The Local Strong and Weak Solutions for a Generalized Pseudoparabolic Equation

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1. Introduction

Davis [1] investigated the pseudoparabolic equation

\[ u_t(t,x) = \frac{\partial}{\partial x} \varphi(u_x) + \alpha u_{xxx}, \quad (1.1) \]

where the constant \( \alpha \geq 0 \), the function \( \varphi \in C^2(-\infty, \infty) \), \( \varphi(0) = 0 \) and \( \varphi'(\xi) > 0 \), and the subscripts \( x \) and \( t \) indicate partial derivatives. Equation (1.1) arises from the study of shearing flows of incompressible simple fluids. The quantity \( \varphi(u_x) + \alpha u_{xxx} \) is viewed as an approximation to the stress functional during such a flow. Much attention has been given to this approximation when the function \( \varphi \) is linear (see [2, 3]). The existence and uniqueness of the global weak solution of the initial value problem for (1.1) were established in [1].
Recently, Chen and Xue [4] investigated the Cauchy problem for the nonlinear generalized pseudoparabolic equation

\begin{equation}
\frac{\partial}{\partial t} u_t - \alpha u_{txx} - \lambda u_{xx} + \gamma u_x + f(u)_x = \frac{\partial}{\partial x} \varphi(u_x) + g(u) - \alpha g(u)_{xx}, \quad x \in \mathbb{R}, \ t > 0,
\end{equation}

where $u(t, x)$ is an unknown function, $\alpha > 0$, $\lambda \geq 0$, $\gamma$ is a real number, $f(s)$, $\varphi(s)$, and $g(s)$ denote given nonlinear functions. The well-posedness of global strong solution in a Sobolev space, the global classical solution and its asymptotic behavior are studied in [4] in which several key assumptions are imposed on the functions $\varphi(s)$ and $g(s)$. In fact, various dynamic properties for many special cases of (1.2) have been established in [5–7]. For example, when $\varphi(s) = g(s) = 0$, (1.2) becomes the generalized regularized long wave Burger equation.

Motivated by the works in [1, 4], we study the problem

\begin{equation}
\frac{\partial}{\partial t} u_t - \alpha u_{txx} = \frac{\partial}{\partial x} \varphi(u_x) + \beta \mu^{2m} u_{xx}, \quad x \in \mathbb{R}, \ t > 0,
\end{equation}

\begin{equation*}
u(0, x) = u_0(x), \quad x \in \mathbb{R},
\end{equation*}

where $\alpha > 0$ and $\beta \geq 0$, $m$ is a nature number, $\varphi(s)$ is a given function, and $u_0(x)$ is a given initial value function. Here we should address that (1.2) does not include the first equation of problem (1.3) due to the term $\beta \mu^{2m} u_{xx}$. Letting $\beta = 0$, the first equation of problem (1.3) reduces to (1.1).

The objectives of this work are threefold. The first objective is to establish the local well-posedness of system (1.3) in the space $C([0, T); H^s(\mathbb{R})] \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ with $s > 3/2$. We should address that the Sobolev index $s \geq 2$ is required to guarantee the local well-posedness of (1.1) and (1.2) in the works of Davis [1] and Chen and Xue [4]. The second aim is to study the existence of local weak solutions for system (1.3). The third aim is to discuss the well-posedness of the global strong solution for problem (1.3). Under the assumptions of the function $\varphi(s)$ and the initial value $u_0(x)$ similar to those presented in [1, 4], problem (1.3) is shown to have a unique global solution in the space $C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}))$.

The organization of this paper is as follows. The well-posedness of local strong solutions for problem (1.3) is investigated in Section 2, and the existence of local weak solutions is established in Section 3. Section 4 deals with the well-posedness of the global strong solution.

2. Local Well-Posedness

Let $L^p = L^p(\mathbb{R})$ ($1 \leq p < +\infty$) be the space of all measurable functions $h$ such that $\|h\|_{L^p}^p = \int_\mathbb{R}|h(t, x)|^p dx < \infty$. We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(\xi) = 0} \sup_{x \in \mathbb{R}} |h(t, x)|$. For any real number $s$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space with the norm defined by

\begin{equation}
\|h\|_{H^s} = \left( \int_\mathbb{R} (1 + |\xi|^2)^s |\hat{h}(t, \xi)|^2 d\xi \right)^{1/2} < \infty,
\end{equation}

where $\hat{h}(t, \xi) = \int_\mathbb{R} e^{-ix\xi} h(t, x) dx$. 


