Research Article

Existence Results for Quasilinear Elliptic Equations with Indefinite Weight

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We provide the existence of a solution for quasilinear elliptic equation

\[-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) + \tilde{a}(x,|\nabla u|)\nabla u = \lambda m(x)|u|^{p-2}u + f(x,u) + h(x) \text{ in } \Omega,\]

under the Neumann boundary condition. Here, we consider the condition that \(\tilde{a}(x,t) = o(t^{p-2})\) as \(t \to \infty\) and \(f(x,u) = o(|u|^{p-1})\) as \(|u| \to \infty\). As a special case, our result implies that the following \(p\)-Laplace equation has at least one solution:

\[-\Delta_p u = \lambda m(x)|u|^{p-2}u + \mu |u|^{r-2}u + h(x) \text{ in } \Omega, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \text{ for every } 1 < r < p < \infty, \lambda \in \mathbb{R}, \mu \neq 0 \text{ and } m, h \in L^\infty(\Omega) \text{ with } \int_{\Omega} m \, dx \neq 0.\]

Moreover, in the nonresonant case, that is, \(\lambda\) is not an eigenvalue of the \(p\)-Laplacian with weight \(m\), we present the existence of a solution of the above \(p\)-Laplace equation for every \(1 < r < p < \infty\), \(\mu \in \mathbb{R}\) and \(m, h \in L^\infty(\Omega)\).

1. Introduction

In this paper, we consider the existence of a solution for the following quasilinear elliptic equation:

\[-\text{div} A(x, \nabla u) = \lambda m(x)|u|^{p-2}u + f(x,u) + h(x) \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega, \quad (P; \lambda, m, h)\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded domain with \(C^2\) boundary \(\partial \Omega\), \(\nu\) denotes the outward unit normal vector on \(\partial \Omega\), \(\lambda \in \mathbb{R}, 1 < p < \infty\) and \(m, h \in L^\infty(\Omega)\). We assume that \(f\) is a Carathéodory function on \(\Omega \times \mathbb{R}\) satisfying

\[\lim_{|t| \to \infty} \frac{f(x,t)}{|t|^{p-2}t} = 0 \text{ uniformly in } x \in \Omega, \quad (1.1)\]
and that \( f(x, t) \) is bounded on a bounded set (admitting \( f \equiv 0 \) in the nonresonant case). Here, \( A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N \) is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption \((A)\)). The equation \((P; \lambda, m, h)\) contains the corresponding \( p \)-Laplacian problem as a special case. Although the operator \( A \) is nonhomogeneous in the second variable in general, we assume that \( A(x, y) \) is asymptotically \((p - 1)\)-homogeneous at infinity in the following sense \((AH)\).

Throughout this paper, we assume that the map \( A \) satisfies the following assumptions \((AH)\) and \((A)\):

\((AH)\) there exist a positive function \( a_\infty \in C^1(\overline{\Omega}, \mathbb{R}) \) and a continuous function \( \bar{a}(x, t) \) on \( \overline{\Omega} \times \mathbb{R} \) such that

\[
A(x, y) = a_\infty(x)|y|^{p-2}y + \bar{a}(x, |y|)y \quad \text{for every } x \in \Omega, y \in \mathbb{R}^N,
\]

\[
\lim_{t \to +\infty} \frac{\bar{a}(x, t)}{|y|^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.
\] (1.2)

\((A)\) \( A(x, y) = a(x, |y|)y \), where \( a(x, t) > 0 \) for all \( x, t \in \overline{\Omega} \times (0, +\infty) \) and

(i) \( A \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N) \);

(ii) there exists \( C_1 > 0 \) such that

\[
|D_y A(x, y)| \leq C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega}, y \in \mathbb{R}^N \setminus \{0\};
\] (1.3)

(iii) there exists \( C_0 > 0 \) such that

\[
D_y A(x, y) \xi \cdot \xi \geq C_0 |y|^{p-2}||\xi||^2 \quad \text{for every } x \in \overline{\Omega}, y \in \mathbb{R}^N \setminus \{0\}, \xi \in \mathbb{R}^N;
\] (1.4)

(iv) there exists \( C_2 > 0 \) such that

\[
|D_x A(x, y)| \leq C_2 \left(1 + |y|^{p-1}\right) \quad \text{for every } x \in \overline{\Omega}, y \in \mathbb{R}^N \setminus \{0\}.
\] (1.5)

A similar hypothesis to \((A)\) is considered in the study of quasilinear elliptic problems (cf. [1, Example 2.2], [2-6]). It is easily seen that many examples as in the above references satisfy the condition \((AH)\). Also, the following example satisfies our hypotheses:

\[
\text{div} \left( \left( |\nabla u|^{p-2} + |\nabla u|^{q-2} \right) \left(1 + |\nabla u|^q\right)^{\frac{p-q}{q}} \nabla u \right) \quad \text{for } 1 < p \leq q < \infty.
\] (1.6)

In particular, for \( A(x, y) = |y|^{p-2}y \), that is, \( \text{div} A(x, \nabla u) \) stands for the usual \( p \)-Laplacian \( \Delta_p u \), we can take \( C_0 = C_1 = p - 1 \) in \((A)\). Conversely, in the case where \( C_0 = C_1 = p - 1 \) holds in \((A)\), by the inequalities in Remark 1.4 (ii) and (iii), we see \( a(x, t) = |t|^{p-2} \) whence \( A(x, y) = |y|^{p-2}y \).

Concerning the weight \( m \), throughout this paper, we assume that

\[
||\{m > 0\}|| := ||\{x \in \Omega; m(x) > 0\}|| > 0
\] (1.7)

holds, where \( |X| \) denotes the Lebesgue measure of a measurable set \( X \).
Because $A(x,y)$ is asymptotically $(p - 1)$-homogeneous at infinity, the solvability of our equation is related to the following homogeneous equation (see Theorem 1.1):

$$-\text{div} \left( a_\infty(x)|\nabla u|^{p-2}\nabla u \right) = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,$$

where $a_\infty$ is the positive function as in $(AH)$. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of $(EV;m)$ if the equation $(EV;m)$ has a nontrivial solution.

There are few existence results of a solution to our equation (and also the $p$-Laplace equation). For example, if $\lambda < 0$ and $m \equiv 1$ hold, then the standard argument guarantees the existence of a solution. For the $p$-Laplacian as a special case of our problem, it is shown in [7] that the equation

$$-\Delta_p u = \lambda m|u|^{p-2}u + h \quad \text{in } \Omega \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega$$

(1.8)

has a unique positive solution provided $0 < \lambda < \lambda^*(m)$, $\int_\Omega m \, dx < 0$ and $0 \neq h \in L^\infty(\Omega)$, where $\lambda^*(m)$ is the principal eigenvalue defined in Section 2.1 with $a_\infty \equiv 1$. In [8], although the resonant case where $\lambda = \lambda_1(m)$ or $\lambda = \lambda_2(m)$ is considered under the assumptions to $f(x,u) = f(u)$, its result does not cover the case of $f(u) = |u|^{r-2}u$ with $1 < r < p$, where $\lambda_i(m)$ ($i = 1, 2$) is $i$th eigenvalue of the $p$-Laplacian with weight $m$. For the Laplace problem under the Neumann boundary condition, we can refer to [9, 10]. Under the Dirichlet boundary condition, the existence results for the Laplace problem are well known when $m \equiv 1$ and $\lambda$ is not an eigenvalue of the Laplacian (cf. [11]). Moreover, under the Dirichlet (or blow-up) boundary condition, many authors study various equations involving the $p$-Laplace (Laplace) operator with (indefinite) weight. For example, we refer to [12] for boundary blow-up problems with Laplacian, [13] for periodic reaction-diffusion problems and [14, 15] for singular quasilinear elliptic problems.

Recently, the present author shows the existence of a solution for our problem in the case where $\lambda$ is between the principal eigenvalue and the second eigenvalue in [6] (for $f \equiv 0$). In addition, a similar situation is treated in [5]. However, existence results are not seen in the case when $\lambda$ is greater than the second eigenvalue for our problem. Therefore, the first purpose of this paper is to present an existence result of a solution in the nonresonant case where $\lambda$ is not an eigenvalue of $(EV;m)$. Then, it studied the existence of at least one solution in the resonant case under assumptions that cover the case $f(u) = \mu|u|^{r-2}u$ with $1 < r < p$ and $\mu \neq 0$.

For the proof of our result, it is necessary to study the weighted eigenvalue problem $(EV;m)$. Thus, in Section 2, we introduce two sequences $\{\lambda_n(m)\}_n$ and $\{\mu_n(m)\}_n$ of an eigenvalue of $(EV;m)$ defined by Ljusternik-Schnirelman theory or Drábek-Robinson’s method (cf. [16]), respectively. Then, we show several properties of above eigenvalues. In Section 3, we give the proof in the nonresonant case by using $\{\mu_n(m)\}_n$. In Sections 4 and 5, we handle the resonant case.
1.1. Statements of Our Existence Results

First, we state the existence result of a solution in the nonresonant case.

**Theorem 1.1.** Assume that $\lambda \in \mathbb{R}$ is not an eigenvalue of $(EV; m)$. Then, $(P; \lambda, m, h)$ has at least one solution.

To state our existence result in the resonant case, we introduce some conditions. Set

$$F(x, u) := \int_0^u f(x, s) \, ds, \quad \tilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t) \, dt,$$  \hspace{1cm} (1.9)

where $\tilde{a}$ is the function as in $(AH)$.

**($H+$)** there exist $0 \leq q \leq p - 1$ and $H_0 > 0$ such that

$$\lim_{|y| \to \infty} \frac{p\tilde{G}(x, y) - \tilde{a}(x, |y|) |y|^2}{|y|^{1+q}} = +\infty \quad \text{uniformly in a.e. } x \in \Omega,$$  \hspace{1cm} (1.10)

$$f(x, t)t - pF(x, t) \geq -H_0 \left(1 + |t|^{1+q}\right) \quad \text{for a.e. } x \in \Omega, \text{ every } t \in \mathbb{R};$$

**($H-$)** there exist $0 \leq q \leq p - 1$ and $H_0 > 0$ such that

$$\lim_{|y| \to \infty} \frac{p\tilde{G}(x, y) - \tilde{a}(x, |y|) |y|^2}{|y|^{1+q}} = -\infty \quad \text{uniformly in a.e. } x \in \Omega,$$  \hspace{1cm} (1.11)

$$f(x, t)t - pF(x, t) \leq H_0 \left(1 + |t|^{1+q}\right) \quad \text{for a.e. } x \in \Omega, \text{ every } t \in \mathbb{R};$$

**($HF+$)** there exist $0 \leq q \leq p - 1$ and $H_0 > 0$ such that

$$p\tilde{G}(x, y) - \tilde{a}(x, |y|) |y|^2 \geq -H_0 \left(1 + |y|^{1+q}\right) \quad \text{for every } x \in \Omega, y \in \mathbb{R}^N,$$

$$\lim_{|t| \to \infty} \frac{f(x, t)t - pF(x, t)}{|t|^{1+q}} = +\infty \quad \text{uniformly in a.e. } x \in \Omega;$$  \hspace{1cm} (1.12)

**($HF-$)** there exist $0 \leq q \leq p - 1$ and $H_0 > 0$ such that

$$p\tilde{G}(x, y) - \tilde{a}(x, |y|) |y|^2 \leq H_0 \left(1 + |y|^{1+q}\right) \quad \text{for every } x \in \Omega, y \in \mathbb{R}^N,$$

$$\lim_{|t| \to \infty} \frac{f(x, t)t - pF(x, t)}{|t|^{1+q}} = -\infty \quad \text{uniformly in a.e. } x \in \Omega.$$  \hspace{1cm} (1.13)
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Theorem 1.2. Assume one of the following conditions:

(i) \( \lambda = 0 \) and \( (HF+) \) or \( (HF-) \) hold;

(ii) \( \lambda \neq 0 \), \( \int_{\Omega} m \, dx \neq 0 \) and one of \( (H+) \), \( (H-) \), \( (HF+) \) and \( (HF-) \) hold;

(iii) \( \lambda \neq 0 \), \( \int_{\Omega} m \, dx = 0 \) and \( (H+) \) or \( (HF+) \) hold;

Then, \((P; \lambda, m, h)\) has at least one solution.

In the special case where \( \tilde{a}(x, t) \equiv 0 \) and \( f(x, u) = \mu |u|^{r-2} u \) for \( 1 < r < p \), we easily see that \((HF+)\) or \((HF-)\) holds with \( 0 \leq q < r - 1 \) provided \( \mu < 0 \) or \( \mu > 0 \), respectively. Therefore, the following result is proved according to Theorem 1.2.

Corollary 1.3. Let \( 1 < r < p < \infty \), \( \mu \neq 0 \) and \( \int_{\Omega} m \, dx \neq 0 \). Then, the following equation has at least one solution:

\[
- \text{div}(a(x)|\nabla u|^{r-2}\nabla u) = \lambda m(x)|u|^{r-2}u + \mu |u|^{r-2}u + h(x) \quad \text{in} \ \Omega,
\]

\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega.
\]

1.2. Properties of the Map \( A \)

In what follows, the norm on \( W^{1,p}(\Omega) \) is given by \( \|u\|_p := \|\nabla u\|_p + \|u\|_p \), where \( \|u\|_q \) denotes the norm of \( L^q(\Omega) \) for \( u \in L^q(\Omega) \) \((1 \leq q \leq \infty)\). Setting \( G(x, y) := \int_0^y a(x, t) \, dt \), then we can easily see that

\[
\nabla_y G(x, y) = A(x, y), \quad G(x, 0) = 0
\]

for every \( x \in \Omega \).

