Research Article

Cocompact Open Sets and Continuity

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Compact subsets of a topological space are used to define coc-open sets as new generalized open sets, and then coc-open sets are used to define (coc)*-open sets as another type of generalized open sets. Several results and examples related to them are obtained; particularly a decomposition of open sets is given. Also, coc-open sets and (coc)*-open sets are used to introduce coc-continuity and (coc)*-continuity, respectively. As a main result, a decomposition theorem of continuity is obtained.

1. Introduction

Throughout this paper by a space we mean a topological space. Let (X, τ) be a space, and let A be a subset of X . A point $x \in X$ is called a condensation point of A if, for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. In 1982, Hdeib defined ω -closed sets and ω -open sets as follows: A is called ω -closed [1] if it contains all its condensation points. The complement of an ω -closed set is called ω -open. In 1989, Hdeib [2] introduced ω -continuity as a generalization of continuity as follows: A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called ω -continuous if the inverse image of each open set is ω -open. The authors in [3] proved that the family of all ω -open sets in a space (X, τ) forms a topology on X finer than τ and that the collection $\{U - C : U \in \tau \text{ and } C \text{ is a countable subset of } X\}$ forms a base for that topology. Recently, in [4], the authors introduced a class of generalized open sets which is stronger ω -open as follows: a subset A of a space (X, τ) is called N -open if for each $x \in A$, there exists $U \in \tau$ such that $x \in U$ and $U - A$ is finite. Throughout this paper, the family of all N -open sets in a space (X, τ) will be denoted by τ^n . The authors in [4] proved that τ^n is a topology on X that is finer than τ and they introduced several concepts related to N -open sets; in particular, they introduced N -continuity as a generalization of continuity and as a stronger form of ω -continuity as follows: a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called N -continuous if the inverse image of each open set is N -open. The authors in [5] continued the study of topological

concepts via N -open sets. In the present work, we will generalize N -open sets by coc-open sets. Several results and concepts related to them will be introduced.

Throughout this paper, we use $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ to denote the set of real numbers, the set of rationals, and the set of natural numbers, respectively. For a subset A of a space (X, τ) , the closure of A and the interior of A will be denoted by $\text{Cl}_\tau(A)$ and $\text{Int}_\tau(A)$, respectively. Also, we write $\tau|_A$ to denote the relative topology on A when A is nonempty. For a nonempty set X , τ_{disc} , τ_{ind} , τ_{cof} , and τ_{coc} will denote, respectively, the discrete topology on X , the indiscrete topology on X , the cofinite topology on X , and the cocountable topology on X . For a subset A of $\mathbb{R}$ we write (A, τ_u) to denote the subspace topology on A relative to the usual topology and we use τ_{lr} to denote the left ray topology on $\mathbb{R}$. For any two spaces (X, τ) and (Y, σ) we use τ_{prod} to denote the product topology on $X \times Y$. The family of all compact subsets of a space (X, τ) will be denoted by $C(X, \tau)$.

2. Cocompact Open Sets

Definition 2.1. A subset A of a space (X, τ) is called co-compact open set (notation: coc-open) if, for every $x \in A$, there exists an open set $U \subseteq X$ and a compact subset $K \in C(X, \tau)$ such that $x \in U - K \subseteq A$. The complement of a coc-open subset is called coc-closed.

The family of all coc-open subsets of a space (X, τ) will be denoted by τ^k and the family $\{U - K : U \in \tau \text{ and } K \in C(X, \tau)\}$ of coc-open sets will be denoted by $\mathcal{B}^k(\tau)$.

Theorem 2.2. Let (X, τ) be a space. Then the collection τ^k forms a topology on X .

Proof. By the definition one has directly that $\emptyset \in \tau^k$. To see that $X \in \tau^k$, let $x \in X$, take $U = X$ and $K = \emptyset$. Then $x \in U - K \subseteq X$.

Let $U_1, U_2 \in \tau^k$, and let $x \in U_1 \cap U_2$. For each $i = 1, 2$, we find an open set V_i and a compact subset K_i such that $x \in V_i - K_i \subseteq U_i$. Take $V = V_1 \cap V_2$ and $K = K_1 \cup K_2$. Then V is open, K is compact, and $x \in V - K \subseteq U_1 \cap U_2$. It follows that $U_1 \cap U_2$ is coc-open.