For $T > 0$ and nonnegative number $s$, $C([0, T); H^s(R))$ denotes the Frechet space of all continuous $H^s$-valued functions on $[0, T)$. We set $\Lambda = (1 - \partial_x^2)^{1/2}$. For simplicity, throughout this paper, we let $c$ denote any positive constant.

The local well-posedness theorem is stated as follows.

**Theorem 2.1.** Provided that $s \geq 3/2$, $u_0 \in H^s(R)$, $\varphi$ is a polynomial of order $N$ with $\varphi(0) = 0$. Then problem (1.3) admits a unique local solution:

$$u(t, x) \in C([0, T); H^s(R)) \bigcap C^1\left([0, T); H^{s-1}(R)\right).$$

(2.2)

**Proof.** In fact, the first equation of problem (1.3) is equivalent to the equation

$$u_t = \Lambda^{-2} \left(\frac{\partial}{\partial x} \varphi(u_x) + \beta u^{2m} u_{xx}\right),$$

which leads to

$$u = u_0 + \int_0^t \Lambda^{-2} \left(\frac{\partial}{\partial x} \varphi(u_x) + \beta u^{2m} u_{xx}\right) d\tau.$$  

(2.4)

Suppose that both $u$ and $v$ are in the closed ball $B_{M_0}(0)$ of radius $M_0 > 1$ about the zero function in $C([0, T]; H^s(R))$ and $A$ is the operator in the right-hand side of (2.4), for fixed $t \in [0, T]$, we get

$$\left\| \int_0^t \Lambda^{-2} \left(\varphi(u_x)_x + \beta u^{2m} u_{xx}\right) d\tau - \int_0^t \Lambda^{-2} \left(\varphi(v_x)_x + \beta v^{2m} v_{xx}\right) d\tau \right\|_{H^s} \leq T \left(\sup_{0 \leq s \leq T} \|\varphi(u_x) - \varphi(v_x)\|_{H^{-1}} + \sup_{0 \leq s \leq T} \left\| u^{2m} u_{xx} - v^{2m} v_{xx} \right\|_{H^{-2}} \right).$$

(2.5)

The algebraic property of $H^{\infty}(R)$ with $s_0 > 1/2$ (see [8–10]) and $s > 3/2$ derives that

$$\left\| u_x^j - v_x^j \right\|_{H^{-1}} = \left\| (u_x - v_x)^j \left( u_x^{j-1} + u_x^{j-2} v_x + \cdots + u_x v_x^{j-2} + v_x^{j-1} \right) \right\|_{H^{-1}} \leq \left\| u_x - v_x \right\|_{H^{-1}} \left\| \left( u_x^{j-1} + u_x^{j-2} v_x + \cdots + u_x v_x^{j-2} + v_x^{j-1} \right) \right\|_{H^{-1}} \leq c \| u_x - v_x \|_{H^{-1}} \sum_{i=0}^{j-1} \left\| u_x \right\|_{H^{-1}}^{j-1-i} \left\| v_x \right\|_{H^{-1}}^{j-1-i} \left\| u - v \right\|_{H^s} \leq c \left( u_x \right)^j - (v_x)^j \left\|_{H^{-1}} \leq c M_0^{j-1} \left\| u - v \right\|_{H^s},$$

(2.6)
In this section, we assume that $u^{2m}u_{xx} = \partial_x(u^{2m}u_x) - 2mu^{2m-1}(u_x)^2$ and $v^{2m}v_{xx} = \partial_x(v^{2m}v_x) - 2mv^{2m-1}(v_x)^2$, we get

$$\left\Vert u^{2m}u_{xx} - v^{2m}v_{xx}\right\Vert_{H^{s-2}} \leq \left\Vert \partial_x\left[u^{2m}u_x - v^{2m}v_x\right]\right\Vert_{H^{s-1}} + c\left\Vert u^{2m-1}u_x^2 - v^{2m-1}v_x^2\right\Vert_{H^{s-1}}$$