Remark 1.4. It is easily seen that the following assertions hold under condition \( (A) \):

(i) for all \( x \in \Omega \), \( A(x, y) \) is maximal monotone and strictly monotone in \( y \);

(ii) \( |A(x, y)| \leq (C_1 / (p - 1))|y|^{p-1} \) for every \( (x, y) \in \Omega \times \mathbb{R}^N \);

(iii) \( A(x, y) y \geq (C_0 / (p - 1))|y|^p \) for every \( (x, y) \in \Omega \times \mathbb{R}^N \);

(iv) \( G(x, y) \) is convex in \( y \) for all \( x \) and satisfies the following inequalities:

\[
A(x, y) y \geq G(x, y) \geq \frac{C_0}{p(p - 1)}|y|^p, \quad G(x, y) \leq \frac{C_1}{p(p - 1)}|y|^p,
\]

for every \( (x, y) \in \Omega \times \mathbb{R}^N \), where \( C_0 \) and \( C_1 \) are the positive constants in \( (A) \).
The following result is proved in [3]. It plays an important role for our proof.

**Proposition 1.5** (see [3, Proposition 1]). Let \( A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^* \) be the map defined by

\[
\langle A(u), v \rangle = \int_{\Omega} A(x, \nabla u) \nabla v \, dx,
\]

for \( u, v \in W^{1,p}(\Omega) \). Then, \( A \) has the \((S)_\nu\) property, that is, any sequence \( \{u_n\} \) weakly convergent to \( u \) with \( \limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0 \) strongly converges to \( u \).

## 2. The Weighted Eigenvalue Problems

### 2.1. Preliminaries

The following lemmas can be easily shown by way of contradiction because \( \int_{\Omega} a_\infty |\nabla u|^p \, dx \) is equivalent to \( ||\nabla u||^p_p \) (note that \( a_\infty \) is positive). Here, we omit the proofs (refer to [7]).

**Lemma 2.1.** Assume \( \int_{\Omega} m \, dx < 0 \). Then, there exists a constant \( c > 0 \) such that \( \int_{\Omega} a_\infty |\nabla u|^p \, dx \geq c||u||^p_p \) for every \( u \in W^{1,p}(\Omega) \) with \( \int_{\Omega} m|u|^p \, dx > 0 \).

**Lemma 2.2.** Assume that \( \int_{\Omega} m \, dx \neq 0 \) and \( \xi > 0 \). Then, there exists a constant \( b(m, \xi) > 0 \) such that

\[
\int_{\Omega} a_\infty |\nabla u|^p \, dx - \xi \int_{\Omega} m|u|^p \, dx \geq b(m, \xi) \int_{\Omega} |u|^p 
\]

for every \( u \in B(m) := \{u \in W^{1,p}(\Omega); \int_{\Omega} m|u|^p \, dx \leq 0\} \).

**Lemma 2.3.** Assume that \( m \geq 0 \) in \( \Omega \). Then, for every \( \xi > 0 \) there existed \( d(m, \xi) > 0 \) such that

\[
\int_{\Omega} a_\infty |\nabla u|^p \, dx - \xi \int_{\Omega} m|u|^p \, dx \geq d(m, \xi) \int_{\Omega} |u|^p 
\]

for every \( u \in W^{1,p}(\Omega) \).

First, we recall the following principle eigenvalue \( \lambda^*(m) \):

\[
\lambda^*(m) := \inf \left\{ \int_{\Omega} a_\infty |\nabla u|^p \, dx; u \in W^{1,p}(\Omega), \int_{\Omega} m|u|^p \, dx = 1 \right\}.
\]

Because of \( \infty > \sup_{x \in \Omega} a_\infty(x) \geq \inf_{x \in \Omega} a_\infty(x) > 0 \), we have the following result as the same argument as in the case of the \( p \)-Laplacian.

**Proposition 2.4** (see [7, Proposition 2.2]). The following assertions hold:

(i) If \( \int_{\Omega} m \, dx \geq 0 \) holds, then \( \lambda^*(m) = 0 \);

(ii) if \( \int_{\Omega} m \, dx < 0 \) holds, then \( \lambda^*(m) > 0 \) is a simple eigenvalue and it admits a positive eigenfunction. In addition, the open interval \((0, \lambda^*(m))\) contains no eigenvalues of \((EV; m)\).
Lemma 2.5. Assume $\int_\Omega m \, dx < 0$. Then, one has $\lambda^*(m + \epsilon) < \lambda^*(m) < \lambda^*(m - \epsilon')$ for every $\epsilon > 0$ and $\epsilon' > 0$ with $|(m - \epsilon') > 0| > 0$.

Proof. We choose a minimizer $u$ for $\lambda^*(m)$ because Proposition 2.4 guarantees the existence of it. Then, for every $\epsilon > 0$, we have

$$
\lambda^*(m + \epsilon) \leq \frac{\int_\Omega a_\omega |\nabla u|^p \, dx}{\int_\Omega (m + \epsilon) |u|^p \, dx} < \frac{\int_\Omega a_\omega |\nabla u|^p \, dx}{\int_\Omega m |u|^p \, dx} = \int_\Omega a_\omega |\nabla u|^p \, dx = \lambda^*(m)
$$

(2.4)

by the definition of $\lambda^*(m + \epsilon)$. By applying the same argument to a minimizer for $\lambda^*(m - \epsilon')$, we obtain $\lambda^*(m) < \lambda^*(m - \epsilon')$ for $\epsilon' > 0$ with $|(m - \epsilon') > 0| > 0$. \qed

2.2. Other Eigenvalues

Here, we introduce two unbounded sequences $\{\lambda_n(m)\}_n$ and $\{\mu_n(m)\}_n$ as follows:

$$
J(u) := \int_\Omega a_\omega |\nabla u|^p \, dx \text{ for } u \in W^{1,p}(\Omega), \quad \tilde{J} := J|_{S(m)},
$$

$$
S(m) := \left\{ u \in W^{1,p}(\Omega); \int_\Omega m |u|^p \, dx = 1 \right\},
$$

$$
\mathcal{S}_n(m) := \{ X \subset S(m); \text{ compact, symmetric and } \gamma(X) \geq n \},
$$

$$
\mathcal{F}_n(m) := \left\{ g \in C \left( S^{n-1}, S(m) \right); g \text{ is odd} \right\},
$$

$$
\lambda_n(m) := \inf_{X \in \mathcal{S}_n(m)} \max_{u \in X} \tilde{J}(u),
$$

$$
\mu_n(m) := \inf_{g \in \mathcal{F}_n(m)} \max_{z \in S^{n-1}} \tilde{J}(g(z)),
$$

where $\gamma(X)$ denotes the Krasnoselskii genus of $X$ (see [17, Definition 5.1] for the definition) and $S^{n-1}$ denotes the usual unit sphere in $\mathbb{R}^n$. We see that $\lambda_n(m)$ is defined by Ljusternik-Schnirelman theory and it is known that the definition of $\mu_n(m)$ is introduced by Drábek and Robinson ([16]) under the $p$-Laplace Dirichlet problem with $m \equiv 1$.

Remark 2.6. The following assertions can be shown easily:

(i) $\lambda_1(m) = \mu_1(m) = \lambda^*(m)$;

(ii) $\mathcal{S}_n(m) \neq \emptyset$ and $\mathcal{F}_n(m) \neq \emptyset$ for every $n \in \mathbb{N}$;

(iii) $g(S^{n-1}) \subset \mathcal{S}_n(m)$ for every $g \in \mathcal{F}_n(m)$;

(iv) $\mu_n(m) \geq \lambda_n(m)$ for every $n \in \mathbb{N}$;

(v) $\lambda_{n+1}(m) \geq \lambda_n(m)$ and $\mu_{n+1}(m) \geq \mu_n(m)$ for every $n \in \mathbb{N}$,

see [18] for the proof of (ii).
Define a $C^1$ functional $\Phi_m$ on $W^{1,p}(\Omega)$ by $\Phi_m(u) := \int_\Omega m|u|^p\,dx$ for $u \in W^{1,p}(\Omega)$. Because $1 \in \mathbb{R}$ is a regular value of $\Phi_m$, it is well known that the norm of the derivative at $u \in S(m)$ of the restriction of $J$ to $S(m)$ is defined as follows:

$$
\|\bar{J}'(u)\|_* := \min\left\{\|J'(u) - t\Phi_m'(u)\|_{W^{1,p}(\Omega)}; t \in \mathbb{R}\right\}
$$

$$
= \sup\{\langle J'(u), v \rangle; v \in T_u(S(m)), \|v\| = 1\},
$$

where $T_u(S(m))$ denotes the tangent space of $S(m)$ at $u$, that is, $T_u(S(m)) = \{v \in W^{1,p}(\Omega); \int_\Omega m|u|^{p-2}uv\,dx = 0\}$. Here, we recall the definition of the Palais-Smale condition for $\bar{J}$.

**Definition 2.7.** $\bar{J}$ is said to satisfy the bounded Palais-Smale condition if any bounded sequence $u_n \in S(m)$ such that $\|\bar{J}(u_n)\|_\infty \to 0$ has a convergent subsequence. Moreover, we say that $\bar{J}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ if any sequence $u_n \in S(m)$ such that $\bar{J}(u_n) \to c$ and $\|\bar{J}(u_n)\|_\infty \to 0$ as $n \to \infty$ has a convergent subsequence. In addition, we say that $\bar{J}$ satisfies the Palais-Smale condition if $\bar{J}$ satisfies the Palais-Smale condition for every $c \in \mathbb{R}$.

The following result can be proved by the same argument as in [19, Proposition 3.3] (which treats the case of the $p$-Laplacian, i.e., $a_\infty \equiv 1$) because of $\infty > \sup_{x \in \Omega} a_\infty(x) \geq \inf_{x \in \Omega} a_\infty(x) > 0$. Here, we omit the proof.

**Lemma 2.8.** The following assertions hold:

(i) $\bar{J}$ satisfies the bounded Palais-Smale condition;

(ii) $\bar{J}$ satisfies the Palais-Smale condition provided $\int_\Omega m\,dx \neq 0$.

**Proposition 2.9.** $\lambda_n(m)$ and $\mu_n(m)$ are eigenvalues of $(EV; m)$ such that

$$
\lim_{n \to \infty} \lambda_n(m) = \lim_{n \to \infty} \mu_n(m) = +\infty.
$$

**Proof.** In the case of $\int_\Omega m\,dx \neq 0$, since $\bar{J}$ satisfies the Palais-Smale condition, we can apply the first deformation lemma on $C^1$ manifold (refer to [20]). Thus, by the standard argument, we can prove that $\lambda_n(m)$ and $\mu_n(m)$ are critical values of $\bar{J}$. This means that $\lambda_n(m)$ and $\mu_n(m)$ are eigenvalues of $(EV; m)$ by the Lagrange multiplier rule. In addition, we can easily show $\lim_{n \to \infty} \lambda_n(m) = +\infty$ by the standard argument via the first deformation lemma on $C^1$ manifold (refer to [21, Proposition 3.14.7], [22] or [17] in the case of a Banach space). Hence, $\lim_{n \to \infty} \mu_n(m) = +\infty$ holds because of $\mu_n(m) \geq \lambda_n(m)$ for every $n \in \mathbb{N}$.

In the case of $\int_\Omega m\,dx = 0$, by the same argument as in [18], our conclusion can be proved. For readers’ convenience, we give a sketch of the proof. For $\varepsilon > 0$, we define $J_{\varepsilon}(u) := J(u) + \varepsilon\|u\|_p^p$ and $\bar{J}_{\varepsilon} := J_{\varepsilon}|_{S(m)}$. Moreover, we set minimax values $\lambda^\varepsilon_n(m)$ and $\mu^\varepsilon_n(m)$ of $\bar{J}_{\varepsilon}$ by

$$
\lambda^\varepsilon_n(m) := \inf_{X \in S_{\varepsilon}(m)} \max_{u \in X} \bar{J}_{\varepsilon}(u), \quad \mu^\varepsilon_n(m) := \inf_{g \in \mathcal{F}_{\varepsilon}(m)} \max_{z \in S^{n-1}} \bar{J}_{\varepsilon}(g(z)).
$$

Because any Palais-Smale sequence of $\bar{J}_{\varepsilon}$ is bounded, it is easily shown that $\bar{J}_{\varepsilon}$ satisfies the Palais-Smale condition (refer to [19, Proposition 3.3]) Hence, it can be proved that $\lambda^\varepsilon_n(m)$
and \( \mu_n^\varepsilon(m) \) are critical values of \( \overline{I}_\varepsilon \). Furthermore, it follows from the argument as in [18, Lemma 3.5] that \( \lambda_n^\varepsilon(m) \to \lambda_n(m) \) and \( \mu_n^\varepsilon(m) \to \mu_n(m) \) as \( \varepsilon \to 0^+ \). Therefore, by noting that \( J_\varepsilon \) is \( p \)-homogeneous, we can obtain a solution \( u_\varepsilon \) with \( \|u_\varepsilon\| = 1 \) for \( -\text{div} (a_\infty |\nabla u|^{p-2}\nabla u) = c_\varepsilon m|u|^{p-2}u \) in \( \Omega \), \( \partial u/\partial \nu = 0 \) on \( \partial\Omega \), where \( c_\varepsilon = \lambda_n^\varepsilon(m) \) or \( \mu_n^\varepsilon(m) \). Because of \( \|u_\varepsilon\| = 1 \), it follows from the standard argument that \( u_\varepsilon \) has a subsequence strongly convergent to a solution \( u \) for

\[
-\text{div} (a_\infty |\nabla u|^{p-2}\nabla u) = cm|u|^{p-2}u \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.9}
\]

where \( c = \lim_{\varepsilon \to 0^+} c_\varepsilon \). Thus, \( \lambda_n(m) \) and \( \mu_n(m) \) are eigenvalues of \( (EV; m) \). To prove \( \lim_{n \to \infty} \lambda_n(m) = +\infty \), by considering a function \( m_\delta(x) := \max\{|m(x), \delta| \} \) for \( \delta > 0 \), we have \( \lambda_n(m_\delta) \leq \lambda_n(m) \) (refer to Proposition 2.10). Because we can apply our first assertion to \( m_\delta \) (note \( \int_\Omega m_\delta \, dx > 0 \)), we obtain \( \lim_{n \to \infty} \mu_n(m) \geq \lim_{n \to \infty} \lambda_n(m) \geq \lim_{n \to \infty} \lambda_n(m_\delta) = +\infty \).

