Research Article

On the Well-Posedness of the Boussinesq Equation in the Triebel-Lizorkin-Lorentz Spaces

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We establish the local well-posedness and obtain a blow-up criterion of smooth solutions for the Boussinesq equations in the framework of Triebel-Lizorkin-Lorentz spaces. The main ingredients of our proofs are Littlewood-Paley decomposition and the paradifferential calculus.

1. Introduction

In this paper, we consider the following inviscid Boussinesq equations:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \theta e_d, & \text{in } \mathbb{R}^d \times (0, T), \\ \partial_t \theta + u \cdot \nabla \theta &= 0, & \text{in } \mathbb{R}^d \times (0, T), \\ \operatorname{div} u &= 0, & \text{in } \mathbb{R}^d \times (0, T), \\ u(x, 0) &= u_0(x), \quad \theta(x, 0) = \theta_0(x), & \text{in } \mathbb{R}^d, \end{aligned} \tag{1.1}$$

with $\operatorname{div} u_0 = 0$ and $e_d = (0, \dots, 0, 1)$. This system describes the motion of lighter or denser incompressible fluid under the influence of gravitational forces, for instance, in the atmospheric sciences, where the vector $u = (u_1, u_2, \dots, u_d)$ is the velocity, the scalar function θ denotes the temperature, and p stands for the pressure in the fluid (see [1]). The term θe_d takes into account the influence of the gravity and the stratification on the motion of the fluid. In case $d = 2$, it is also used as a simplified model for the 3D axisymmetric Euler equations with swirl away from the symmetric axis $r = 0$.

The regularity or singularity questions of (1.1) are an outstanding open problem in the mathematical fluid mechanics. Over the past few years, (1.1) has been studied extensively both theoretically (see [1–6] and references therein) and numerically (see [7–9]). In particular, the local well-posedness and some blow-up criteria of (1.1) are established by Chae and Nam [3] in the Sobolev spaces, Liu et al. [5, 6] and Xiang [10] in the Besov spaces, and Cui et al. [11] in the Hölder spaces. The purpose of this paper is to establish the local well-posedness for the Boussineq equations (1.1) and to obtain a blow-up criterion of the smooth solutions in the framework of Triebel-Lizorkin-Lorentz spaces, which contain the classical Triebel-Lizorkin spaces and Lorentz spaces. Indeed, it is more natural to consider the well-posedness in the Triebel-Lizorkin-type spaces than in the Besov spaces in the sense that $F_{p,2}^s = W^{s,p}$ for $1 < p < \infty$.

Before proceeding further, we mention several local well-posedness results for other fluid equations in the Triebel-Lizorkin spaces. Recently, Chae [12] introduced a family trajectory mapping $\{X_j(\alpha, t)\}$ satisfying

$$\begin{aligned} \frac{d}{dt}X_j(\alpha, t) &= S_{j-2}v(X_j(\alpha, t), t), \\ X_j(\alpha, 0) &= \alpha, \end{aligned} \quad (1.2)$$

where S_{j-2} is a frequency projection to the ball $\{\xi \in \mathbb{R}^2 \mid |\xi| \lesssim 2^j\}$ and used the following equivalent relation:

$$\left\| 2^{js} \|\Delta_j \theta(X_j(\alpha, t), t)\|_{L^q(\mathbb{Z})} \right\|_{L^p(d\alpha)} \sim \left\| 2^{js} \|\Delta_j \theta(x, t)\|_{L^q(\mathbb{Z})} \right\|_{L^p(dx)} = \|\theta(t)\|_{F_{p,q}^s}, \quad (1.3)$$

to estimate the frequency-localized solutions of the Euler equations in the Triebel-Lizorkin spaces. However, it seems difficult to give a strict proof for (1.3)-type equivalent relation due to the lack of the uniform change of the coordinates independent of j . To avoid this trouble, Chen et al. [13] introduced a particle trajectory mapping $X(\alpha, t)$ independent of j defined by

$$\begin{aligned} \frac{d}{dt}X(\alpha, t) &= v(X(\alpha, t), t), \\ X(\alpha, 0) &= \alpha, \end{aligned} \quad (1.4)$$

and then they established a new commutator estimate to obtain the local well-posedness of the ideal MHD equations in the Triebel-Lizorkin spaces.

