/a; q)_\infty} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{a^s(y/a, y_1 t_1 a, \dots, y_k t_k a; q)_\infty}{(x/a, a_1 t_1 a, x_1 t_1 a, \dots, a_k t_k a, x_k t_k a; q)_\infty} \\
&\quad \times {}_{2k+1}\phi_{2k} \left[\begin{matrix} \frac{y}{x}, a_1 t_1 a, x_1 t_1 a, \dots, a_k t_k a, x_k t_k a \\ \frac{aq}{x}, y_1 t_1 a, \dots, y_k t_k a \end{matrix} ; q, q^{1+s} \right] \\
&\quad + \frac{x^s(y/x, y_1 t_1 x, \dots, y_k t_k x; q)_\infty}{(a/x, a_1 t_1 x, x_1 t_1 x, \dots, a_k t_k x, x_k t_k x; q)_\infty}, \\
&\quad \times {}_{2k+1}\phi_{2k} \left[\begin{matrix} \frac{y}{a}, a_1 t_1 x, x_1 t_1 x, \dots, a_k t_k x, x_k t_k x \\ \frac{xq}{a}, y_1 t_1 x, \dots, y_k t_k x \end{matrix} ; q, q^{1+s} \right],
\end{aligned} \tag{5.4}$$

which is the right hand side of (5.2) after simplification. The proof is complete. $\square$

6. Some Miscellaneous Generating Functions for Al-Salam-Carlitz Polynomials

In this section, we deduce the following terminating generating functions for Al-Salam-Carlitz polynomials.

Theorem 6.1. For $n \in \mathbb{N}$, one has

$$\begin{aligned}
&\sum_{k=0}^n U_k(cq^{k-1}, d, b \mid q) \frac{(-1)^k q^{nk - \binom{k}{2}} (q^{-n}, a; q)_k \left(\frac{c}{abd}\right)^k}{(q, c/d, c/b; q)_k} \\
&= \frac{(c/(ad); q)_n}{(c/d, c/b; q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \frac{dq^{1-n}}{c} \\ \frac{adq^{1-n}}{c} \end{matrix} ; q, \frac{cq^n}{b} \right].
\end{aligned} \tag{6.1}$$

Theorem 6.2. For $n \in \mathbb{N}$, one has

$$\begin{aligned}
&\sum_{k=0}^n V_{n-k}(aq^k, d, b \mid q) \frac{(q^{-n}, a/d, a/b; q)_k \left(\frac{bcdq^n}{a}\right)^k}{(q, c; q)_k} \\
&= \frac{(cd/a; q)_n b^n}{(c; q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \frac{a}{d} \\ \frac{aq^{1-n}}{cd} \end{matrix} ; q, \frac{aq}{bc} \right].
\end{aligned} \tag{6.2}$$

Corollary 6.3. For $n \in \mathbb{N}$, one has

$$\begin{aligned} \sum_{k=0}^n g_k(d|q) \frac{(q^{-n};q)_k}{(q;q)_k} \left(\frac{cq^n}{d}\right)^k &= \left(\frac{c}{d};q\right)_n {}_1\phi_1 \left[\begin{matrix} q^{-n} \\ \frac{dq^{1-n}}{c} \end{matrix} ; q, dq \right], \\ \sum_{k=0}^n h_{n-k}(d|q) \frac{(q^{-n};q)_k}{(q;q)_k} (cdq^n)^k &= (cd;q)_n {}_2\phi_1 \left[\begin{matrix} q^{-n}, 0 \\ \frac{q^{1-n}}{(cd)} \end{matrix} ; q, \frac{q}{c} \right]. \end{aligned} \quad (6.3)$$

Remark 6.4. Letting $b \rightarrow \infty$ in Theorems 6.1 and 6.2, noting the fact that

$$\lim_{b \rightarrow \infty} (-1)^k q^{-\binom{k}{2}} b^{-k} U_k(cq^{k-1}, d, b|q) = \lim_{b \rightarrow \infty} b^{k-n} V_{n-k}(aq^k, d, b|q) = 1. \quad (6.4)$$

Formula (6.1) and (6.2) reduce to q -Chu-Vandermonde formula (2.2), respectively. Replacing c by ac and letting $(a, b) = (0, 1)$ in Theorems 6.1 and 6.2, formula (6.1) and (6.2) reduce to (6.3), respectively.

Proof of Theorems 6.1 and 6.2. Replacing c by $c\mu/(bd)$ on both sides of the second formula in (2.2), we have

$$\sum_{k=0}^n \frac{(q^{-n}, a; q)_k}{(q; q)_k} \left(\frac{cq^n}{abd}\right)^k \cdot \mu^k \left(\frac{c\mu q^k}{(bd)}; q\right)_\infty = \frac{(c\mu/(abd), c\mu q^n/(bd); q)_\infty}{(c\mu q^n/(bd); q)_\infty}. \quad (6.5)$$

Applying moments $\int_{-\infty}^{\infty} d\beta^{(b,d)}(\mu)$ in (6.5), using (2.10) and (2.13), we gain

$$\begin{aligned} \sum_{k=0}^n \frac{(q^{-n}, a; q)_k}{(q; q)_k} \left(\frac{cq^n}{abd}\right)^k (-1)^k q^{-\binom{k}{2}} \left(\frac{cq^k}{d}, \frac{cq^k}{b}; q\right)_\infty U_k(cq^{k-1}, d, b|q) \\ = \frac{(c/(ad), cq^n/d, cq^n/b; q)_\infty}{(cq^n/(ad); q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \frac{dq^{1-n}}{c} \\ \frac{adq^{1-n}}{c} \end{matrix} ; q, \frac{cq^n}{b} \right], \end{aligned} \quad (6.6)$$

which is (6.1) after simplification. Similarly, setting a by $aw/(bd)$ on both sides of the second formula in (2.2), we obtain

$$\begin{aligned} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q, c; q)_k} \left(\frac{bcdq^n}{a}\right)^k \cdot \frac{w^{n-k}}{(awq^k/(bd); q)_\infty} \\ = \frac{(-1)^n q^{\binom{n}{2}} (bcd/a)^n}{(c; q)_n} \cdot \frac{(awq^{1-n}/(bcd); q)_\infty}{(awq/(bcd), aw/(bd); q)_\infty}. \end{aligned} \quad (6.7)$$

Utilizing moments $\int_{-\infty}^{\infty} d\alpha^{(b,d)}(w)$ in (6.7), by (2.11) and (2.12), we have

$$\begin{aligned} & \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q, c; q)_k} \left(\frac{bcdq^n}{a} \right)^k \frac{1}{(aq^k/d, aq^k/b; q)_{\infty}} V_{n-k}(aq^k, d, b | q) \\ &= \frac{(-1)^n q^{\binom{n}{2}} (bcd/a)^n}{(c; q)_n} \frac{(aq^{1-n}/(cd); q)_{\infty}}{(aq/(cd), a/d, a/b; q)_{\infty}} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \frac{a}{d} \\ \frac{aq^{1-n}}{(cd)} \end{matrix} ; q, \frac{aq}{bc} \right], \end{aligned} \quad (6.8)$$

which becomes (6.2) after simplification. The proof is complete. $\square$

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