Choosing $T$ sufficiently small such that $\theta < 1$, we know that $A$ is a contractive mapping. Applying the above inequality and (2.4) yields

$$\left\Vert Au\right\Vert_{H^s} \leq \left\Vert u\right\Vert_{H^s} + \theta \left\Vert u\right\Vert_{H^s}.$$  

Choosing $T$ sufficiently small such that $\theta M_0 + \left\Vert u_0\right\Vert_{H^s} < M_0$, we know that $A$ maps $B_{M_0}(0)$ to itself. It follows from the contractive mapping principle that the mapping $A$ has a unique fixed point $u$ in $B_{M_0}(0)$. This completes the proof of Theorem 2.1.

### 3. Existence of Local Weak Solutions

In this section, we assume that $\varphi(\eta) = \eta^{2N+1}$ where $N$ is a nature number. In order to establish the existence of local weak solution, we need the following lemmas.

**Lemma 3.1** (see Kato and Ponce [8]). If $r \geq 0$, then $H^r \cap L^\infty$ is an algebra. Moreover,

$$\left\Vert uv\right\Vert_r \leq c\left(\left\Vert u\right\Vert_{L^r}\left\Vert v\right\Vert_r + \left\Vert u\right\Vert_r\left\Vert v\right\Vert_{L^\infty}\right),$$  

where $c$ is a constant depending only on $r$.

**Lemma 3.2** (see Kato and Ponce [8]). Let $r > 0$. If $u \in H^r \cap W^{1,\infty}$ and $v \in H^{r-1} \cap L^\infty$, then

$$\left\Vert [\Lambda', u]v\right\Vert_{L^2} \leq c\left(\left\Vert \partial_xu\right\Vert_{L^r}\left\Vert \Lambda^{-1}v\right\Vert_{L^2} + \left\Vert \Lambda' u\right\Vert_{L^2}\left\Vert v\right\Vert_{L^\infty}\right).$$  

$$\left\Vert \left[u^{2m}u_{xx} - v^{2m}v_{xx}\right]\right\Vert_{H^{s-2}} \leq \left\Vert \partial_x\left[u^{2m}u_x - v^{2m}v_x\right]\right\Vert_{H^{s-1}} + c\left\Vert u^{2m-1}u_x^2 - v^{2m-1}v_x^2\right\Vert_{H^{s-1}}$$

(2.7)
Lemma 3.3. Let $s \geq 3/2$, $q(u_s) = u_2^{2N+1}$, and the function $u(t, x)$ is a solution of problem (1.3) and the initial data $u_0(x) \in H^s$. Then the following results hold.

For $q \in (0, s - 1]$, there is a constant $c$ such that

$$
\int_{\mathbb{R}} (\Lambda^{q+1} u)^2 \, dx \leq \int_{\mathbb{R}} \left( \left(\Lambda^{q+1} u_0\right)^2 \right) \, dx + c \int_0^t \left( \|u_x\|^2_{L^2} + \|u\|^2_{L^\infty} \right) \, dt.
$$

(3.3)

For $q \in [0, s - 1]$, there is a constant $c$ such that

$$
\|u_t\|_{H^q} \leq c \|u\|_{H^{q+1}} \left( \|u\|^2_{L^\infty} + \|u_x\|^2_{L^2} \right).
$$

(3.4)

Proof. For $q \in (0, s - 1]$, applying $(\Lambda^{q} u) \Lambda^{q}$ to both sides of the first equation of system (1.3) and integrating with respect to $x$ by parts, we have the identity

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\Lambda^{q} u)^2 + \alpha (\Lambda^{q} u_x)^2 \, dx = \int_{\mathbb{R}} (\Lambda^{q} u) \Lambda^{q} (\varphi(u_s)) \, dx + \beta \int_{\mathbb{R}} \Lambda^{q} u \Lambda^{q} \left[ u^{2m} u_{xx} \right] \, dx.
$$

(3.5)