In this paper, we will adapt the method of Chen et al. [13] to establish the local well-posedness and to obtain a blow-up criterion of the smooth solutions for the Boussineq equations (1.1) in the framework of Triebel-Lizorkin-Lorentz spaces. Precisely, we first define the particle trajectory mapping $X(\alpha, t)$ by (1.4). Then to show the well-posedness in the new framework, we have to establish the commutator estimate (Proposition 2.5) and the product estimate (Proposition 2.6). For the blow-up criterion, we also need the logarithmic inequality (Proposition 2.7). Fortunately, these preliminary estimates and inequalities have been obtained in our recent work [14].

Our main results can be formulated in the following way.

Theorem 1.1. (i) *Local Existence.* Let $1 < r \leq p < \infty$, $1 < q < \infty$, and $s > d/p + 1$. If $(u_0, \theta_0) \in (F_{p,q}^{s,r}(\mathbb{R}^d))^{d+1}$, then there exists $T_1 = T_1(\|u_0\|_{F_{p,q}^{s,r}}, \|\theta_0\|_{F_{p,q}^{s,r}}) > 0$ such that the Boussinesq equation (1.1) has a unique solution $(u, \theta) \in (C([0, T_1]; F_{p,q}^{s,r}(\mathbb{R}^d)))^{d+1}$.

(ii) *Blow-Up Criterion.* The local-in-time solution $(u, \theta) \in (C([0, T_1]; F_{p,q}^{s,r}(\mathbb{R}^d)))^{d+1}$ constructed in (i) blows up at finite time $T^* > T_1$, that is,

$$\limsup_{t \rightarrow T^*-} (\|u(t)\|_{F_{p,q}^{s,r}} + \|\theta(t)\|_{F_{p,q}^{s,r}}) = \infty, \quad (1.5)$$

if and only if

$$\int_0^{T^*} (\|u(t)\|_{\dot{F}_{\infty,\infty}^1} + \|\theta(t)\|_{\dot{F}_{\infty,\infty}^1}) dt = \infty. \quad (1.6)$$

Remark 1.2. In case $\theta_0 \equiv 0$, we have $\theta \equiv 0$ and thus get the local well-posedness and blow-up criterion for the incompressible Euler equations in the Triebel-Lizorkin-Lorentz spaces, which generalize the corresponding results in [12] because of $F_{p,q}^{s,p} = F_{p,q}^s$. For general θ_0 , we also extend the results of [3] from Sobolev spaces to Triebel-Lizorkin-Lorentz spaces.

Remark 1.3. For the MHD equations, we can also generalize the local well-posedness and blow-up criterion of [13] from the Triebel-Lizorkin spaces to the Triebel-Lizorkin-Lorentz spaces by using similar arguments.

The rest of this paper is organized as follows. We state some preliminaries on functional settings in Section 2 and then prove the local well-posedness and blow-up criterion in Section 3.

2. Preliminaries

In this section, we give some definitions and basic estimates (see [14, 15] for details). First of all, we let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions. For a fixed radially symmetric bump function $\chi \in C_0^\infty(B(0, 4/3))$ with value 1 over the ball $B(0, 3/4)$, we set $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$ and then have the following dyadic partition of unity:

$$\begin{aligned} \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbb{R}^d, \\ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \end{aligned} \quad (2.1)$$

The frequency localization operators Δ_j and S_j can be defined as follows:

$$\begin{aligned} \Delta_j u &:= \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}(u)) = 2^{jd}(\mathcal{F}^{-1}\varphi(2^j\cdot)) * u, \quad j \in \mathbb{Z}, \\ S_j u &:= \sum_{j' \leq j-1} \Delta_{j'} u = \chi(2^{-j}D)u = 2^{jd}(\mathcal{F}^{-1}\chi(2^j\cdot)) * u, \quad j \in \mathbb{Z}, \end{aligned} \quad (2.2)$$

where $\mathcal{F}$ is the Fourier transform and D is the Fourier multiplier with symbol $|\xi|$. The frequency localization operators Δ_j and S_j have nice almost orthogonal properties in L^2 :

$$\Delta_j \Delta_{j'} u \equiv 0, \quad \text{if } |j - j'| \geq 2, \quad \Delta_j (S_{j'-1} u \Delta_{j'} v) \equiv 0, \quad \text{if } |j - j'| \geq 5 \quad (2.3)$$

for any $u, v \in \mathcal{S}'(\mathbb{R}^d)$.