Let $\{U_\alpha : \alpha \in \Delta\}$ be a collection of coc-open subsets of (X, τ) and $x \in \bigcup_{\alpha \in \Delta} U_\alpha$. Then there exists $\alpha_0 \in \Delta$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is coc-open, then there exists an open set V and a compact subset K , such that $x \in V - K \subseteq U_{\alpha_0}$. Therefore, we have $x \in V - K \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Hence, $\bigcup_{\alpha \in \Delta} U_\alpha$ is coc-open. $\square$

The following result follows directly from Definition 2.1.

Proposition 2.3. Let (X, τ) be a space. Then the collection $\mathcal{B}^k(\tau)$ forms a base for τ^k .

Corollary 2.4. Let (X, τ) be a space. Then the collection $\tau \cup \{X - K : K \in C(X, \tau)\}$ forms a subbase for τ^k .

For a space (X, τ) , the following example shows that the collection $\mathcal{B}^k(\tau)$ is not a topology on X in general.

Example 2.5. Let $X = \mathbb{R}$, and let $\tau = \tau_{\text{lr}}$. Consider the collection of elements of $\mathcal{B}^k(\tau)$, $G_n = (-\infty, n + 1/2) - \{1, 2, \dots, n\}$, $n \in \mathbb{N}$. Then $\bigcup \{G_n : n \in \mathbb{N}\} = \mathbb{R} - \mathbb{N}$ which is not in $\mathcal{B}^k(\tau)$.

Theorem 2.6. Let (X, τ) be a space. Then $\tau \subseteq \tau^n \subseteq \tau^k$.

Proof. Obvious. $\square$

Remark 2.7. Each of the two inclusions in Theorem 2.6 is not equality in general; to see this, let $X = \mathbb{N}$ and $\tau = \tau_{\text{ind}}$. Then $\tau^n = \tau_{\text{cof}}$ and $\tau^k = \tau_{\text{disc}}$, and therefore $\tau \neq \tau^n$ and $\tau^n \neq \tau^k$.

In Remark 2.7, the space (X, τ) is an example on a compact space (X, τ) for which (X, τ^k) is not compact.

Definition 2.8 (see [6]). A space (X, τ) is called CC if every compact set in X is closed.

It is well known that every T_2 space is CC, but not conversely.

Theorem 2.9. Let (X, τ) be a space. Then the following are equivalent:

- (a) (X, τ) is CC,
- (b) $\tau = \mathcal{B}^k(\tau)$,
- (c) $\tau = \tau^n = \tau^k$.

Proof. (a) $\Rightarrow$ (b) As $\emptyset$ is a compact subset of X , then, for every $U \in \tau$, $U - \emptyset = U \in \mathcal{B}^k(\tau)$. Hence, we have $\tau \subseteq \mathcal{B}^k(\tau)$. Now let $U - K \in \mathcal{B}^k(\tau)$, where $U \in \tau$ and K is a compact subset of X . As (X, τ) is CC, then K is closed and hence $U - K \in \tau$. Therefore, we have $\mathcal{B}^k(\tau) \subseteq \tau$.

(b) $\Rightarrow$ (c) By Theorem 2.6, it is sufficient to see that $\tau^k \subseteq \tau$. Since by (b) $\tau = \mathcal{B}^k(\tau)$ and as $\mathcal{B}^k(\tau)$ is a base for τ^k , then $\tau^k \subseteq \tau$.

(c) $\Rightarrow$ (a) Let $K \in C(X, \tau)$. Then $X - K \in \tau^k$, and by (c), $X - K \in \tau$. Therefore, K is closed in X . $\square$

Corollary 2.10. If (X, τ) is a T_2 -space, then $\tau = \tau^n = \tau^k$.

Theorem 2.11. For any space (X, τ) , (X, τ^k) is CC.

Proof. Let $K \in C(X, \tau^k)$. As $\tau \subseteq \tau^k$, then $C(X, \tau^k) \subseteq C(X, \tau)$ and hence $K \in C(X, \tau)$. Thus, we have $X - K \in \tau^k$, and hence K is closed in the space (X, τ^k) . $\square$

Corollary 2.12. For any space (X, τ) , $(\tau^k)^k = \tau^k$.