We will estimate the two terms on the right-hand side of (3.5), respectively. For the first term, by using the Cauchy-Schwartz inequality and Lemmas 3.1 and 3.2, we have

$$
\left| \int_{\mathbb{R}} (\Lambda^{q} u) \Lambda^{q} (\varphi(u_s)) \, dx \right| = \left| \int_{\mathbb{R}} (\Lambda^{q} u_x) \Lambda^{q} (\varphi(u_s)) \, dx \right| \leq c \|\Lambda^{q} u_x\|_{L^2} \left\| \Lambda^{q} (u_s)^{2N+1} \right\|_{L^2} \leq c \|u\|_{H^{q+1}} \left\| \Lambda^{q} u_x \right\|_{L^2}^{2N}.
$$

(3.6)

For the second term, we have

$$
\int_{\mathbb{R}} \Lambda^{q} u \Lambda^{q} \left[ u^{2m} u_{xx} \right] \, dx = \int_{\mathbb{R}} \Lambda^{q} u \Lambda^{q} \left[ \left( u^{2m} u_x \right)_x - 2mu^{2m-1} u_x^2 \right] \, dx
$$

$$
= \int_{\mathbb{R}} \Lambda^{q} u_x \Lambda^{q} \left( u^{2m} u_x \right) \, dx - 2m \int_{\mathbb{R}} \Lambda^{q} u \Lambda^{q} \left[ u^{2m-1} u_x^2 \right] \, dx = K_1 + K_2.
$$

(3.7)

For $K_1$, applying Lemma 3.1 derives

$$
|K_1| \leq c \|u\|_{H^{q+1}}^2 \left( \|u\|_{L^\infty}^{2m} + \|u_x\|_{L^2} \|u\|_{L^\infty}^{2m-1} \right).
$$

(3.8)
For $K_2$, we get
\[
|K_2| \leq c\|u\|_{H^1}\|u^{2m-1}u_x^2\|_{H^1}
\leq c\|u\|_{H^1}\left(\|u^{2m-1}u_x\|_{L^\infty} + \|u^{2m-1}u_x\|_{H^1}\right)
\leq c\|u\|_{H^1}^2\left(\|u_x\|_{L^\infty} + \|u\|_{L^\infty}^2\right).
\]  
(3.9)

It follows from (3.5)–(3.9) that there exists a constant $c$ such that
\[
\frac{1}{2} \int_R \left((\Lambda^q u)^2 + (\Lambda^q u_x)^2\right) dx
\leq c\|u\|_{H^1}^2\left(\|u_x\|_{L^\infty}^2 + \|u\|_{L^\infty}^2\right).
\]  
(3.10)

Integrating both sides of the above inequality with respect to $t$ results in inequality (3.3).

To estimate the norm of $u_t$, we apply the operator $(1 - \partial_x^2)^{-1}$ to both sides of the first equation of system (1.3) to obtain the equation
\[
u_t = \Lambda^{-2}\left(\frac{\partial}{\partial x}\varphi(u_x) + \beta u^{2m}u_{xxx}\right). \tag{3.11}
\]

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of (3.11) for $q \in [0, s - 1]$ gives rise to
\[
\int_R (\Lambda^q u_t)^2 dx = \int_R (\Lambda^q u_t)\Lambda^q\left[\partial_x\varphi(u_x) + u^{2m}u_{xxx}\right] d\tau. \tag{3.12}
\]