**Proposition 2.10.** Let \( 1 < r < \infty \) if \( N \leq p \) and \( p^*/(p^* - p) < r < \infty \) if \( N > p \). Then, the following assertions hold:

(i) if \( m' \geq m \) in \( \Omega \), then \( \mu_k(m') \leq \mu_k(m) \);

(ii) if \( \lim_{n \to \infty} m_n = m \) in \( L'(\Omega) \), then \( \sup_{n \to \infty} \mu_k(m_n) \leq \mu_k(m) \);

(iii) if \( \int_\Omega m \, dx \neq 0 \) and \( \lim_{n \to \infty} m_n = m \) in \( L'(\Omega) \), then \( \lim_{n \to \infty} \mu_k(m_n) = \mu_k(m) \).

Moreover, the same conclusion holds for \( \lambda_k(m) \).

**Proof.** We only treat \( \mu_k(m) \) because we can give the proof for \( \lambda_k(m) \) similarly.

(i) Let \( m' \geq m \) in \( \Omega \). Fix an arbitrary \( \varepsilon > 0 \). Then, by the definition of \( \mu_k(m) \), there exists a \( g \in \mathcal{F}_k(m) \) such that \( \max_{z \in S^{k-1}} J(g(z)) < \mu_k(m) + \varepsilon \). Set \( \tilde{g}(z) := g(z)/ (\int_\Omega m' |g(z)|^p \, dx)^{1/p} \) for \( z \in S^{k-1} \) (note \( \int_\Omega m' |g(z)|^p \, dx \geq \int_\Omega m |g(z)|^p \, dx = 1 \), then \( \tilde{g} \in \mathcal{F}_k(m') \) holds. Therefore, by the definition of \( \mu_k(m') \), we have

\[
\mu_k(m') \leq \max_{z \in S^{k-1}} J(\tilde{g}(z)) = \max_{z \in S^{k-1}} \frac{J(g(z))}{\int_\Omega m' |g(z)|^p \, dx} \leq \max_{z \in S^{k-1}} J(g(z)) < \mu_k(m) + \varepsilon. \tag{2.10}
\]

because of \( \int_\Omega m' |g(z)|^p \, dx \geq \int_\Omega m |g(z)|^p \, dx = 1 \) for every \( z \in S^{k-1} \). Since \( \varepsilon > 0 \) is arbitrary, we obtain \( \mu_k(m') \leq \mu_k(m) \).

(ii) Let \( \lim_{n \to \infty} m_n = m \) in \( L'(\Omega) \) and fix an arbitrary \( \varepsilon > 0 \). By the definition of \( \mu_k(m) \), there exists a \( g \in \mathcal{F}_k(m) \) such that \( \max_{z \in S^{k-1}} J(g(z)) < \mu_k(m) + \varepsilon/2 \). Since \( g(S^{k-1}) \) is compact and \( pr' := pr/(r - 1) \leq p^* \), we set \( M := \max_{z \in S^{k-1}} \|u\|_{pr'} \). Then, due to Holder’s inequality and \( m \to m \) in \( L'(\Omega) \), there exists an \( n_0 \in \mathbb{N} \) such that

\[
\int_\Omega m_n |u|^p \, dx = 1 + \int_\Omega (m_n - m) |u|^p \, dx \geq 1 - \|m_n - m\|_p M^p > 0 \tag{2.11}
\]

for every \( u \in g(S^{k-1}) \) and \( n \geq n_0 \). Therefore, by a similar argument to (i), we obtain

\[
\mu_k(m_n) \leq \max_{z \in S^{k-1}} J(g(z)) \leq \frac{\mu_k(m) + \varepsilon/2}{1 - \|m_n - m\|_p M^p} < \mu_k(m) + \varepsilon \tag{2.12}
\]
for sufficiently large $n$. Hence, $\limsup_{n \to \infty} \mu_k(m_n) \leq \mu_k(m) + \varepsilon$ follows. Since $\varepsilon > 0$ is arbitrary, our conclusion is proved.

(iii) Let $\lim_{n \to \infty} m_n = m$ in $L^r(\Omega)$ and $\int_\Omega m\,dx \neq 0$. We fix an arbitrary $\varepsilon > 0$. Due to our assertion (ii), there exists an $n_1 \in \mathbb{N}$ such that $\mu_k(m_n) \leq \mu_k(m) + \varepsilon/2$. For every $n \geq n_1$, by the definition of $\mu_k(m_n)$, we can take $g_n \in \mathcal{F}_k(m_n)$ satisfying $\max_{z \in S^{k-1}} J(g_n(z)) < \mu_k(m_n) + \varepsilon/2$.

Here, we will prove

$$\sup_{n \geq n_1} \max \left\{ ||u||_p; u \in g_n(S^{k-1}) \right\} < \infty. \tag{2.13}$$

If $u \in g_n(S^{k-1})$ satisfies $\int_\Omega m|u|^p\,dx \leq 0$, then we obtain

$$b(m,1)||u||_p^p \leq J(u) - \int_\Omega m|u|^p\,dx = J(u) - \int_\Omega m_n|u|^p\,dx + \int_\Omega (m_n - m)|u|^p\,dx$$

$$\leq \mu_k(m_n) + R - 1 + ||m_n - m||_r||u||_{p'}^p$$

$$\leq \mu_k(m) + \varepsilon + C||m_n - m||_r||u||_{p'}^p + \frac{CJ(u)||m_n - m||_r}{\inf_{\Omega} a_{\infty}}$$

$$\leq \left( 1 + \frac{C||m_n - m||_r}{\inf_{\Omega} a_{\infty}} \right)(\mu_k(m) + \varepsilon) + C||m_n - m||_r||u||_{p'}^p \tag{2.14}$$

by Lemma 2.2 and Hölder’s inequality (note $||u||_{p'}^p \leq J(u)/\inf_{\Omega} a_{\infty}$ and $\mu_k(m_n) \leq \mu_k(m) + \varepsilon/2$), where $C > 0$ is a constant (independent of $n$ and $u$) obtained by the continuity of $W^{1,p}(\Omega)$ into $L^{p'}(\Omega)$. Therefore, if we take an $n_2 \geq n_1$ satisfying $C||m_n - m||_r \leq b(m,1)/2$ for every $n \geq n_2$, then we obtain

$$||u||_{p'}^p \leq \frac{2}{b(m,1)} \left( 1 + \frac{b(m,1)}{2\inf_{\Omega} a_{\infty}} \right)(\mu_k(m) + \varepsilon) \tag{2.15}$$

for every $u \in g_n(S^{k-1})$ provided $\int_\Omega m|u|^p\,dx \leq 0$ and $n \geq n_2$. Similarly, in the case where $m$ changes sign, for every $u \in g_n(S^{k-1})$ satisfying $\int_\Omega m|u|^p\,dx > 0$, we have

$$b(-m,1)||u||_p^p \leq J(u) - \int_\Omega (-m)|u|^p\,dx$$

$$\leq \left( 1 + \frac{C||m_n - m||_r}{\inf_{\Omega} a_{\infty}} \right)(\mu_k(m) + \varepsilon) + 1 + C||m_n - m||_r||u||_{p'}^p. \tag{2.16}$$

Hence, by taking a sufficiently large $n_3 \geq n_2$, we get the inequality

$$||u||_{p'}^p \leq \frac{2}{b(-m,1)} \left( 1 + \frac{b(-m,1)}{2\inf_{\Omega} a_{\infty}} \right)(\mu_k(m) + \varepsilon + 1), \tag{2.17}$$
for every $u \in g_n(S^{k-1})$ with $\int_{\Omega} m|u|^p \, dx > 0$ and $n \geq n_3$. In the case of $m \geq 0$ in $\Omega$, by using Lemma 2.3 instead of Lemma 2.2, we have a similar inequality

$$
\|u\|_p^p \leq \frac{2}{d(m,1)} \left(1 + \frac{d(m,1)}{2 \inf_{\Omega} a_\infty}\right)(\mu_k(m) + \varepsilon + 1),
$$

(2.18)

for every $u \in g_n(S^{k-1})$ provided $n \geq n_4$ (some sufficiently large $n_4 \geq n_3$). Consequently, our claim follows from (2.15), (2.17), and (2.18).

Let us return to the proof of (iii). Because

$$
\sup\{\|u\|_{pr}; u \in g_n(S^{k-1}), n \geq n_3\} =: R < +\infty
$$

(2.19)

holds by (2.13), $J(u) \leq \mu_k(m) + \varepsilon/2$ and the continuity of $W^{1,p}(\Omega)$ into $L^{pr'}(\Omega)$, we see the inequality

$$
\int_{\Omega} m|u|^p \, dx = 1 - \int_{\Omega} (m_n - m)|u|^p \, dx > 1 - \|m_n - m\|_{pr} > 0,
$$

(2.20)

for every $u \in g_n(S^{k-1})$ and $n \geq n_5$ (some sufficiently large $n_5 \geq n_4$). By considering $\tilde{g}_n(\cdot) := g_n(\cdot)/\left(\int_{\Omega} m|g_n(\cdot)|^p \, dx\right)^{1/p} \in \mathcal{F}_k(m)$, we obtain

$$
\mu_k(m) \leq \max_{z \in S^{k-1}} J(\tilde{g}_n(z)) \leq \frac{\max_{z \in S^{k-1}} J(g_n(z))}{1 - \|m_n - m\|_{pr}} \leq \frac{\mu_k(m_n) + \varepsilon/2}{1 - \|m_n - m\|_{pr}}.
$$

(2.21)

Because of $\|m_n - m\|_r \to 0$, we get $\mu_k(m_n) \geq \mu_k(m) - \varepsilon$ for sufficiently large $n$, and hence our conclusion holds.

Finally, we recall the second eigenvalue of $(EV;m)$ obtained by the mountain pass theorem.

$$
\Sigma(m) := \{\eta \in C([0,1], S(m)); \eta(0) \in P, \eta(1) \in (-P)\},
$$

$$
c(m) := \inf_{\eta \in \Sigma(m)} \max_{t \in [0,1]} \tilde{f}(\eta(t)),
$$

(2.22)

where $P := \{u \in W^{1,p}(\Omega); u(x) \geq 0 \text{ a.e. } x \in \Omega\}$.

Since $\infty > \sup_{x \in \Omega} a_\infty(x) \geq \inf_{x \in \Omega} a_\infty(x) > 0$ holds, the following result can be shown by the same argument as in [19] (although they handle the asymmetry case, it is sufficient to consider the case of $m \equiv n$ in this paper). See [19, Theorem 3.2] for the proof.

**Theorem 2.11.** $c(m)$ is an eigenvalue of $(EV;m)$ which satisfies $\lambda^\ast(m) < c(m)$. Moreover, there is no eigenvalues of $(EV;m)$ between $\lambda^\ast(m)$ and $c(m)$. 

Now, we have the following result.

**Proposition 2.12.**

\[ \lambda_2(m) = \mu_2(m) = c(m) \]  \hspace{1cm} (2.23)

holds, where \( c(m) \) is a minimax value defined by (2.22).

**Proof.** First, we prove the inequality \( c(m) \geq \mu_2(m) \). Because \( c(m) \) is an eigenvalue (note that the following equation is homogeneous), we can choose a solution \( u \in W^{1,p}(\Omega) \) with

\[ \int_{\Omega} m|u|^p \, dx = 1 \]

for

\[-\text{div} \left( a_{\infty}(x)|\nabla u|^{p-2}\nabla u \right) = c(m)m(x)|u|^{p-2}u \quad \text{in} \ \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega. \]  \hspace{1cm} (2.24)

Note that \( u \) is a sign-changing function because any eigenfunction associated with any eigenvalue greater than the principal eigenvalue changes sign (refer to [18, Proposition 4.3]). Thus, we have

\[ 0 < \int_{\Omega} a_{\infty} |\nabla u_+|^p \, dx = c(m) \int_{\Omega} mu_+^p \, dx \]  \hspace{1cm} (2.25)

by taking \( \pm u_+ \) as test function (recall that \( u_+ := \max\{\pm u, 0\} \)). Hence, we may assume that \( \int_{\Omega} mu_+^p \, dx = 1 \) by the normalization. Set \( X := \{ su_+ - tu_-; |s|^p + |t|^p = 1 \} \subset S(m) \). Then, because \( X \) is homeomorphic to \( S^1 \), there exists \( g \in \mathcal{F}_2(m) \) such that \( g(S^1) = X \). Since the value of \( J \) is equal to \( c(m) \) on \( X \), we obtain

\[ \mu_2(m) \leq \max_{z \in S^1} \tilde{f}(g(z)) = c(m) \]  \hspace{1cm} (2.26)

by the definition of \( \mu_2(m) \) and \( X \).