Proof. Theorems 2.9 and 2.11. $\square$

Theorem 2.13. If (X, τ) is a hereditarily compact space, then $\tau^k = \tau_{\text{disc}}$.

Proof. For every $x \in X$, $X - \{x\}$ is compact and so $\{x\} = X - (X - \{x\}) \in \mathcal{B}^k(\tau) \subseteq \tau^k$. Therefore, $\tau^k = \tau_{\text{disc}}$. $\square$

Each of the following three examples shows that the converse of Theorem 2.13 is not true in general.

Example 2.14. Let $X = \mathbb{R}$ and $\tau = \tau_{\text{lr}}$. For every $x \in X$, take $K = (-\infty, x + 1] - \{x\}$ and $U = (-\infty, x + 1)$. Then $K \in C(X, \tau)$, $U = (-\infty, x + 1) \in \tau$, and $\{x\} = U - K$. This shows that $\tau^k = \tau_{\text{disc}}$. On the other hand, it is well known that (X, τ) is not hereditarily compact.

Example 2.15. Let $X = \mathbb{N}$ and $\tau = \{\emptyset, \mathbb{N}\} \cup \{U_n : n \in \mathbb{N}\}$, where $U_n = \{1, 2, \dots, n\}$. Then the compact subsets of (X, τ) are the finite sets. For every $n \in \mathbb{N}$, $U_n \in \tau$, U_{n-1} is compact, and $\{n\} = U_n - U_{n-1}$. Therefore, $\tau^k = \tau_{\text{disc}}$.

Example 2.16. Let $X = \mathbb{N}$ and τ be the topology on $\mathbb{N}$ having the family $\{\{2n-1, 2n\} : n \in \mathbb{N}\}$ as a base. Then the compact subsets of (X, τ) are the finite sets. If $x \in \mathbb{N}$ with x is odd, then $\{x\} = \{x, x+1\} - \{x+1\}$ and as $\{x, x+1\} \in \tau$ and $\{x+1\}$ is compact, then $\{x\} \in \tau^k$. Similarly, if x is even then $\{x\} \in \tau^k$. Therefore, $\tau^k = \tau_{\text{disc}}$.

The following question is natural: Is there a space (X, τ) for which $\tau^k \neq \tau$ and $\tau^k \neq \tau_{\text{disc}}$?
The following example shows that the answer of the above question is yes.

Example 2.17. Let $X = \mathbb{R}$ and $\tau = \{X\} \cup \{U \subseteq X : 1 \notin U\}$. Then $C(X, \tau) = \{K \subseteq X : 1 \in K\} \cup \{K \subseteq X : 1 \notin K \text{ and } K \text{ is finite}\}$, hence $\tau^k = \tau \cup \{U \subseteq X : 1 \in U \text{ and } X - U \text{ is finite}\}$. Note that $\tau^k \neq \tau$ and $\tau^k \neq \tau_{\text{disc}}$.

Theorem 2.18. Let (X, τ) be a space and A a nonempty subset of X . Then $(\tau|_A)^k \subseteq \tau^k|_A$.

Proof. Let $B \in (\tau|_A)^k$ and $x \in B$. Then there exists $V \in \tau|_A$ and a compact subset $K \subseteq A$ such that $x \in V - K \subseteq B$. Since $V \in \tau|_A$, then we can write $V = U \cap A$, where U is open in X . Since $U - K \in \tau^k$, $(U - K) \cap A \in \tau^k|_A$. Hence, $B \in \tau^k|_A$. $\square$

Question 1. Let (X, τ) be a space and A a nonempty subset of X . Is it true that $(\tau|_A)^k = \tau^k|_A$?

The following result is a partial answer for Question 1.

Theorem 2.19. Let (X, τ) be a space and A be a nonempty closed set in (X, τ) . Then $(\tau|_A)^k = \tau^k|_A$.