For the right-hand of (3.12), we have
\[
\left|\int_R (\Lambda^q u_t)\left(1 - \partial_x^2\right)^{-1}\Lambda^q\partial_x\varphi(u_x) dx\right|
\leq c\|u_t\|_{H^1}\left(\int_R \left(1 + \xi^2\right)^{q-1}\left[\int_R \left[u^{2N} \varphi(u_x)\right] d\eta\right]^2\right)^{1/2}
\leq c\|u_t\|_{H^1}\left(\int_R \left(1 + \xi^2\right)^{q-1}\left[\int_R \left[u^{2m} \varphi(u_x)\right] d\eta\right]^2\right)^{1/2},
\]
\[
\left|\int_R (\Lambda^q u_t)\left(1 - \partial_x^2\right)^{-1}\Lambda^q\partial_x\left(u^{2m}u_x\right) dx\right|
\leq c\|u_t\|_{H^1}\left(\int_R \left(1 + \xi^2\right)^{q-1}\left[\int_R \left[u^{2m} \varphi(u_x)\right] d\eta\right]^2\right)^{1/2},
\]
\[
\left|\int_R (\Lambda^q u_t)\left(1 - \partial_x^2\right)^{-1}\Lambda^q\left(u^{2m-1}u_x^2\right) dx\right|
\leq c\|u_t\|_{H^1}\left(\int_R \left(1 + \xi^2\right)^{q-1}\left[\int_R \left[u^{2m-1} \varphi(u_x)\right] d\eta\right]^2\right)^{1/2}
\leq c\|u_t\|_{H^1}\left(\int_R \left(1 + \xi^2\right)^{q-1}\left[\int_R \left[u^{2m-1} \varphi(u_x)\right] d\eta\right]^2\right)^{1/2}.
\]  
(3.13)
Applying (3.13) into (3.12) yields the inequality
\[
\|u_t\|_{H^s} \leq c\|u\|_{H^s} \|u\|_{H^{s+1}} \left(\|u\|_{L^\infty}^{2m-1} + \|u_x\|_{L^\infty}^{2N-1}\right) \tag{3.14}
\]
for a constant \(c > 0\). This completes the proof of Lemma 3.3.

**Lemma 3.4.** If \(u(t,x)\) is a solution of problem (1.3), \(\alpha > 0\), and \(\varphi(\eta) = \eta^{2N+1}\), then
\[
\|u\|_{L^\infty} \leq c\|u\|_{H^s(R)} \leq c\|u_0\|_{H^s(R)}, \tag{3.15}
\]
where \(c\) is a constant.

**Proof.** Multiplying both sides of the first equation of (1.3) by \(u(t,x)\) and integrating with respect to \(x\) over \(R\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_R \left[ u(t,x)^2 + a(u_x(t,x))^2 \right] dx = \int_R \varphi(u_x)_x u(t,x) dx + \beta \int_R u^{2m+1} u_{xx} dx. \tag{3.16}
\]

Since
\[
\int_R \varphi(u_x)_x u(t,x) dx + \beta \int_R u^{2m+1} u_{xx} dx = - \int_R u_x^{2N+2} dx - \beta(2m + 1) \int_R u^{2m} u_x^2 dx < 0, \tag{3.17}
\]
we derive that
\[
\frac{1}{2} \frac{d}{dt} \int_R \left[ u(t,x)^2 + a(u_x(t,x))^2 \right] dx < 0, \tag{3.18}
\]
which results in
\[
\int_R \left[ u(t,x)^2 + a(u_x(t,x))^2 \right] < \int_R \left[ u(0,x)^2 + a(u_x(0,x))^2 \right] \leq c\|u_0\|_{H^s}^2. \tag{3.19}
\]
From (3.19), we know that (3.15) holds. This completes the proof.

Defining
\[
\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases} \tag{3.20}
\]
and setting \(\phi_\varepsilon(x) = \varepsilon^{-1/4} \phi(\varepsilon^{-1/4}x)\) with \(0 < \varepsilon < 1/4\) and \(u_0 = \phi_\varepsilon \ast u_0\), we know that \(u_0 \in C^\infty\) for any \(u_0 \in H^s(R)\) and \(s > 0\).
It follows from Theorem 2.1 that for each \( \varepsilon \) the Cauchy problem

\[
\begin{align*}
    u_t - u_{txx} &= \frac{\partial}{\partial x} q(u_x) + \beta u^{2m} u_{xx}, \\
    u(0, x) &= u_\varepsilon(x), \quad x \in \mathbb{R}
\end{align*}
\]  

(3.21)

has a unique solution \( u_\varepsilon(t, x) \in C^\infty([0, T); H^\infty) \).

**Lemma 3.5.** Under the assumptions of problem (3.21), the following estimates hold for any \( \varepsilon \) with \( 0 < \varepsilon < 1/4 \), \( u_\varepsilon \in H^s(\mathbb{R}) \) and \( s > 0 \):

\[
\begin{align*}
    \|u_{\varepsilon 0}\|_{L^\infty} &\leq c_1 \|u_{\varepsilon 0}\|_{L^\infty}, \\
    \|u_\varepsilon\|_{H^{s+1}} &\leq c_1, \quad \text{if } q \leq s,
\end{align*}
\]

(3.22)

where \( c_1 \) is a constant independent of \( \varepsilon \).