Next, we will prove the inequality \( c(m) \leq \lambda_2(m) \) by dividing into two cases: \( \int_{\Omega} m \, dx \neq 0 \) and \( \int_{\Omega} m \, dx = 0 \).

Case of \( \int_{\Omega} m \, dx \neq 0 \): by way of contradiction, we assume that \( \lambda_2(m) < c(m) \). Then, \( \lambda^*(m) = \lambda_1(m) = \lambda_2(m) \) follows from Theorem 2.11. Note that \( \tilde{f} \) satisfies the Palais-Smale condition in this case (see Lemma 2.8), and hence we can apply the first deformation lemma to \( \tilde{f} \). Therefore, by the standard argument (cf. [22], [17, Lemma 5.6]), we see that \( \gamma(K) \geq 2 \), where \( K := \{ u \in S(m); \tilde{f}(u) = 0, \tilde{f}(u) = \lambda^*(m) \} \). This means that \( K \) is an infinite set, that is, the following equation has infinite many solutions:

\[-\text{div} \left( a_{\infty}(x)|\nabla u|^{p-2}\nabla u \right) = \lambda^*(m)m(x)|u|^{p-2}u \quad \text{in} \ \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega \]  \hspace{1cm} (2.27)

due to the Lagrange multiplier’s rule. This contradicts to the fact described as in Proposition 2.4 that \( \lambda^*(m) \) is simple. As a result, we have shown that \( c(m) = \lambda_2(m) = \mu_2(m) \) holds in the case of \( \int_{\Omega} m \, dx \neq 0 \) (note \( \lambda_n(m) \leq \mu_n(m) \)).
Due to bracketleftmath\begin{equation}
\begin{split}
I_{\lambda,m}(u) &= \int_{\Omega} G(x, \nabla u) \, dx - \frac{\lambda}{p} \int_{\Omega} m|u|^p \, dx - \int_{\Omega} F(x, u) \, dx - \int_{\Omega} hu \, dx \\
&= \frac{1}{p} \int_{\Omega} a_{\omega} |\nabla u|^p \, dx + \int_{\Omega} \tilde{G}(x, \nabla u) \, dx - \frac{\lambda}{p} \int_{\Omega} m|u|^p \, dx \\
&\quad - \int_{\Omega} F(x, u) \, dx - \int_{\Omega} hu \, dx
\end{split}
\end{equation}
for u ∈ W^{1,p}(\Omega) ((1.15) or (1.9) for the definition of G, \tilde{G}, and F). It is easily seen that \( I_{\lambda,m} \) is well defined and class of \( I_{\lambda,m} \in C^1 \) on \( W^{1,p}(\Omega) \) by (1.1), (1.16) and the continuity of \( W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \).

Remark 3.1. Let \( u \in W^{1,p}(\Omega) \) be a critical point of \( I_{\lambda,m} \), namely, \( u \) satisfies the equality
\begin{equation}
\int_{\Omega} A(x, \nabla u) \nabla \varphi \, dx = \lambda \int_{\Omega} m|u|^{p-2} u \varphi \, dx + \int_{\Omega} f(x, u) \varphi \, dx + \int_{\Omega} h \varphi \, dx
\end{equation}
for every \( \varphi \in W^{1,p}(\Omega) \). Then, \( u \in L^\infty(\Omega) \) by the Moser iteration process (refer to Theorem C in [4]). Therefore, \( u \in C^{1,\alpha}(\overline{\Omega}) \) (0 < \( \alpha < 1 \)) follows from the regularity result in [23]. Furthermore, due to [24, Theorem 3], \( u \) satisfies \( (P; \lambda, m, h) \) in the distribution sense and the boundary condition
\begin{equation}
0 = \frac{\partial u}{\partial \nu_A} = A(\cdot, \nabla u)\nu = a(\cdot, |\nabla u|) \frac{\partial u}{\partial \nu} \quad \text{in} \ W^{-1,q}(\partial\Omega)
\end{equation}
for every 1 < \( q < \infty \) (see [24] for the definition of \( W^{-1,q}(\partial\Omega) \)). Since \( u \in C^{1,\alpha}(\overline{\Omega}) \) and \( a(x, t) > 0 \) for every \( t \neq 0 \), \( u \) satisfies the Neumann boundary condition, that is, \( (\partial u/\partial \nu)(x) = 0 \) for every \( x \in \partial\Omega \).

3.1. The Palais-Smale Condition in the Nonresonant Case

First, we recall the definition of the Palais-Smale condition.

Definition 3.2. A \( C^1 \) functional \( \Psi \) on a Banach space \( X \) is said to satisfy the Palais-Smale condition at \( c \in \mathbb{R} \) if a Palais-Smale sequence \( \{u_n\} \subset X \) at level \( c \), namely,
\begin{equation}
\Psi(u_n) \longrightarrow c, \quad \|\Psi'(u_n)\|_{X'} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty
\end{equation}
has a convergent subsequence. We say that \( \Psi \) satisfies the Palais-Smale condition if \( \Psi \) satisfies the Palais-Smale condition at any \( c \in \mathbb{R} \). Moreover, we say that \( \Psi \) satisfies the bounded Palais-Smale condition if any bounded sequence \( \{u_n\} \) such that \( \{\Psi(u_n)\} \) is bounded and \( \|\Psi'(u_n)\|_X \to 0 \) as \( n \to \infty \) has a convergent subsequence.

Concerning the Palais-Smale condition, we state the following result developed from [6, Proposition 7].

**Proposition 3.3.** If \( \lambda \) is not an eigenvalue of \((EV;m)\), then \( I_{\lambda,m} \) satisfies the Palais-Smale condition.

**Proof.** Let \( \{u_n\} \) be a Palais-Smale sequence of \( I_{\lambda,m} \), namely,

\[
I_{\lambda,m}(u_n) \to c, \quad \left\| I'_{\lambda,m}(u_n) \right\|_{W^{1,p}(\Omega)^*} \to 0 \quad \text{as} \quad n \to \infty
\]

for some \( c \in \mathbb{R} \). It is sufficient to prove only the boundedness of \( \|u_n\| \) because of the operator \( A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^* \) described in Proposition 1.5 has the (S)+ property.

To prove the boundedness of \( \|u_n\| \), it suffices to show that \( \|u_n\|_p \) is bounded because of the inequality \( |f(x,u)| \leq C(|u|^{p-1} + 1) \) (obtained by (1.1)) and the following inequality:

\[
\left( I'_{\lambda,m}(u_n), u_n \right) + \lambda \int_{\Omega} m|u_n|^p \, dx + \int_{\Omega} f(x,u_n)u_n \, dx + \int_{\Omega} hu_n \, dx,
\]

\[
= \int_{\Omega} A(x, \nabla u_n) \nabla u_n \, dx \geq \frac{C_0}{p-1} \|\nabla u_n\|_p^p,
\]

where we use Remark 1.4 (iii) in the last inequality. By way of contradiction, we may assume that \( \|u_n\|_p \to \infty \) as \( n \to \infty \) by choosing a subsequence if necessary. Set \( v_n := u_n / \|u_n\|_p \). Then, since the inequality (3.6) guarantees that \( \{v_n\} \) is bounded in \( W^{1,p}(\Omega) \), we may suppose, by choosing a subsequence, that \( v_n \to v_0 \) in \( W^{1,p}(\Omega) \) and \( v_n \to v_0 \) in \( L^p(\Omega) \) for some \( v_0 \).

Here, we will prove that

\[
\lim_{n \to \infty} \frac{\|f(\cdot, u_n)\|_p}{\|u_n\|_p^{p-1}} = 0,
\]

where \( p' = p/(p-1) \). Fix an arbitrary \( \varepsilon > 0 \). It follows from (1.1) that there exists a \( C_\varepsilon > 0 \) such that

\[
|f(x,u)| \leq \varepsilon |u|^{p-1} + C_\varepsilon \quad \text{for every} \quad u \in \mathbb{R}, \quad \text{a.e.} \quad x \in \Omega.
\]

Then, we obtain

\[
\int_{\Omega} \left| f(x,u_n) \right|^{p'} \, dx \leq 2^{p'} \int_{\Omega} \left( \varepsilon^{p'} |u_n|^{p'} + C_\varepsilon^{p'} \right) \, dx \leq 2^{p'} \varepsilon^{p'} \|u_n\|_p^{p'} + 2^{p'} C_\varepsilon^{p'} |\Omega|,
\]

(3.9)
Since we are assuming that \( \| u_n \|_p \to \infty \) as \( n \to \infty \), there exists \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \)

\[
\frac{\| f(\cdot, u_n) \|_{p'}}{\| u_n \|_{p}^{p-1}} \leq 4\varepsilon
\]  

(3.10)

holds. This shows that \( \lim_{n \to \infty} \| f(\cdot, u_n) \|_{p'}/\| u_n \|_{p}^{p-1} = 0 \) because \( \varepsilon > 0 \) is arbitrary.

Here, we recall the following result proved in [6]:

\[
\lim_{n \to -\infty} \int_{\Omega} \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla (v_n - v_0)}{\| u_n \|_{p}^{p-1}} \, dx = \lim_{n \to -\infty} \int_{\Omega} \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla \varphi}{\| u_n \|_{p}^{p-1}} \, dx = 0,
\]  

(3.11)

for every \( \varphi \in W^{1,p}(\Omega) \). Thus, by considering

\[
o(1) = \frac{\langle I_{\lambda,m}^{'}(u_n), v_n - v_0 \rangle}{\| u_n \|_{p}^{p-1}} = \int_{\Omega} a_{\infty} |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v_0) \, dx + o(1),
\]  

(3.12)

we see that \( v_n \) strongly converges to \( v_0 \) in \( W^{1,p}(\Omega) \) (note that \( p \)-Laplacian has the \((S)\) property). Therefore, by taking a limit in \( o(1) = \langle I_{\lambda,m}^{'}(u_n), \varphi \rangle/\| u_n \|_{p}^{p-1} \) for any \( \varphi \in W^{1,p}(\Omega) \) and by noting (3.7) and (3.11), we know that \( v_0 \) is a nontrivial solution (note \( \| v_0 \|_p = 1 \) of

\[
-\text{div}(a_{\infty} |\nabla u|^{p-2} \nabla u) = \lambda m |u|^{p-2} u \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega.
\]  

(3.13)

This means that \( \lambda \) is an eigenvalue of \((EV; m)\). This is a contradiction. Hence, \( \| u_n \|_p \) is bounded. \( \square \)

### 3.2. Key Lemmas

To show the linking lemma, we define

\[
Y(\mu, m) := \left\{ u \in W^{1,p}(\Omega); \int_{\Omega} a_{\infty} |\nabla u|^{p} \, dx \geq \mu \int_{\Omega} m |u|^{p} \, dx \right\}
\]  

(3.14)

for \( \mu \in \mathbb{R} \).

**Lemma 3.4.** Let \( g_0 \in C(S^{k-1}, W^{1,p}(\Omega) \setminus \{0\}) \) be odd and \( 0 < \mu \leq \mu_{k+1}(m) \). Then, \( g(S_\mu^+) \cap Y(\mu, m) \neq \emptyset \) for every \( g \in C(S_\mu^+, W^{1,p}(\Omega)) \) with \( g|_{S_\mu^+} = g_0 \), where \( Y(\mu, m) \) is the set introduced in (3.14) and \( S_\mu^+ \) is the upper hemisphere in \( \mathbb{R}^{k+1} \) with boundary \( S_k \).
Proof. Fix any \( g \in C(S^k, W^{1,p}(\Omega)) \) such that \( g|_{S^{k-1}} = g_0 \). If \( u \in C(S^k) \) satisfies \( \int_D m|u|^p \, dx \leq 0 \), then \( u \in Y(\mu, m) \) holds. So, we may assume that \( \int_D m|u|^p \, dx > 0 \) for every \( u \in C(S^k) \). Define \( \bar{g} \in \mathcal{F}_{k+1}(m) \) as follows:

\[
\bar{g}(z) := \begin{cases} 
\frac{g(z)}{\int_D m|g(z)|^p \, dx} & \text{if } z \in S^k, \\
\frac{g(z)}{\int_D m|g(z)|^p \, dx} & \text{if } z \in S^k.
\end{cases}
\] (3.15)

By the definition of \( \mu_{k+1}(m) \), there exists \( z_0 \in S^k \) such that \( \bar{f}(\bar{g}(z_0)) \geq \mu_{k+1}(m) \). Since \( \bar{g} \) is odd and \( J \) is even, we may suppose \( z_0 \in S^k \). So, this yields the inequality \( J(\bar{g}(z_0)) \geq \mu_{k+1}(m) \int_D m|g(z_0)|^p \, dx \geq \mu \int_D m|g(z_0)|^p \, dx \), whence \( g(z_0) \in Y(\mu, m) \) holds.

\[\square\]

**Lemma 3.5.** Let \( \mu_k(m) < \lambda \). Then, there exists \( g_0 \in \mathcal{F}_k(m) \) such that

\[
\max_{z \in S^{k-1}} J(g_0(z)) < \lambda, \quad \max_{z \in S^{k-1}} I_{\lambda,m}(T g_0(z)) \to -\infty \quad \text{as } |T| \to \infty,
\] (3.16)

where \( \mu_k(m) \) is defined by (2.5).