Proof. By Theorem 2.18, $(\tau|_A)^k \subseteq \tau^k|_A$. Conversely, let $B \in \tau^k|_A$ and $x \in B$. Choose $H \in \tau^k$ such that $B = H \cap A$. As $H \in \tau^k$, there exists $U \in \tau$ and $K \in C(X, \tau)$ such that $x \in U - K \subseteq H$. Thus, we have $x \in (U \cap A) - (K \cap A) \subseteq B$, $U \cap A \in \tau|_A$, and $K \cap A \in C(A, \tau|_A)$. It follows that $B \in (\tau|_A)^k$. $\square$

Definition 2.20. Let (X, τ) be a space, and let $A \subseteq X$. The coc-closure of A in (X, τ) is denoted by $\text{coc-Cl}_\tau(A)$ and defined as follows:

$$\text{coc-Cl}_\tau(A) = \cap \{B : B \text{ is coc-closed in } (X, \tau) \text{ and } A \subseteq B\}. \quad (2.1)$$

Remark 2.21. Let (X, τ) be a space, and let $A \subseteq X$. Then $\text{coc-Cl}_\tau(A) = \text{Cl}_{\tau^k}(A)$ and $\text{coc-Cl}_\tau(A) \subseteq \text{Cl}_\tau(A)$.

Definition 2.22. A space (X, τ) is called antilocally compact if any compact subset of X has empty interior.

For any infinite set X , (X, τ_{coc}) is an anti-locally compact space. Also, $(\mathbb{Q}, \tau_u)$ is an example of an anti-locally compact space.

Theorem 2.23. Let (X, τ) be an anti-locally compact space. If $A \in \tau$ then $\text{coc-Cl}_\tau(A) = \text{Cl}_\tau(A)$.

Proof. According to Remark 2.21, only we need to show that $\text{Cl}_\tau(A) \subseteq \text{Cl}_{\tau^k}(A)$. Suppose to the contrary that there is $x \notin \text{Cl}_\tau(A) - \text{Cl}_{\tau^k}(A)$. As $x \notin \text{Cl}_{\tau^k}(A)$, there exists $G \supseteq A$ coc-closed such that $x \in G$ and $G \cap A = \emptyset$. Take $U \in \tau$ and $K \in C(X, \tau)$ such that $x \in U - K \subseteq G$. Thus we have $U \cap A \subseteq K$. Since $x \in \text{Cl}_\tau(A)$, it follows that $U \cap A \neq \emptyset$ and hence $\text{Int}(K) \neq \emptyset$. This contradicts the assumption that (X, τ) is anti-locally compact. $\square$

In Theorem 2.23 the assumption “anti-locally compact” on the space cannot be dropped. As an example let $X = \mathbb{R}$ and $\tau = \{\emptyset, X, \{0\}\}$, then $\{0\} \in \tau$, $\text{coc-Cl}_\tau(\{0\}) = \{0\}$ while $\text{Cl}_\tau(\{0\}) = \mathbb{R}$.

Theorem 2.24. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is injective, open, and continuous, then $f : (X, \tau^k) \rightarrow (Y, \sigma^k)$ is open.*

Proof. Let $G = U - K$ where $U \in \tau$ and $K \in C(X, \tau)$ be a basic element for τ^k . As f is injective, $f(G) = f(U) - f(K)$. Also, as $f : (X, \tau) \rightarrow (Y, \sigma)$ is open, $f(U) \in \sigma$. And as $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous, $f(K) \in C(Y, \sigma)$. This ends the proof. $\square$

Remark 2.25. In Theorem 2.24, the continuity condition cannot be dropped. Take $f : (\mathbb{R}, \tau_{\text{ind}}) \rightarrow (\mathbb{R}, \tau_u)$, where $f(x) = \tan^{-1}x$. Then f is injective and open. On the other hand, as $(\mathbb{R}, \tau_{\text{ind}})$ is hereditarily compact we have $(\tau_{\text{ind}})^k = \tau_{\text{disc}}$, and as $(\mathbb{R}, \tau_u)$ is T_2 we have $(\tau_u)^k = \tau_u$. Thus, $f : (\mathbb{R}, (\tau_{\text{ind}})^k) \rightarrow (\mathbb{R}, (\tau_u)^k)$ is not open.

3. (coc)*-Open Sets

Definition 3.1. A subset A of a space (X, τ) is called (coc)*-open if $\text{Int}_{\tau^k}(A) = \text{Int}_\tau(A)$.

The family of all (coc)*-open subsets of a space (X, τ) will be denoted by $\mathcal{B}^{k*}(\tau)$.