**Proof.** Using the definition of \( u_{\varepsilon 0} \) and \( u_{\varepsilon 0x} \) results in the conclusion of the lemma. \( \square \)

**Lemma 3.6.** Suppose that \( u_\varepsilon(x) \in H^s(\mathbb{R}) \) with \( s \in [1, 3/2] \) such that \( \|u_{\varepsilon 0}\|_{L^\infty} < \infty \). Let \( u_\varepsilon \) be defined as in system (3.21) and let \( q(\eta) = \eta^{2N+1} \). Then there exist two positive constants \( T \) and \( c \), independent of \( \varepsilon \), such that the solution \( u_\varepsilon \) of problem (3.21) satisfies \( \|u_{\varepsilon x}\|_{L^\infty} \leq c \) for any \( t \in [0, T) \).

**Proof.** Using notation \( u = u_\varepsilon \) and differentiating both sides of the first equation of problem (3.11) with respect to \( x \) give rise to

\[
u_{tx} = -\varphi(u_x) - \beta u^{2m} u_x + \Lambda^{-2} \left[ \varphi(u_x) + \beta u^{2m} u_x - 2m\beta \left( u^{2m-1} u_x \right)_x \right].
\]

(3.23)

Letting \( p \) be an integer and multiplying the above equation by \( u_x^{2p+1} \) and then integrating the resulting equation with respect to \( x \) yield the equality

\[
\frac{1}{2p+2} \frac{d}{dt} \int_{\mathbb{R}} (u_x)^{2p+2} dx = -\int_{\mathbb{R}} \varphi(u_x) u_x^{2p+1} dx - \beta \int_{\mathbb{R}} u^{2m} u_x^{2p+1} dx + \int_{\mathbb{R}} J u_x^{2p+1} dx,
\]

(3.24)

where

\[
J = \Lambda^{-2} \left[ \varphi(u_x) + \beta u^{2m} u_x - 2m\beta \left( u^{2m-1} u_x \right)_x \right].
\]

(3.25)

Applying the Hölder’s inequality to (3.24) and noting Lemmas 3.4 and 3.5, we obtain

\[
\frac{1}{2p+2} \frac{d}{dt} \int_{\mathbb{R}} (u_x)^{2p+2} dx \leq c \|u_x\|_{L^\infty}^{2N} \int_{\mathbb{R}} |u_x|^{2p+2} dx + c \int_{\mathbb{R}} u_x^{2p+2} dx
\]

\[
+ \left( \int_{\mathbb{R}} |J|^{2p+2} dx \right)^{1/(2p+2)} \left( u_x^{2p+2} dx \right)^{2(p+1)/(2p+2)}.
\]

(3.26)
or
\[
\frac{d}{dt} \left( \int_R (u_x)^{2(p+2)} \, dx \right)^{1/(2p+2)} \leq c \|u_x\|_{L^2}^2 \left( \int_R |u_x|^{2(p+2)} \, dx \right)^{1/(2p+2)} + c \left( \int_R |u_x|^{2(p+2)} \, dx \right)^{1/(2p+2)}.
\]
(3.27)

Since \(\|f\|_{L^p} \to \|f\|_{L^\infty} \) as \(p \to \infty\) for any \(f \in L^\infty \cap L^2\), integrating both sides of the inequality (3.27) with respect to \(t\) and taking the limit as \(p \to \infty\) result in the estimate
\[
\|u_x\|_{L^\infty} \leq \|u_{0x}\|_{L^\infty} + \int_0^t c \left( \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^{2N+1} + \|f\|_{L^\infty} \right) \, d\tau.
\]
(3.28)