*Proof.* Choose \( \epsilon_0 > 0 \) such that \( \mu_k(m) + \epsilon_0 < \lambda \). By the definition of \( \mu_k(m) \), there exists \( g_0 \in \mathcal{F}_k(m) \) such that

\[
\max_{z \in S^{k-1}} J(g_0(z)) < \mu_k(m) + \epsilon_0.
\] (3.17)

Due to the compactness of \( g_0(S^{k-1}) \), we put \( M := \max_{z \in S^{k-1}} \|g_0(z)\|_p \). By the property of the function \( \tilde{a} \) as in (AH) and Young’s inequality, for every \( \epsilon > 0 \) there exist constants \( C_\epsilon > 0 \) and \( C'_\epsilon > 0 \) such that

\[
|\tilde{G}(x, y)| \leq \frac{\epsilon}{2} |y|^p + C_\epsilon |y| \leq \epsilon |y|^p + C'_\epsilon \leq \frac{\epsilon}{\inf_{\Omega} a_\infty} a_\infty(x) |y|^p + C'_\epsilon
\] (3.18)

for every \( x \in \Omega \) and \( y \in \mathbb{R}^N \). Moreover, the hypothesis (1.1) ensures that for every \( \epsilon' > 0 \) there exist constants \( D_\epsilon' > 0 \) satisfying

\[
|F(x, u)| \leq \frac{\epsilon'}{2} |u|^p + D_\epsilon' |u| \leq \epsilon' |u|^p + D_\epsilon'
\] (3.19)

for every \( u \in \mathbb{R} \) and a.e. \( x \in \Omega \). Hence, we have

\[
I_{\lambda,m}(Tu) \leq \frac{T^p}{p} \left( 1 + \frac{\lambda \epsilon}{a} \right) \int_{\Omega} a_\infty |\nabla u|^p \, dx - \frac{T^p (\lambda - p \epsilon' M^p)}{p} + T \|h\|_{\infty} ||u||_1 + C
\]

\[
\leq \frac{T^p}{p} \left( \left( 1 + \frac{\lambda \epsilon}{a} \right) (\mu_k(m) + \epsilon_0) - \lambda + p M^p \epsilon' \right) + TM \|h\|_{\infty} \|\Omega\|^{(p-1)/p} + C
\] (3.20)
for every $T > 0$, $u \in g_0(S^{k-1})$, $\varepsilon > 0$ and $\varepsilon' > 0$ since $g_0(S^{k-1}) \subset S(m)$, (3.17), (3.18) and (3.19), where $C = (C'_\varepsilon + D'_\varepsilon)|\Omega|$ and $g = \inf_{x \in \Omega} a_\infty(x) > 0$. By taking $\varepsilon > 0$ and $\varepsilon' > 0$ satisfying $(1 + p/\varepsilon)(\mu_k(m) + \varepsilon_0) - \lambda + \varepsilon'M_p\varepsilon' < 0$, we show that $\max_{z \in S^{k-1}} I_{1,m}(Tg_0(z)) \to -\infty$ as $T \to +\infty$. Thus, our conclusion follows because $g_0(S^{k-1})$ is symmetric. 

\section{The Case $\int_\Omega m \, dx \neq 0$}

\textbf{Lemma 3.6.} Let $Tg_0(Tg_0(z)) \to -\infty$ as $T \to +\infty$. Thus, our conclusion follows because $g_0(S^{k-1})$ is symmetric.

\begin{equation}
I_{1,m}(u) \geq \frac{a - \varepsilon p}{pa} \int_\Omega a_\infty |\nabla u|^p \, dx - \frac{\lambda}{p} \int_\Omega m |u|^p \, dx - \varepsilon' \|u\|_p^p
\end{equation}

for every $u \in W^{1,p}(\Omega)$ and $\varepsilon' > 0$.

Let $u \in W^{1,p}(\Omega)$ satisfy $\int_\Omega m |u|^p \, dx \leq 0$. Then, the following inequality follows from Lemma 2.2:

\begin{equation}
D_0 \int_\Omega a_\infty |\nabla u|^p \, dx - \lambda \int_\Omega m |u|^p \, dx \geq D_0 \frac{1}{2} \int_\Omega a_\infty |\nabla u|^p \, dx + b(m, \xi) \|u\|_p^p,
\end{equation}

where $b(m, \xi)$ is a positive constant independent of $u$ with $\xi = 2\lambda / D_0$ and $D_0 = (a - \varepsilon p) / a$.

For every $u \in W^{1,p}(\Omega)$ such that $\int_\Omega m |u|^p \, dx > 0$, we obtain

\begin{equation}
D_0 \int_\Omega a_\infty |\nabla u|^p \, dx - \lambda \int_\Omega m |u|^p \, dx \geq \left(D_0 - \frac{\lambda}{\lambda^*(m)}\right) \int_\Omega a_\infty |\nabla u|^p \, dx
\geq \frac{1}{2} \left(D_0 - \frac{\lambda}{\lambda^*(m)}\right) \int_\Omega a_\infty |\nabla u|^p \, dx + \frac{c}{2} \left(D_0 - \frac{\lambda}{\lambda^*(m)}\right) \|u\|_p^p
\end{equation}

by the definition of $\lambda^*(m)$, Lemma 2.1 and $D_0 - \lambda / \lambda^*(m) > 0$, where $c > 0$ is a constant obtained by Lemma 2.1.

Consequently, if we choose a $\varepsilon' > 0$ satisfying $\varepsilon' < \min\{b(m, \xi) / p, c(D_0 - \lambda / \lambda^*(m)) / (2p)\}$, then we obtain positive constants $d_1$ and $d_2$ (independent of $u$) such that

\begin{equation}
I_{1,m}(u) \geq d_1 \int_\Omega a_\infty |\nabla u|^p \, dx + d_2 \|u\|_p^p - \|h\|_{\infty} \|u\|_p \|\Omega\|^{(p-1)/p} - (C'_\varepsilon + D'_\varepsilon)\|\Omega\|
\geq \min\{ad_1, d_2\} \|u\|^p - \|h\|_{\infty} \|u\|_p \|\Omega\|^{(p-1)/p} - (C'_\varepsilon + D'_\varepsilon)\|\Omega\|
\end{equation}
for every \( u \in W^{1,p}(\Omega) \) by (3.21), (3.22), and (3.23). Because of \( p > 1 \), our conclusion is shown.

**Lemma 3.7.** Let \( m \geq 0 \) in \( \Omega \) and \( m \neq 0 \). If \( \lambda < 0 \) holds, then \( I_{\lambda,m} \) is bounded from below, coercive and w.l.s.c. on \( W^{1,p}(\Omega) \).

**Proof.** First, as the same reason in Lemma 3.6, it follows that \( I_{\lambda,m} \) is w.l.s.c. on \( W^{1,p}(\Omega) \). By a similar argument to Lemma 3.6, for every \( \epsilon' > 0 \) and \( 0 < \epsilon < a/p \) where \( a = \inf_{\Omega} a_{\infty} \), we obtain

\[
I_{\lambda,m}(u) \geq \frac{a - \epsilon p}{p a} \int_{\Omega} a_{\infty}|\nabla u|^p dx + \frac{|\lambda|}{p} \int_{\Omega} m|u|^p dx - \epsilon'\|u\|_p^p - \|h\|_\infty\|u\|_p^p \tag{3.25}
\]

for every \( u \in W^{1,p}(\Omega) \) (note \( \lambda < 0 \)). Here, from Lemma 2.3,

\[
D_0 \int_{\Omega} a_{\infty}|\nabla u|^p dx + |\lambda| \int_{\Omega} m|u|^p dx \geq \frac{D_0}{2} \int_{\Omega} a_{\infty}|\nabla u|^p dx + \frac{D_0}{2} b(\xi, m)\|u\|_p^p \tag{3.26}
\]

for every \( u \in W^{1,p}(\Omega) \) follows, where \( D_0 := (a - \epsilon p)/a, \xi := 2|\lambda|/D_0 \) and \( b(\xi, m) \) is a constant obtained in Lemma 2.3. Therefore, by choosing a \( \epsilon' \) such that \( 0 < \epsilon' < D_0 b(\xi, m)/2 \), we can prove our conclusion.

**Lemma 3.8.** Let \( m dx \neq 0 \) and \( 0 < \lambda < \mu \). Then, \( I_{\lambda,m} \) is bounded from below on \( Y(\mu, m) \), where \( Y(\mu, m) \) is the set introduced in (3.14).

**Proof.** Due to the same inequalities concerning \( G \) and \( F \) as in Lemma 3.5, for every \( \epsilon > 0 \) and \( \epsilon' > 0 \), there exists \( C = C(\epsilon, \epsilon') > 0 \) such that

\[
I_{\lambda,m}(u) \geq \frac{a - \epsilon p}{p a} \int_{\Omega} a_{\infty}|\nabla u|^p dx - \lambda \int_{\Omega} m|u|^p dx - \epsilon'\|u\|_p^p - \|h\|_\infty\|u\|_1 - C|\Omega| \tag{3.27}
\]

for every \( u \in W^{1,p}(\Omega) \), where \( a := \inf_{x \in \Omega} a_{\infty}(x) \). Choose positive constants \( \epsilon \) and \( \delta \) such that \( D_0 := 1 - \epsilon p/a > \delta > \lambda/\mu \) (note \( \lambda/\mu < 1 \)).

First, we consider the case of \( m \geq 0 \) in \( \Omega \). For every \( u \in Y(\mu, m) \), we obtain

\[
D_0 \int_{\Omega} a_{\infty}|\nabla u|^p dx - \lambda \int_{\Omega} m|u|^p dx \tag{3.28}
\]

\[
\geq (D_0 - \delta) \int_{\Omega} a_{\infty}|\nabla u|^p dx + (\delta \mu - \lambda) \int_{\Omega} m|u|^p dx \geq d(m, \xi_1)(D_0 - \delta)\|u\|_p^p
\]

by Lemma 2.3 with \( \xi_1 = (\delta \mu - \lambda)/(D_0 - \delta) \) (note \( \delta \mu - \lambda > 0 \) and \( D_0 - \delta > 0 \)).
Next, we handle with the case where \( m \) changes sign. Let \( u \in W^{1,p}(\Omega) \) satisfy \( \int_{\Omega} m|u|^p \, dx \leq 0 \). Then, we have for such \( u \)
\[
D_0 \int_{\Omega} a_{\infty} \nabla u|^p \, dx - \lambda \int_{\Omega} m|u|^p \, dx \geq b(m, \xi_\lambda) D_0 \|u\|_p^p
\]
(3.29)
by Lemma 2.2, where \( D_0 = 1 - p\varepsilon / a \) and \( \xi_\lambda := \lambda / D_0 \).

On the other hand, for \( u \in \overline{Y(\mu, m)} \) with \( \int_{\Omega} m|u|^p \, dx > 0 \), the following inequality follows from Lemma 2.2:
\[
D_0 \int_{\Omega} a_{\infty} \nabla u|^p \, dx - \lambda \int_{\Omega} m|u|^p \, dx \geq (D_0 - \delta) \int_{\Omega} a_{\infty} \nabla u|^p \, dx - (\delta \mu - \lambda) \int_{\Omega} (-m)|u|^p \, dx
\]
(3.30)
\[
\geq b(-m, \xi_\lambda) (D_0 - \delta) \|u\|_p^p.
\]
Consequently, by (3.27), (3.29), (3.28), and (3.30), there exists \( d > 0 \) independent of \( u \) such that
\[
I_{\lambda,m}(u) \geq (d - \varepsilon') \|u\|_p^p - \|u\|_\infty \|u\|_p \|\Omega\|^{(p-1)/p} - C|\Omega|
\]
(3.31)
for every \( u \in Y(\mu, m) \). Hence, our conclusion is shown by taking \( \varepsilon' > 0 \) satisfying \( \varepsilon' < d \). \( \square \)

**Proof of Theorem 1.1 in the Case \( \int_{\Omega} m \, dx \neq 0 \).** First, if either \( m \geq 0 \) on \( \Omega \) and \( \lambda < 0 \) or \( 0 < \lambda < \lambda^*(m) = \mu_1(m) \) (i.e., \( \int_{\Omega} m \, dx < 0 \)) holds, then Lemma 3.7 or Lemma 3.6 guarantees the existence of a global minimizer of \( I_{\lambda,m} \), respectively (cf. [25, Theorem 1.1]). Hence, \( (P; \lambda, m, h) \) has a solution.