Theorem 3.2. *Let (X, τ) be a space. Then $\tau \subseteq \mathcal{B}^{k*}(\tau)$.*

Proof. Let $A \in \tau$. Then $A = \text{Int}_\tau(A) \subseteq \text{Int}_{\tau^k}(A) \subseteq A$. Thus $\text{Int}_\tau(A) = \text{Int}_{\tau^k}(A)$ and hence $A \in \mathcal{B}^{k*}(\tau)$. $\square$

Theorem 3.3. *If (X, τ) is a CC space, then every subset of X is (coc)*-open.*

Proof. Let $A \subseteq X$. Since (X, τ) is CC, then $\tau = \tau^k$, and so $\text{Int}_{\tau^k}(A) = \text{Int}_\tau(A)$. Therefore, A is (coc)*-open. $\square$

Corollary 3.4. *If (X, τ) is a T_2 space, then every subset of X is (coc)*-open.*

According to Corollary 3.4, the inclusion in Theorem 3.2 is not equality in any T_2 space that is not discrete for example, in $(\mathbb{R}, \tau_u)$ for the set $A = (0, 1) \cup \{2\}$, we have $\text{Int}_{(\tau_u)^k}(A) = \text{Int}_{\tau_u}(A) = (0, 1)$ and thus $A \in \mathcal{B}^{k*}(\tau_u) - \tau_u$.

Theorem 3.5. *If (X, τ) is a hereditarily compact space, then $\tau = \mathcal{B}^{k*}(\tau)$.*

Proof. By Theorem 3.2, we need only to show that $\mathcal{B}^{k*}(\tau) \subseteq \tau$. Let $A \in \mathcal{B}^{k*}(\tau)$. Then $\text{Int}_{\tau^k}(A) = \text{Int}_\tau(A)$. Since (X, τ) is hereditarily compact, then, by Theorem 2.13, $\tau^k = \tau_{\text{disc}}$ and thus $\text{Int}_\tau(A) = \text{Int}_{\tau^k}(A) = A$. Therefore, $A \in \tau$. $\square$

The following result is a new decomposition of open sets in a space.

Theorem 3.6. *Let (X, τ) be a space. Then $\tau = \tau^k \cap \mathcal{B}^{k*}(\tau)$.*

Proof. By Theorems 2.6 and 3.2, it follows that $\tau \subseteq \tau^k \cap \mathcal{B}^{k*}(\tau)$. Conversely, let $A \in \tau^k \cap \mathcal{B}^{k*}(\tau)$. As $A \in \tau^k$, then $\text{Int}_{\tau^k}(A) = A$. Also, since $A \in \mathcal{B}^{k*}(\tau)$, then $\text{Int}_{\tau^k}(A) = \text{Int}_\tau(A)$. It follows that $\text{Int}_\tau(A) = A$ and hence $A \in \tau$. $\square$

Theorem 3.7. *For a space (X, τ) , one has the following:*

- (a) $\emptyset, X \in \mathcal{B}^{k^*}(\tau)$,
- (b) if $A, B \in \mathcal{B}^{k^*}(\tau)$, then $A \cap B \in \mathcal{B}^{k^*}(\tau)$.

Proof. (a) The proof follows directly from Theorem 3.2.

(b) Let $A, B \in \mathcal{B}^{k^*}(\tau)$. Then $\text{Int}_{\tau^k}(A) = \text{Int}_{\tau}(A)$ and $\text{Int}_{\tau^k}(B) = \text{Int}_{\tau}(B)$. Thus we have

$$\begin{aligned} \text{Int}_{\tau}(A \cap B) &= \text{Int}_{\tau}(A) \cap \text{Int}_{\tau}(B) \\ &= \text{Int}_{\tau^k}(A) \cap \text{Int}_{\tau^k}(B) \\ &= \text{Int}_{\tau^k}(A \cap B). \end{aligned} \tag{3.1}$$

It follows that $A \cap B \in \mathcal{B}^{k^*}(\tau)$. □

The following example shows that arbitrary union of k^* -open sets need not to be k^* -open in general.