Using the algebra property of \(H^m(R)\) with \(m > 1/2\) yields \(\|u_x\|_{H^{1/2}}\) means that there exists a sufficiently small \(\delta > 0\) such that \(\|u_x\|_{H^{1/2}} = \|u_x\|_{H^{1/2}}\)
\[
\|f\|_{H^{1/2}} \leq c \|f\|_{H^{1/2}} \leq c \Lambda^{-2} \left[ \varphi(u_x) + \beta u_x^{2m} - 2m \beta \left( u_x^{2m-1} u_x \right)_x \right]_{H^{1/2}},
\]
\[
\leq c \left( \|\varphi(u_x)\|_{H^0} + \|u_x\|_{H^1} + \|u_x^{2m-1} u_x\|_{H^0} \right),
\]
(3.29)

in which Lemmas 3.4 and 3.5 are used. From (3.28) and (3.29), one has
\[
\|u_x\|_{L^\infty} \leq \|u_{0x}\|_{L^\infty} + c \int_0^t \left[ 1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{2N} + \|u_x\|_{L^\infty}^{2N+1} \right] \, d\tau.
\]
(3.30)

From Lemma 3.5, it follows from the contraction mapping principle that there is a \(T > 0\) such that the equation
\[
\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c \int_0^t \left[ 1 + \|W\|_{L^\infty} + \|W\|_{L^\infty}^{2N} + \|W\|_{L^\infty}^{2N+1} \right] \, d\tau
\]
(3.31)

has a unique solution \(W \in C[0,T]\). Using the result presented on page 51 in [11] yields that there are constants \(T > 0\) and \(c > 0\) independent of \(\varepsilon\) such that \(\|u_x\|_{L^\infty} \leq \|W(t)\|_{L^\infty} \leq c\) for arbitrary \(t \in [0,T]\), which leads to the conclusion of Lemma 3.6.

Using Lemmas 3.3–3.6, notation \(u_\varepsilon = u\) and Gronwall’s inequality result in the inequalities
\[
\|u_\varepsilon\|_{H^0} \leq C_T e^{C_T},
\]
\[
\|u_\varepsilon\|_{H^1} \leq C_T e^{C_T},
\]
(3.32)
where \( q \in (0, s] \), \( r \in (0, s - 1] \) (\( 1 \leq s \leq 3/2 \)) and \( C_T \) depends on \( T \). It follows from the Aubin’s compactness theorem that there is a subsequence of \( \{u_{\epsilon_n}\} \), denoted by \( \{u_{\epsilon_n}\} \), such that \( \{u_{\epsilon_n}\} \) and their temporal derivatives \( \{u_{\epsilon_n,t}\} \) are weakly convergent to a function \( u(t, x) \) and its derivative \( u_t \) in \( L^2([0, T], H^r) \) and \( L^2([0, T], H^{r-1}) \), respectively. Moreover, for any real number \( R_1 > 0 \), \( \{u_{\epsilon_n}\} \) is convergent to the function \( u \) strongly in the space \( L^2([0, T], H^r(-R_1, R_1)) \) and \( \{u_{\epsilon_n,t}\} \) converges to \( u_t \) strongly in the space \( L^2([0, T], H^{r-1}(-R_1, R_1)) \) for \( r \in [0, s - 1] \). Thus, we can prove the existence of a weak solution to (1.3).

**Theorem 3.7.** Suppose that \( u_0(x) \in H^s \) with \( 1 \leq s \leq 3/2 \), \( \|u_0\|_{H^s} < \infty \) and \( \varphi(\eta) = \eta^{2N+1} \). Then there exists a \( T > 0 \) such that (1.3) subject to initial value \( u_0(x) \) has a weak solution \( u(t, x) \in L^2([0, T], H^s) \) in the sense of distribution and \( u_\epsilon \in L^\infty([0, T] \times R) \).

**Proof.** From Lemma 3.6, we know that \( \{u_{\epsilon_n}\} (\epsilon_n \to 0) \) is bounded in the space \( L^\infty \). Thus, the sequences \( \{u_{\epsilon_n}\}, \{u_{\epsilon_n,t}\}, \{u_{\epsilon_n,x}^2\}, \) and \( \{u_{\epsilon_n,x}^2,N+1\} \) are weakly convergent to \( u_\epsilon, u_t, u_x^2, \) and \( u_x^2,N+1 \) in \( L^2([0, T], H^r(-R, R)) \) for any \( r \in [0, s - 1] \), separately. Therefore, \( u \) satisfies the equation