Since \( \lambda \) is an eigenvalue of \( (EV; m) \) if and only if \(-\lambda \) is one of \( (EV; -m) \), it suffices to consider the case of \( \lambda > \lambda^*(m) \geq 0 \). Furthermore, by Proposition 2.9, Remark 2.6 (i), and our hypothesis that \( \lambda \) is not an eigenvalue of \( (EV; m) \), we may assume that there exists a \( k \in \mathbb{N} \) such that \( \mu_k(m) < \lambda < \mu_{k+1}(m) \). By Lemmas 3.5 and 3.8, we can choose \( T > 0 \) and \( g_0 \in \mathcal{F}_k(m) \) satisfying
\[
\max_{z \in \mathcal{S}^{k-1}} I_{\lambda,m}(Tg_0(z)) < \inf \{ I_{\lambda,m}(u); u \in Y(\mu_{k+1}(m), m) \} =: \alpha.
\]
(3.32)

Put
\[
\Sigma := \left\{ g \in C\left( \mathcal{S}^k, W^{1,p}(\Omega) \right); g|_{\mathcal{S}^{k-1}} = Tg_0 \right\},
\]
\[
c := \inf_{g \in \Sigma} \max_{z \in \mathcal{S}^k} I_{\lambda,m}(g(z)).
\]
(33)

Then, it follows from Lemma 3.4 and (3.32) that \( c \geq \alpha > \max_{z \in \mathcal{S}^{k-1}} I_{\lambda,m}(Tg_0(z)) \) holds. Since \( I_{\lambda,m} \) satisfies the Palais-Smale condition by Proposition 3.3, the minimax theorem guarantees (cf. [25, Theorem 4.6]) that \( c \) is a critical value of \( I_{\lambda,m} \). Hence, \((P; \lambda, m, h)\) has at least one solution. \( \square \)
3.4. The Case $\int_\Omega m \, dx = 0$

First, we introduce an approximate functional $I^*_{\lambda,m,n}$ as follows:

$$I^*_{\lambda,m,n}(u) := I_{\lambda,m}(u) + \frac{1}{pn} \|u\|_p^p = I_{\lambda,m-1/(\lambda n)}(u) \quad \text{for } u \in W^{1,p}(\Omega). \quad (3.34)$$

**Lemma 3.9.** Let $0 < \lambda < \mu$. Then, there exists an $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $I^*_{\lambda,m,n}$ is bounded from below on $Y(\mu, m - 1/\lambda n)$, where $Y(\mu, m - 1/\lambda n)$ is the set introduced in (3.14).

**Proof.** Choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \lambda \leq \sup_{x \in \Omega} m(x)/2$. Then, for every $n \geq n_0$, Lemma 3.8 guarantees that $I^*_{\lambda,m,n} = I_{\lambda,m-1/(\lambda n)}$ bounded from below on $Y(\mu, m - 1/(\lambda n))$ because of $\int_\Omega (m - 1/(\lambda n)) \, dx < 0$ and $|m - 1/(\lambda n)| > 0 > 0$.

**Proof of Theorem 1.1 in the Case $\int_\Omega m \, dx = 0$**. By noting that $\lambda m = (-\lambda)(-m)$ and $\mu_1 (m) = \lambda^*(m) = 0$, we may assume that $\mu_k(m) < \lambda < \mu_{k+1}(m)$ for some $k \in \mathbb{N}$. Let $n_0$ be a natural number obtained by Lemma 3.9. Due to Proposition 2.10 (i) and (ii), there exists an $n_1 \geq n_0$ such that

$$\mu_k (m) \leq \mu_k \left( m - \frac{1}{n\lambda} \right) \leq \mu_k \left( m - \frac{1}{n_1\lambda} \right) < \lambda < \mu_{k+1}(m) \leq \mu_{k+1} \left( m - \frac{1}{n\lambda} \right) \quad (3.35)$$

for every $n \geq n_1$. Thus, for every $n \geq n_1$, we can take $T_n > 0$ and $g_n \in F_k(m - 1/(n\lambda))$ satisfying

$$\max_{z \in S^{k-1}} I^*_{\lambda,m,n}(T_n g_n(z)) < \inf \left\{ I_{\lambda,m,n}(u) ; u \in Y \left( \mu_{k+1} \left( m - \frac{1}{(n\lambda)} \right) , m - \frac{1}{(n\lambda)} \right) \right\} \quad (3.36)$$

by applying Lemmas 3.5 and 3.9 to $I^*_{\lambda,m,n} = I_{\lambda,m-1/(n\lambda)}$ (note (3.35)). Set

$$\Sigma_n := \left\{ g \in C(S^k, W^{1,p}(\Omega)) ; g|_{S^{k-1}} = T_n g_n \right\},$$

$$c_n := \inf_{g \in \Sigma_n} \max_{z \in S^k} I^*_{\lambda,m,n}(g(z)) \quad (3.37)$$

for each $n \geq n_1$. Then, for each $n \geq n_1$, we can obtain $u_n$ satisfying

$$\left| I^*_{\lambda,m,n}(u_n) - c_n \right| < \frac{1}{n}, \quad \left\| \left( I^*_{\lambda,m,n} \right)'(u_n) \right\|_{W^{1,p}(\Omega)} < \frac{1}{n} \quad (3.38)$$

by applying Ekeland’s variational principle to each $I^*_{\lambda,m,n}$ (refer to [25, Theorem 4.3]). In addition, we can see that $\{ u_n \}$ is bounded in $W^{1,p}(\Omega)$. Indeed, if there exists a subsequence $\{ u_{n_l} \}$ satisfying $\|u_{n_l}\|_p \to \infty$ as $l \to \infty$, then we can show that $\lambda$ is an eigenvalue of $(EV; m)$ by the same argument as in Proposition 3.3. This contradicts to our assumption that $\lambda$ is not an eigenvalue of $(EV; m)$. Moreover, the boundedness of $\|\nabla u_n\|_p$ follows from a similar inequality to (3.6) as in Proposition 3.3 under the boundedness of $\|u_n\|_p$. 
Therefore, we may assume, by choosing a subsequence that \( \{u_n\} \) is a Palais-Smale sequence of \( I_{\lambda,m} \) since \( I_{\lambda,m} \) is bounded on a bounded set and according to the following inequality:

\[
\left\| I'_{\lambda,m}(u_n) \right\|_{(W^{1,r}(|\Omega|))'} \leq \left\| I'_{\lambda,m}(u_n) - \left( I^{+}_{\lambda,m,n} \right)'(u_n) \right\|_{(W^{1,r}(|\Omega|))'} + \frac{1}{n} \leq \frac{1}{n} \|u_n\|^{p-1} + \frac{1}{n}. \tag{3.39}
\]

Therefore, because \( I_{\lambda,m} \) satisfies the Palais-Smale condition by Proposition 3.3, \( I_{\lambda,m} \) has a critical point, whence \((P;\lambda,m,h)\) has at least one solution. \( \square \)

4. Proof of Theorem 1.2

First, we will prove the following result concerning the Palais-Smale condition under the additional hypothesis \((H \pm)\) or \((HF \pm)\).

**Proposition 4.1.** Assume that one of the following conditions hold:

(i) \( \lambda = 0 \) and \((HF+)\) or \((HF-)\);

(ii) \( \lambda \neq 0 \) and one of \((H+)\), \((H-)\), \((HF+)\) and \((HF-)\).

Then, \( I_{\lambda,m} \) satisfies the Palais-Smale condition.

**Proof.** As the same reason in Proposition 3.3, it suffices to prove the boundedness of a Palais-Smale sequence \( \{u_n\} \) such that \( I_{\lambda,m}(u_n) \rightarrow c \) (for some \( c \in \mathbb{R} \)) and \( \|I_{\lambda,m}(u_n)\|_{W_q} \rightarrow 0 \) as \( n \rightarrow \infty \). By way of contradiction, we may assume that \( \|u_n\|_p \rightarrow \infty \) as \( n \rightarrow \infty \) by choosing a subsequence. Set \( v_n := u_n/\|u_n\|_p \). Then, by the same argument as in Proposition 3.3, \( \{v_n\} \) has a subsequence strongly convergent to \( v_0 \) being a nontrivial solution of

\[
-\text{div} \left( a_{\infty}(x) |\nabla u|^{p-2} \nabla u \right) = \lambda m(x) |u|^{p-2} u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial v} = 0 \quad \text{on } \partial \Omega. \tag{4.1}
\]

To simplify the notation, we denote the above subsequence strongly convergent to \( v_0 \) by \( \{v_n\} \), again. Thus, \( |u_n(x)| \rightarrow \infty \) as \( n \rightarrow \infty \) for a.e. \( x \in \Omega_0 \) := \{ \( x' \in \Omega; v_0(x') \neq 0 \) \} (note \( \|v_0\|_p = 1 \)).

Assume \((HF+)\) or \((HF-)\). Then, we can obtain

\[
(I) := \int_{\Omega} \frac{f(x,u_n)u_n - pF(x,u_n)}{\|u_n\|^{1+q}} dx \rightarrow \pm \infty \quad \text{if } (HF \pm), \text{ respectively}. \tag{4.2}
\]

Indeed, it follows from \((HF+)\) that there exist \( R > 0 \) and \( C > 0 \) independent of \( n \) such that \( f(x,t) - pF(x,t) \geq 0 \) if \( |t| \geq R \) and a.e. \( x \in \Omega \), and \( |f(x,t) - pF(x,t)| \leq C \) for every \( |t| \leq R \) and a.e. \( x \in \Omega \). Therefore, since \( |u_n(x)| \rightarrow \infty \) a.e. \( x \in \Omega_0 \) and \( |\Omega_0| > 0 \) (note \( \|v_0\|_p = 1 \)), we have \( (I) \) if \((HF+)\) holds, by applying Fatou’s lemma to the following inequality:

\[
(I) \geq \int_{\Omega_0} \frac{f(x,u_n)u_n - pF(x,u_n)}{\|u_n\|^{1+q}} |v_n|^{1+q} dx - \frac{C|\Omega \setminus \Omega_0|}{\|u_n\|^{1+q}}. \tag{4.3}
\]
In the case of \((HF-)\), by considering \(-f\) instead of \(f\) as in the above argument, we can show our claim (4.2).

Furthermore, by Hölder’s inequality, we have

\[
(II) := \int_{\Omega} \frac{p\tilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^{1+q}} \, dx \\
\leq H_0 \int_{\Omega} \left( |\nabla v_n|^{1+q} + \frac{1}{\|u_n\|_p^{1+q}} \right) \, dx \\
\leq H_0 \|\nabla v_0\|_p^{1+q} |\Omega|^{(p-1-q)/p} + o(1)
\]  

(4.4)

in the case of \((HF-)\) because \(v_n \rightarrow v_0\) in \(W^{1,p}(\Omega)\), where \(q \in [0, p-1]\) and \(H_0 > 0\) are constants as in \((HF-)\). Similarly, we obtain

\[
(II) \geq -H_0 \|\nabla v_0\|_p^{1+q} |\Omega|^{(p-1-q)/p} + o(1)
\]  

(4.5)
in the case of \((HF+)\).

Hence, we have a contradiction because of (4.2), (4.4), or (4.5) by taking a limit inferior or superior in the following equality:

\[
o(1) = \frac{pI_{1,m}(u_n) - \langle I_{1,m}(u_n), u_n \rangle}{\|u_n\|_p^{1+q}} = (II) + (I) + (1-p) \int_{\Omega} \frac{hv_n}{\|u_n\|_p^p} \, dx,
\]  

(4.6)

where we use the fact that \(\|u_n\|/\|u_n\|_p^{1+q} = \|v_n\|/\|u_n\|_p^p\) is bounded because of \(q \geq 0\).

Assume \(\lambda \neq 0\) and \((H+)\) or \((H-)\): because \(v_0\) is a nontrivial solution of (4.1) with \(\lambda \neq 0\), \(v_0\) is not a constant function, that is, \(\|\nabla v_0\|_p > 0\). Therefore, we have \(|\nabla u_n(x)| \rightarrow \infty\) as \(n \rightarrow \infty\) for a.e. \(x \in \Omega_0 := \{ x' \in \Omega; |\nabla v_0(x')| \neq 0\}\). Because of \(|\Omega_0| > 0\), we can show

\[
\int_{\Omega} \frac{p\tilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^{1+q}} \, dx \rightarrow \pm \infty \quad \text{if} \quad (H\pm), \text{ respectively},
\]  

(4.7)

by a similar argument to one for \(f\) in the above. In addition, we can easily obtain the following inequality:

\[
\pm \int_{\Omega} \frac{f(x, u_n)u_n - pF(x, u_n)}{\|u_n\|_p^{1+q}} \, dx \geq -H_0 \|v_n\|_1^{1+q} + o(1) = -H_0 \|v_0\|_1^{1+q} + o(1)
\]  

(4.8)
in the case of \((H\pm)\), respectively. Hence, we have a contradiction by considering \(o(1) = (pI_{1,m}(u_n) - \langle I_{1,m}(u_n), u_n \rangle) / \|u_n\|_p^{1+q}\). 

\[\Box\]
Proof. First, we note that the boundedness of \( I_{\lambda,m} \) on \( W^{1,p} \):

\[
I_{\lambda,m,n}^{\pm}(u) := I_{\lambda,m}(u) \pm \frac{1}{pn} \|u\|_{p}^p \quad \text{for } u \in W^{1,p}.
\]

Note \( I_{\lambda,m,n}^{\pm}(u) = I_{\lambda,m+1/(\lambda m)}(u) \) on \( W^{1,p} \) provided \( \lambda \neq 0 \).

**Proposition 4.2.** If either \( \lambda \neq 0 \) and \( (H^+) \) or \( (HF^+) \) (resp., either \( \lambda \neq 0 \) and \( (H^-) \) or \( (HF^-) \)) and \( \{u_n\} \) satisfies

\[
\sup_{n \in \mathbb{N}} I_{\lambda,m,n}^{\pm}(u_n) < +\infty, \quad \lim_{n \to \infty} \left\| \left(I_{\lambda,m,n}^{+}\right)'(u_n) \right\|_{W^{1,p}^*} = 0, \quad (4.10)
\]

\[
\left( \text{resp. } \inf_{n \in \mathbb{N}} I_{\lambda,m,n}^{\pm}(u_n) > -\infty, \lim_{n \to \infty} \left\| \left(I_{\lambda,m,n}^{-}\right)'(u_n) \right\|_{W^{1,p}^*} = 0 \right),
\]

then \( \{u_n\} \) is bounded in \( W^{1,p} \).