Example 3.8. Consider the space defined in Example 2.17. For every natural number $n \geq 3$, put $A_n = \mathbb{R} - \{n, n+1, n+2, \dots\}$. Then, for each $n \geq 3$, $\text{Int}_{\tau^k}(A_n) = \text{Int}_{\tau}(A_n) = A_n - \{1\}$, and thus A_n is k^* -open. On the other hand, $\text{Int}_{\tau^k}(\bigcup_{n \geq 3} A_n) = \bigcup_{n \geq 3} A_n$ while $\text{Int}_{\tau}(\bigcup_{n \geq 3} A_n) = (\bigcup_{n \geq 3} A_n) - \{1\}$.

4. coc-Continuous Functions

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called coc-continuous at a point $x \in X$, if for every open set V containing $f(x)$ there is a coc-open set U containing x such that $f(U) \subseteq V$. If f is coc-continuous at each point of X , then f is said to be coc-continuous.

The following theorem follows directly from the definition.

Theorem 4.2. *A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is coc-continuous if and only if $f : (X, \tau^k) \rightarrow (Y, \sigma)$ is continuous.*

Theorem 4.3. *Every N -continuous function is coc-continuous.*

Proof. Straightforward. □

The identity function $I : (\mathbb{R}, \tau_{\text{ind}}) \rightarrow (\mathbb{R}, \tau_{\text{disc}})$ is a coc-continuous function that is not N -continuous.

The proof of the following result follows directly from Theorem 2.9.

Theorem 4.4. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function for which (X, τ) is CC, then the following are equivalent.*

- (a) f is continuous.
- (b) f is N -continuous.
- (c) f is coc-continuous.

The following example shows that the composition of two N -continuous functions need not to be even coc-continuous.

Example 4.5. Let $X = \mathbb{R}$, $Y = \{0, 1, 2\}$, $Z = \{a, b\}$, τ be as in Example 2.17, $\sigma = \{\emptyset, Y, \{0\}, \{0, 1\}\}$, and $\mu = \{\emptyset, Z, \{a\}\}$. Define the function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(x) = 2$ if $x \in \{0, 1\}$ and $f(x) = 1$ otherwise, and define the function $g : (Y, \sigma) \rightarrow (Z, \mu)$ by $g(0) = g(2) = a$ and $g(1) = b$. Then f and g are N -continuous functions, but $g \circ f$ is not coc-continuous since $(g \circ f)^{-1}(\{a\}) = \{0, 1\} \notin \tau^k$.

Theorem 4.6. (a) If $f : (X, \tau) \rightarrow (Y, \sigma)$ is N -continuous and if $g : (Y, \sigma) \rightarrow (Z, \mu)$ is continuous, then $g \circ f : (X, \tau) \rightarrow (Z, \mu)$ is N -continuous.

(b) If $f : (X, \tau) \rightarrow (Y, \sigma)$ is coc-continuous and if $g : (Y, \sigma) \rightarrow (Z, \mu)$ is continuous, then $g \circ f : (X, \tau) \rightarrow (Z, \mu)$ is coc-continuous.

Proof. (a) It follows by noting that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is N -continuous if and only if $f : (X, \tau^n) \rightarrow (Y, \sigma)$.

(b) The proof follows directly from Theorem 4.2. $\square$

Theorem 4.7. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is coc-continuous and A is a nonempty closed set in (X, τ) , then the restriction of f to A , $f|_A : (A, \tau|_A) \rightarrow (Y, \sigma)$ is a coc-continuous function.

Proof. Let V be any open set in Y . Since f is coc-continuous, then $f^{-1}(V)$ is coc-open in X and by Theorem 2.19, $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ is coc-open in A . Therefore $f|_A$ is coc-continuous. $\square$

Theorem 4.8. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is coc-continuous and $X = A \cup B$, where A and B are coc-closed subsets in (X, τ) and $f|_A : (A, \tau|_A) \rightarrow (Y, \sigma)$, $f|_B : (B, \tau|_B) \rightarrow (Y, \sigma)$ are coc-continuous functions, then f is coc-continuous.