\[
\int_0^T \int_R u(g_t - g_{xx}) \, dx \, dt = \int_0^T \int_R \left[ u_x^{N+1} g_x + \beta u^2 u_x g_x - 2m\beta u^{2m-1} u_x^2 g \right] \, dx \, dt, \tag{3.33}
\]

with \( u(0, x) = u_0(x) \) and \( g \in C_0^\infty \). Since \( X = L^1([0, T] \times R) \) is a separable Banach space and \( \{u_{\epsilon_n}\} \) is a bounded sequence in the dual space \( X^* = L^\infty([0, T] \times R) \) of \( X \), there exists a subsequence of \( \{u_{\epsilon_n}\} \), still denoted by \( \{u_{\epsilon_n}\} \), weakly star convergent to a function \( v \) in \( L^\infty([0, T] \times R) \). It derives from the weakly convergence of \( \{u_{\epsilon_n}\} \) to \( u_\epsilon \) in \( L^2([0, T], H^r(-R, R)) \) that \( u_\epsilon = v \) almost everywhere. Thus, we obtain \( u_\epsilon \in L^\infty([0, T] \times R) \).

4. Well-Posedness of Global Solutions

**Lemma 4.1.** If \( u(t, x) \) is a solution of problem (1.3), \( \alpha > 0 \), \( \varphi(\eta) = \eta^{2N+1} \), then

\[
\|u_\epsilon\|_{L^\infty} \leq A^{1/2}, \tag{4.1}
\]

where

\[
A = \int_R \left[ \frac{1 + \alpha}{\alpha} (u_0'(x))^2 + (1 + \alpha)(u_0''(x))^2 \right] \, dx. \tag{4.2}
\]

**Proof.** Multiplying each side of the first equation of problem (1.3) by \( u_{\epsilon,t} \) and integrating over \([0, t] \times R \) yields

\[
\int_0^t \int_R \left( u_{\epsilon,xx} q'(u_\epsilon) + u^{2m} u_{\epsilon,xx} + \frac{\alpha}{2} \frac{\partial}{\partial t} (u_{\epsilon,xx}^2) \right) \, dx \, dt = \int_0^t \int_R u_t u_{\epsilon,xx} \, dx \, dt. \tag{4.3}
\]

Integrating the right-hand side of the above identity by parts and using \( u_\epsilon(\pm \infty) = 0 \), we get

\[
2 \int_0^t \int_R u_t u_{\epsilon,xx} \, dx \, dt = \int_R [u_0'(x)]^2 \, dx - \int_R u_\epsilon^2(t, x) \, dx. \tag{4.4}
\]
Abstract and Applied Analysis

From (4.3), (4.4) and the assumption of this lemma, we have

\[ \alpha \|u_{xx}\|_{L^2}^2 + \|u_x\|_{L^2} \leq \int_R \left[ (u_0'(x))^2 + \alpha (u_0''(x))^2 \right] dx, \]  

(4.5)

from which we obtain (4.1). □

**Theorem 4.2.** Suppose that \( s \geq 2, u_0 \in H^s(R), \varphi(u_x) = u_x^{2N+1} \) with positive integer \( N \). Then problem (1.3) has a unique global solution:

\[ u(t,x) \in C([0, \infty); H^s(R)) \cap C^1 \left( [0, \infty); H^{s-1}(R) \right). \]  

(4.6)

**Proof.** Using the Gronwall inequality and Lemma 3.3 and choosing \( s = q + 1 \), we have

\[ \|u\|_{H^s} \leq c \|u_0\|_{H^s} e^{\int_0^t \left( \|u_0\|_{H^s}^2 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|u_{xxx}\|_{L^2} + \|u_{xxxx}\|_{L^2} \right) dt}. \]  

(4.7)

From Lemma 4.1, we have

\[ \|u_x\| \leq A^{1/2} = \left( \int_R \left[ \frac{1 + \alpha}{\alpha} (u_0'(x))^2 + (1 + \alpha) (u_0''(x))^2 \right] dx \right)^{1/2} \leq c \|u_0\|_{H^2(R)}. \]  

(4.8)

Using (4.7) and (4.8) derives

\[ \|u\|_{H^s} \leq c \|u_0\|_{H^s} e^{ct}, \]  

(4.9)

which completes the proof of Theorem 4.2. □

**References**


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