**Proof.** First, we note that the boundedness of \( \|u_n\|_p \) guarantees that \( \|u_n\| \) is bounded by \( \lim_{n \to \infty} \|(I_{\lambda,m,n}^+)'(u_n)\|_{W^{1,p}^*} = 0 \) (refer to (3.6) as in the proof of Proposition 3.3). Moreover, because of the following equality:

\[
\frac{pI_{\lambda,m,n}^{\pm}(u_n) - \left\langle \left(I_{\lambda,m,n}^{\pm}\right)'(u_n), u_n \right\rangle}{\|u_n\|_p^{1+q}} = (1-p) \int_{\Omega} \frac{h v_n}{\|u_n\|_p^q} \, dx,
\]

\[
+ \int_{\Omega} \frac{p \tilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|) |\nabla u_n|^2}{\|u_n\|_p^{1+q}} \, dx + \int_{\Omega} \frac{f(x, u_n) u_n - p F(x, u_n)}{\|u_n\|_p} \, dx,
\]

we can prove the boundedness of \( \|u_n\|_p \) by the same argument as in Proposition 4.1. \( \square \)

**Proof of Theorem 1.2.** Because of \( \lambda m = (-\lambda)(-m) \), we may assume \( \lambda \geq 0 \). In the case where \( \int_{\Omega} m \, dx \neq 0 \) and \( \mu_k(m) < \lambda < \mu_{k+1}(m) \) for some \( k \in \mathbb{N} \), the proof of Theorem 1.1 implies the existence of a critical point of \( I_{\lambda,m} \) because \( I_{\lambda,m} \) satisfies the Palais-Smale condition by Proposition 4.1. Concerning other cases, in the next section, we will prove the existence of a bounded sequence \( \{u_n\} \) satisfying \( (I_{\lambda,m,n}^+)'(u_n) \to 0 \) or \( (I_{\lambda,m,n}^-)'(u_n) \to 0 \) in \( W^{1,p} \) as \( n \to \infty \). Because \( I_{\lambda,m} \) is bounded on a bounded set, we may assume that \( I_{\lambda,m}(u_n) \) converges to some \( c \in \mathbb{R} \) by choosing a subsequence. In addition, by noting the inequality \( \|(I_{\lambda,m,n}^+)'(u_n)\|_{W^{1,p}} \leq \|(I_{\lambda,m,n}^+)'(u_n)\|_{W^{1,p}^*} + \|u_n\|_{p-1}^{-1} / n \), we easily see that \( \{u_n\} \) is a bounded Palais-Smale sequence of \( I_{\lambda,m} \). Therefore, \( I_{\lambda,m} \) has a critical point since \( I_{\lambda,m} \) satisfies the Palais-Smale condition by Proposition 4.1. \( \square \)
5. Construction of a Bounded Palais-Smale Sequence

In this section, due to the reason stated in the proof of Theorem 1.2, we will construct a bounded sequence \( \{u_n\} \) satisfying \((I_{\lambda,m,n}^*)'(u_n) \to 0\) or \((I_{\lambda,m,n}^-)'(u_n) \to 0\) in \(W^{1,p}(\Omega)^*\) as \(n \to \infty\). It implies the existence of a bounded Palais-Smale sequence of \(I_{\lambda,m}\).

5.1. The Case \(\lambda = 0\)

Assume \((HF+)\)

In this case, we can show that for each \(n \in \mathbb{N}\), \(I_{\lambda,m,n}^+\) has a global minimizer \(u_n\). Indeed, for \(0 < \varepsilon < 1/(pn)\), there exists \(C_\varepsilon > 0\) such that \(I_{\lambda,m,n}^+(u) \geq C_\varepsilon \|\nabla u\|_p^p/(p(p - 1)) + (1/(pn) - \varepsilon)\|u\|_p^p - ||h||_\infty\|u\|_1 - C_\varepsilon\) for every \(u \in W^{1,p}(\Omega)\) by (1.1), (1.16) and \(\lambda = 0\) (refer to the inequality as in the proof of Lemma 3.5). This means that \(I_{\lambda,m,n}^+\) is coercive and bounded from below on \(W^{1,p}(\Omega)\). Therefore, \(I_{\lambda,m,n}^+\) has a global minimizer \(u_n\) since \(I_{\lambda,m,n}^+\) is w.l.s.c. on \(W^{1,p}(\Omega)\) as the same reason in Lemma 3.6.

Furthermore, because of \((I_{\lambda,m,n}^+)'(u_n) = 0\) in \(W^{1,p}(\Omega)^*\) and \(I_{\lambda,m,n}^+(u_n) = \min_{W^{1,p}(\Omega)} I_{\lambda,m,n}^+(0) = 0\), it follows from Proposition 4.2 that \(\{u_n\}\) is bounded.

Assume \((HF-)\)

Choose \(n_0 \in \mathbb{N}\) such that \(1/n_0 < c(1) = \mu_2(1)\), where \(c(1)\) is the second eigenvalue of \((EV; 1)\) (so the weight function \(m = 1\) and see (2.22) for the definition). Then, by noting that \(I_{0,m,n_0}^+ = I_{1/n_0,1}^+\), we have

\[
\alpha := \inf \left\{ I_{0,m,n_0}^+(u); u \in Y(c(1), 1) \right\} > -\infty
\]  

(5.1)

by Lemma 3.8, where \(Y(c(1), 1)\) is a subset defined by (3.14) with the weight \(m = 1\), that is,

\[
Y(c(1), 1) := \left\{ u \in W^{1,p}(\Omega); \int_\Omega a_\infty |\nabla u|^p dx \geq c(1) \|u\|_p^p \right\}.
\]

(5.2)

Moreover, \(\inf \{I_{0,m,n}^+(u); u \in Y(c(1), 1)\} \geq \alpha\) for every \(n \geq n_0\) holds because \(I_{0,m,n}^+(u) \geq I_{0,m,n_0}^+(u)\) for every \(u \in W^{1,p}(\Omega)\). Since \(\int_\Omega F(x, u)dx = o(1)\|u\|_p^p\) as \(\|u\|_p \to \infty\) by (1.1), there exists \(T_n > 0\) such that \(I_{0,m,n}^-(\pm T_n) = -T_n^p(|\Omega|/(np)) - o(1) < \alpha - 2\).

Define

\[
\Sigma_n := \left\{ g \in C\left([0, 1], W^{1,p}(\Omega)\right); g(0) = T_n, g(1) = -T_n \right\},
\]

(5.3)

\[
c_n := \inf_{g \in \Sigma_n} \max_{t \in [0, 1]} I_{0,m,n}^-(g(t))
\]

for \(n \geq n_0\). By the definition of \(c(1)\), we easily see that \(g([0, 1]) \cap Y(c(1), 1) \neq \emptyset\) for every \(g \in \Sigma_n\) (refer to [6] or Lemma 3.4). Hence,

\[
c_n \geq \inf \left\{ I_{0,m,n}^-(u); u \in Y(c(1), 1) \right\} \geq \alpha > I_{0,m,n}^-(\pm T_n)
\]

(5.4)
holds, whence \( c_n \) is bounded from below. Moreover, by applying Ekeland's variational principle to each \( I_{0,m,n}^- \), we can obtain a sequence \( \{u_n\} \) satisfying \( I_{0,m,n}^-(u_n) - c_n < 1/n \) and \( \| (I_{0,m,n}^-)'(u_n) \|_{W^{1,p}(\Omega)} < 1/n \). Since \( c_n \) is bounded from below, it follows from Proposition 4.2 that \( \{u_n\} \) is bounded. As a result, we can construct a bounded sequence \( \{u_n\} \) satisfying \( (I_{0,m,n}^-)'(u_n) \to 0 \) as \( n \to \infty \) in \( W^{1,p}(\Omega)^* \).

### 5.2. The Case \( \lambda = \lambda^*(m) = \mu_1(m) \) with \( \int_{\Omega} m dx < 0 \)

**Assume (H+) or (HF+)**

Since we see that \( I_{\lambda,m,n}^+ = I_{\lambda,m-1/(n\lambda)} \) and \( \lambda^*(m - 1/(n\lambda)) > \lambda^*(m) = \lambda > 0 \) (according to Lemma 2.5), \( I_{\lambda,m,n}^+ \) is coercive, bounded from below and w.l.o.g. on \( W^{1,p}(\Omega) \) by Lemma 3.6. Thus, we obtain a global minimizer \( u_n \) of \( I_{\lambda,m,n}^+ \) for sufficiently large \( n \) such that \( \| (m - 1/(n\lambda)) > 0 \| > 0 \). Because of \( I_{\lambda,m,n}^+(u_n) \leq I_{\lambda,m,n}^+(0) = 0 \) for every \( n \), Proposition 4.2 guarantees that \( \{u_n\} \) is bounded.

**Assume (H-) or (HF-)**

First, we note that \( I_{\lambda,m,n}^+ = I_{\lambda,m+1/(n\lambda)} \) and \( 0 < \lambda^*(m + 1/(n\lambda)) < \lambda^*(m) = \lambda \) by Lemma 2.5 for sufficiently large \( n \) such that \( \int_{\Omega} (m + 1/(n\lambda)) dx < 0 \). Moreover, it follows from Proposition 2.10 and \( \mu_1(m) < \mu_2(m) \) that there exists an \( n_0 \in \mathbb{N} \) satisfying \( \int_{\Omega} m + 1/(n_0\lambda) dx < 0 \) and

\[
\lambda^*\left(m + \frac{1}{n\lambda}\right) < \lambda = \mu_1(m) < \mu_2\left(m + \frac{1}{n_0\lambda}\right) \leq \mu_2\left(m + \frac{1}{n\lambda}\right) \leq \mu_2(m)
\]  

(5.5)

for every \( n \geq n_0 \). By applying Theorem 1.1 to each case of a weight \( m + 1/(n\lambda) \) (note that \( \lambda \) is not an eigenvalue of \( (EV; m + 1/(n\lambda)) \)) by (5.5), there exists \( u_n \) satisfying \( (I_{\lambda,m,n}^-)'(u_n) = 0 \) (note \( I_{\lambda,m,n}^- = I_{\lambda,m+1/(n\lambda)}^- \)) and

\[
I_{\lambda,m,n}^-(u_n) = c_n \geq \inf\left\{ I_{\lambda,m,n}^-(u); u \in Y(\mu_2(m_{n_0}), m_{n_0}) \right\},
\]

(5.6)

where the last inequality follows from Lemma 3.4 with \( m_{n_0} := m + 1/(n_0\lambda) \). On the other hand, because \( I_{\lambda,m,n}^-(u) \geq I_{\lambda,m,n}^-(u) = I_{\lambda,m,n}^-(u) \) for every \( u \in W^{1,p}(\Omega) \) and \( n \geq n_0 \), we have

\[
c_n \geq \inf\left\{ I_{\lambda,m,n}^-(u); u \in Y(\mu_2(m_{n_0}), m_{n_0}) \right\} > -\infty
\]

(5.7)

for every \( n \geq n_0 \), where the last inequality follows from Lemma 3.8. Thus, \( c_n \) is bounded from below. Hence, Proposition 4.2 guarantees the boundedness of \( \{u_n\} \).
5.3. The Case $\lambda = \mu_{k+1}(m)$ with $\int_{\Omega} m \, dx \neq 0$

Assume $(H+)$ or $(HF+)$

We may assume $\mu_k(m) < \mu_{k+1}(m) = \lambda$ by taking $k$ anew if necessary (note that we have already proved the case of $\mu_k(m) < \lambda < \mu_{k+1}(m)$ in Section 4). Here, we can choose an $n_0 \in \mathbb{N}$ such that $\int_{\Omega} (m - 1/\langle n \lambda \rangle) \, dx \neq 0$, $\|m - 1/\langle n \lambda \rangle\| > 0$ and

$$
\mu_k \left( m - \frac{1}{n \lambda} \right) \leq \mu_k \left( m - \frac{1}{n_0 \lambda} \right) < \lambda - \frac{1}{n \|m\|_{\infty}} < \lambda = \mu_{k+1}(m) \leq \mu_{k+1} \left( m - \frac{1}{n \lambda} \right)
$$

(5.8)

for every $n \geq n_0$ by $\int_{\Omega} m \, dx \neq 0$ and Proposition 2.10 (i), (iii). Note the following inequality:

$$
I_{\lambda,m,n_0}^+(u) \geq I_{\lambda,m,n}^+(u) \geq I_{\lambda-1/(n \|m\|_{\infty}),m}(u)
$$

(5.9)

for every $u \in W^{1,p}(\Omega)$ and $n \geq n_0$, where the last inequality is obtained by $\|u\|^{p*}_p \geq \int_{\Omega} m |u|^p \, dx / \|m\|_{\infty}$. Let $n \geq n_0$. It follows from Lemma 3.8 and (5.8) that $I_{\lambda-1/(n \|m\|_{\infty}),m}$ is bounded from below on $Y(\lambda, m)$. Hence, (5.9) yields that $I_{\lambda,m,n}^+$ is also bounded from below on $Y(\lambda, m)$, namely,

$$
\alpha_n := \inf \left\{ I_{\lambda,m,n}^+(u); u \in Y(\lambda, m) \right\} > -\infty.
$$

(5.10)

On the other hand, because of $\mu_k(m - 1/(n_0 \lambda)) < \lambda$ (see (5.8)), Lemma 3.5 guarantees the existence of $g_0 \in \mathcal{F}_k(m - 1/(n_0 \lambda))$ satisfying

$$
\max_{z \in S^{s-1}} I_{\lambda,m,n_0}^+(Tg_0(z)) = \max_{z \in S^{s-1}} I_{\lambda,m-1/(n_0 \lambda),m}(Tg_0(z)) \to -\infty \quad \text{as } |T| \to \infty.
$$

(5.11)

Thus, for each $n \geq n_0$, we can take $T_n > 0$ such that

$$
\max_{z \in S^{s-1}} I_{\lambda,m,n}^+(T_ng_0(z)) \leq \max_{z \in S^{s-1}} I_{\lambda,m,n_0}^+(T_ng_0(z)) \leq \alpha_n - 1,
$$

(5.12)

(note (5.9) for the first inequality). Set

$$
\Sigma_n := \left\{ g \in C\left( S^1, W^{1,p}(\Omega) \right); g|_{S^{s-1}} = T_ng_0 \right\},
$$

$$
c_n^+ := \inf_{g \in \Sigma_n} \max_{z \in S^1} I_{\lambda,m,n}^+(g(z))
$$

(5.13)

for $n \geq n_0$. Since $g(S^1) \cap Y(\lambda, m) \neq \emptyset$ for every $g \in \Sigma_n$ by Lemma 3.4 and $\lambda = \mu_{k+1}(m)$, we have $c_n^+ \geq \alpha_n > \max_{z \in S^{s-1}} I_{\lambda,m,n_0}^+(T_ng_0(z))$. Therefore, Ekeland's variational principle (refer to [25, Theorem 4.3]) guarantees the existence of $u_n$ satisfying $I_{\lambda,m,n}^+(u_n) - c_n < 1/n$ and $\|(I_{\lambda,m,n}^+)'(u_n)\|_{W^{1,p}(\Omega)} < 1/n$. 