Proof. By Theorem 4.2 it is sufficient to show that $f : (X, \tau^k) \rightarrow (Y, \sigma)$ is continuous. Let C be a closed subset of (Y, σ) . Then $f^{-1}(C) = f^{-1}(C) \cap X = f^{-1}(C) \cap (A \cup B) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B)$. Since $f|_A : (A, \tau|_A) \rightarrow (Y, \sigma)$ is coc-continuous, then $f^{-1}(C) \cap A = (f|_A)^{-1}(C)$ is coc-closed in $(A, \tau|_A)$, and as A is coc-closed in (X, τ) , it follows that $f^{-1}(C) \cap A$ is coc-closed in (X, τ) ; similarly one can conclude that $f^{-1}(C) \cap B$ is coc-closed in (X, τ) . It follows that $f^{-1}(C)$ is closed in (X, τ^k) and hence $f : (X, \tau^k) \rightarrow (Y, \sigma)$ is continuous. $\square$

The following result follows directly from Theorem 4.2.

Theorem 4.9. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (X, \tau) \rightarrow (Z, \mu)$ be two functions. Then the function $h : (X, \tau) \rightarrow (Y \times Z, \tau_{\text{prod}})$ defined by $h(x) = (f(x), g(x))$ is coc-continuous if and only if f and g are coc-continuous.

Corollary 4.10. A function $w : (X, \tau) \rightarrow (Y, \sigma)$ is coc-continuous if and only if the graph function $h : (X, \tau) \rightarrow (X \times Y, \tau_{\text{prod}})$, given by $h(x) = (x, w(x))$ for every $x \in X$, is coc-continuous.

Theorem 4.11. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. If there is a coc-open subset A of (X, τ) containing $x \in X$ such that the restriction of f to A , $f|_A : (A, \tau|_A) \rightarrow (Y, \sigma)$ is coc-continuous at x , then f is coc-continuous at x .

Proof. Let $V \in \sigma$ with $f(x) \in V$. Since $f|_A$ is coc-continuous at x , there is $U \in (\tau|_A)^k \subseteq \tau^k$ such that $x \in U$ and $(f|_A)(U) = f(U) \subseteq V$. $\square$

Corollary 4.12. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function, and let $\{A_\alpha : \alpha \in \Delta\} \subseteq \tau^k$ be a cover of X such that, for each $\alpha \in \Delta$, $f|_{A_\alpha} : (A_\alpha, \tau|_{A_\alpha}) \rightarrow (Y, \sigma)$ is coc-continuous, then f is coc-continuous.*

Proof. Let $x \in X$. We show that $f : (X, \tau) \rightarrow (Y, \sigma)$ is coc-continuous at x . Since $\{A_\alpha : \alpha \in \Delta\}$ is a cover of X , then there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha_0}$. Therefore, by Theorem 4.11, it follows that f is coc-continuous at x . $\square$

Definition 4.13. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $(\text{coc})^*$ -continuous if the inverse image of each open set is $(\text{coc})^*$ -open.

Theorem 4.14. *Every continuous function is $(\text{coc})^*$ -continuous.*

Proof. The proof follows directly from Theorem 3.2. $\square$

Theorem 4.15. *If (X, τ) is CC, then every function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $(\text{coc})^*$ -continuous.*

Proof. The proof follows directly from Theorem 3.3. $\square$

Corollary 4.16. *If (X, τ) is T_2 , then every function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $(\text{coc})^*$ -continuous.*

By Corollary 4.16, it follows that the function $f : (\mathbb{R}, \tau_u) \rightarrow (\mathbb{R}, \tau_u)$ where $f(x) = 0$ for x is rational and $f(x) = 1$ for x is irrational is $(\text{coc})^*$ -continuous. On the other hand, it is well known that this function is discontinuous every where. Also, by Theorem 4.4, f is not coc-continuous.

Theorem 4.17. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function with (X, τ) being a hereditarily compact space. Then f is continuous if and only if f is $(\text{coc})^*$ -continuous.*

Proof. The proof follows directly from Theorem 3.5. $\square$

By Theorem 4.17, it follows that the identity function $I : (\mathbb{R}, \tau_{\text{ind}}) \rightarrow (\mathbb{R}, \tau_{\text{disc}})$ is not $(\text{coc})^*$ -continuous. Therefore, this is an example of a coc-continuous function that is not $(\text{coc})^*$ -continuous.

We end this section by the following decomposition of continuity via coc-continuity and $(\text{coc})^*$ -continuity.

Theorem 4.18. *A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous if and only if it is coc-continuous and $(\text{coc})^*$ -continuous.*

Proof. The proof follows directly from Theorem 3.6. $\square$

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