Finally, to show the boundedness of \(|u_n|\) due to Proposition 4.2, we will prove that \(c_n^*\) is bounded from above. For each \(n \geq n_0\), we define a continuous map \(g_n\) from \(S^k\) to \(W^{1,p}(\Omega)\) by

\[
g_n(z) := \begin{cases} 
(1 - z_{k+1}) T_n g_0 \left( \frac{z'}{\sqrt{1 - z_{k+1}^2}} \right) & \text{for } z = (z', z_{k+1}) \in S^k_+ \text{ with } 0 \leq z_{k+1} < 1, \\
0 & \text{for } z = (z', z_{k+1}) \in S^k_+ \text{ with } z_{k+1} = 1.
\end{cases}
\]

(5.14)

Then, \(g_n \in \Sigma_n\) holds. This leads to

\[
c_n^* \leq \sup_{t \geq 0, z \in S^{k-1}} I_{\lambda,m,n}^+(tg_0(z)) \leq \sup_{t \geq 0, z \in S^{k-1}} I_{\lambda,m,n}^+(tg_0(z)) < +\infty
\]

(5.15)

because of (5.9), (5.11), and the compactness of \(g_0(S^{k-1})\).

Assume \((H-)\) or \((HF-)\)

Because the case of \(\mu_1(m) = \lambda^*(m)\) is already shown (see Sections 5.1 and 5.2), We may assume \((0 < \mu_k(m) = \lambda < \mu_{k+1}(m)\) for some \(k \geq 2\) by taking \(k\) anew if necessary. Here, we can choose an \(n_0 \in \mathbb{N}\) such that \(\int_{\Omega} (m + 1/(n\lambda))dx \neq 0\) and

\[
\mu_k \left( m + \frac{1}{n\lambda} \right) \leq \mu_k(m) = \lambda < \mu_{k+1} \left( m + \frac{1}{n_0\lambda} \right) \leq \mu_{k+1} \left( m + \frac{1}{n\lambda} \right) \leq \mu_{k+1}(m)
\]

(5.16)

for every \(n \geq n_0\) by \(\int_{\Omega} m \, dx \neq 0\) and Proposition 2.10 (i), (iii). Moreover, we note the following inequality:

\[
I_{\lambda,m,n_0}^-(u) \leq I_{\lambda,m,n_0}^+ (u) = I_{\lambda,m+1/(n\lambda)}(u) \leq I_{\lambda+1/(n|m|_\infty),m}(u)
\]

(5.17)

for every \(u \in W^{1,p}(\Omega)\) and \(n \geq n_0\). It follows from Lemma 3.8 and (5.16) (note (5.17) also) that \(I_{\lambda,m,n_0}^- = I_{\lambda,m_0}\) is bounded from below on \(Y(\mu_{k+1}(m_0), m_0)\) with \(m_0 := m + 1/(n_0\lambda)\). Hence, (5.17) implies

\[
\inf \left\{ I_{\lambda,m,n}^-(u); u \in Y(\mu_{k+1}(m_0), m_0) \right\} \\
\geq \inf \left\{ I_{\lambda,m_0}^-(u); u \in Y(\mu_{k+1}(m_0), m_0) \right\} =: \alpha_0 > -\infty
\]

(5.18)

for every \(n \geq n_0\). Because of \(\lambda + 1/(n|m|_\infty) > \lambda = \mu_k(m)\), there exist \(g_n \in \Psi_k(m)\) and \(T_n > 0\) such that

\[
\max_{z \in S^{k-1}} I_{\lambda,m,n}^-(T_n g_n(z)) \leq \max_{z \in S^{k-1}} I_{\lambda+1/(n|m|_\infty),m}(T_n g_n(z)) < \alpha_0 - 1
\]

(5.19)
by Lemma 3.5. Define

\[
\Sigma_n := \left\{ g \in C \left( S^k_+, W^{1,p}(\Omega) \right); \ g|_{S^{k-1}} = T_ng_n \right\},
\]

\[
c_n^- := \inf_{g \in \Sigma_n} \max_{z \in S^k_+} I_{\lambda,m,n}^- (g(z))
\]

for \( n \geq n_0 \). Then, \( c_n^- \geq a_0 \) occurs (see (5.18)) since \( g(S^k) \cap Y(\mu_{k+1}(m_0), m_0) \neq \emptyset \) for every \( g \in \Sigma_n \) by Lemma 3.4. This means that \( c_n^- \) is bounded from below. Consequently, we can obtain a desired bounded sequence by the same argument as in Sections 5.1 and 5.2.

### 5.4. The Case (iii) as in Theorem 1.2

First, note that we are assuming the hypothesis \((H+)\) or \((HF+)\) in this case (iii). In addition, as the reason in the proof of Theorem 1.2, it suffices to handle with \( \lambda > 0 \).

Let \( k \in \mathbb{N} \) satisfy \( \mu_k(m) < \lambda \leq \mu_{k+1}(m) \). According to Proposition 2.10 (i) and (ii), we can take an \( n_0 \in \mathbb{N} \) such that \( |m - 1/(n\lambda)| > 0 \) and

\[
\mu_k \left( m - \frac{1}{2n\lambda} \right) \leq \mu_k \left( m - \frac{1}{n_0\lambda} \right) < \lambda - \frac{1}{2n\|m\|_\infty} < \lambda \leq \mu_{k+1}(m) \leq \mu_{k+1} \left( m - \frac{1}{2n\lambda} \right)
\]

for every \( n \geq n_0 \). The following inequality follows from the easy estimates:

\[
I_{\lambda,m,n_0}^+(u) \geq I_{\lambda,m,n}^+(u) = I_{\lambda,m-1/(n\lambda)}^+(u) \geq I_{\lambda-1/(2n\|m\|_\infty),m-1/(2n\lambda)}^+(u)
\]

for every \( u \in W^{1,p}(\Omega) \) and \( n \geq n_0 \). Let \( n \geq n_0 \) and set \( m_n := m - 1/(2n\lambda) \). Because of (5.21), Lemma 3.8 implies that \( I_{\lambda-1/(2n\|m\|_\infty),m_n} \) is bounded from below on \( Y(\mu_{k+1}(m_n), m_n) \) with (note \( \int_\Omega m_n \, dx \neq 0 \)). Hence, (5.22) yields that

\[
a_n := \inf \left\{ I_{\lambda,m,n}^+(u); u \in Y(\mu_{k+1}(m_n), m_n) \right\} > -\infty
\]

for each \( n \geq n_0 \). On the other hand, because of \( \mu_k(m - 1/(n_0\lambda)) < \lambda \) (see (5.21)), Lemma 3.5 guarantees the existence of \( g_0 \in \mathcal{F}_k (m - 1/(n_0\lambda)) \) satisfying

\[
\max_{z \in S^{k-1}} I_{\lambda,m,n_0}^+(Tg_0(z)) = \max_{z \in S^{k-1}} I_{\lambda,m-1/(n_0\lambda)}^+(Tg_0(z)) \to -\infty \quad \text{as } T \to \infty.
\]

Therefore, for each \( n \geq n_0 \), we can choose \( T_n > 0 \) such that

\[
\max_{z \in S^{k-1}} I_{\lambda,m,n}^+(T_n g_0(z)) \leq \max_{z \in S^{k-1}} I_{\lambda,m,n_0}^+(T_n g_0(z)) \leq a_n - 1,
\]
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(note (5.22) for the first inequality). Set

\[
\Sigma_n := \left\{ g \in C\left( S_+^k, W^{1,p}(\Omega) \right); g|_{\partial \Omega} = T_0g_0 \right\},
\]

\[
c_n^* := \inf_{g \in \Sigma_n} \max_{z \in S_+^k} I_{\lambda,m,n}^+(g(z))
\]

for \( n \geq n_0 \). Since \( g(S_+^k) \cap Y(\mu_{k+1}(m_n), m_n) \neq \emptyset \) for every \( g \in \Sigma_n \) by Lemma 3.4, we have \( c_n^* \geq \alpha_n > \max_{z \in S_+^k} I_{\lambda,m,n}^+ (T_0g_0(z)) \). Moreover, by the same argument as in Section 5.3 (note (5.24)), we have

\[
c_n^* \leq \sup_{t \geq 0, z \in S_+^k} I_{\lambda,m,n}^+(tg_0(z)) \leq \sup_{t \geq 0, z \in S_+^k} I_{\lambda,m,n}^+(tg_0(z)) < +\infty,
\]

and hence our conclusion is shown.

**Remark 5.1.** If \( \int_{\Omega} m\,dx = 0 \) holds, then we can not show the continuity of \( \mu_k(m) \) with respect to \( m \) (refer to Proposition 2.10). Hence, we are not able to construct a bounded Palais-Smale sequence under \((H-)\) or \((HF-)\). However, if we have the additional information about the existence of a suitable \( m' \in L^{\infty}(\Omega) \) such that \( m' \geq m \) in \( \Omega \), \( \int_{\Omega} m'\,dx \neq 0 \) and \( \mu_k(m) \leq \mu_{k+1}(m') \) when \( \mu_k(m) \leq \mu_{k+1}(m') \) occurs, then we can still easily prove that equation \((P;\lambda,m,h)\) has a solution in the case also where \( \lambda \neq 0, \int_{\Omega} m\,dx = 0 \) and \((H-)\) or \((HF-)\). In fact, let \( 0 < \mu_k(m) \leq \mu_{k+1}(m') \) for some \( k \geq 2 \). Note the following inequality:

\[
I_{\lambda,m,n}^+(u) \geq I_{\lambda,m,n}^+(u) - \frac{1}{np} \|u\|_p^p = I_{\lambda,m,n}^+(u)
\]

for every \( u \in W^{1,p}(\Omega) \) and \( n \). Fix \( n_0 \in \mathbb{N} \) such that \( \int_{\Omega} m' \, dx > 0 \) and \( \int_{\Omega} m' \, dx > 0 \). Set \( m'_0 := m' - 1/(n_0 \lambda) \). Because of \( \lambda < \mu_{k+1}(m') \leq \mu_{k+1}(m'_0) \) (the last inequality follows from Proposition 2.10 (i)), Lemma 3.8 implies that \( I_{\lambda,m'_0} \) is bounded from below on \( Y(\mu_{k+1}(m'_0), m'_0) \) (note \( \int_{\Omega} m'_0\,dx > 0 \)). By combining this fact and (5.28), we have

\[
\inf_{n \geq n_0} \inf \left\{ I_{\lambda,m_n}(u); u \in Y(\mu_{k+1}(m'_0), m'_0) \right\} \\
\geq \inf \left\{ I_{\lambda,m'_0}(u); u \in Y(\mu_{k+1}(m'_0), m'_0) \right\} > -\infty.
\]

Because of \( \lambda + 1/(n\|m\|_{\infty}) \geq \lambda \geq \mu_k(m) \), for each \( n \geq n_0 \), we can take a \( g_n \in \mathcal{F}_k(m) \) satisfying

\[
\max_{z \in S_+^k} I_{\lambda,m,n}(Tg_n(z)) \leq \max_{z \in S_+^k} I_{\lambda,m,n}(Tg_n(z)) \rightarrow -\infty
\]

as \( T \rightarrow \infty \) by Lemma 3.5.

Since any extension \( g \in C(S_+^k, W^{1,p}(\Omega)) \) of \( Tg_n \) \((T > 0)\) links \( Y(\mu_{k+1}(m'_0), m'_0) \) by Lemma 3.4, we can construct a desired sequence by the same argument as in Section 5.3 under \((H-)\) or \((HF-)\).
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