Next we recall several function spaces used in this paper. The first one is *the Triebel-Lizorkin spaces*. For $s \in \mathbb{R}$, $(p, q) \in [1, +\infty) \times [1, +\infty]$, we define the homogeneous Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^d)$ as the set of tempered distributions $u \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^d)$ such that

$$\|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^d)} := \left\| \left\| 2^{js} |\Delta_j u| \right\|_{l^q(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^d)} < \infty, \quad (2.4)$$

where $\mathcal{P}$ is the polynomial space. Then for $s > 0$, $(p, q) \in [1, +\infty) \times [1, +\infty]$, we can define the inhomogeneous Triebel-Lizorkin spaces $F_{p,q}^s(\mathbb{R}^d)$ as the set of tempered distributions u such that

$$\|u\|_{F_{p,q}^s(\mathbb{R}^d)} := \|u\|_{L^p(\mathbb{R}^d)} + \|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^d)} < \infty. \quad (2.5)$$

The inhomogeneous Triebel-Lizorkin spaces $F_{p,q}^s$ are Banach spaces with norms $\|\cdot\|_{F_{p,q}^s}$.

The second one is *the Lorentz spaces*. Let $u_*(\tau)$ be the distribution function of a function $u(x)$ and let $u^*(t)$ be its nonincreasing rearrangement, that is,

$$u_*(\tau) = \text{meas}\{x \mid |u(x)| > \tau\}, \quad u^*(\tau) = \inf\{\tau > 0 \mid u_*(\tau) \leq t\}. \quad (2.6)$$

Then for $1 < p, r < \infty$, the Lorentz spaces $L^{p,r}(\mathbb{R}^d)$ are defined as the set of those measurable functions u satisfying

$$\|u\|_{L^{p,r}} = \left(\frac{r}{p} \int_0^\infty \left(t^{1/p} u^{**}(t) \right)^r \frac{dt}{t} \right)^{1/r} < \infty, \quad (2.7)$$

where $u^{**}(t) := (1/t) \int_0^t u^*(s) ds$ is the average rearrangement. The Lorentz spaces $L^{p,r}$ are Banach spaces with norms $\|\cdot\|_{L^{p,r}}$. Since $u_*(\tau)$ is nonincreasing in τ , it is easy to verify that the following equivalent characterization holds:

$$\|u\|_{L^{p,r}} \sim \left\| 2^j u_*^{1/p}(2^j) \right\|_r. \quad (2.8)$$

Moreover, it is well known that there exist $1 \leq p_1$ and $p_2 < \infty$ such that $L^{p,r}$ are the interpolations spaces between L^{p_1} and L^{p_2} . Thus we can deduce the following basic fact, which is standard in the usual L^p spaces.

Lemma 2.1. Assume $u \in L^{p,r}$ for $1 < p, r < \infty$. If $X : \alpha \rightarrow X(\alpha)$ is a volume-preserving diffeomorphism, then

$$\|u(\alpha)\|_{L^{p,r}} = \|u(X(\alpha))\|_{L^{p,r}}. \quad (2.9)$$

The last one is called *the Triebel-Lizorkin-Lorentz spaces* and is introduced recently by Yang et al. [16] to unify the Triebel-Lizorkin spaces and the Lorentz spaces.

Definition 2.2. For $s \in \mathbb{R}$ and $1 < p, q, r < \infty$, a distribution $u \in \mathcal{S}'$ is said to be in the homogeneous Triebel-Lizorkin-Lorentz spaces $\dot{F}_{p,q}^{s,r}(\mathbb{R}^d)$ if

$$\left\| 2^k \left(\text{meas} \left\{ x \in \mathbb{R}^d \mid \|2^{js} |\Delta_j u|\|_{l_j^q(\mathbb{Z})} > 2^k \right\} \right)^{1/p} \right\|_{l_k^r(\mathbb{Z})} < \infty. \quad (2.10)$$

Indeed we can deduce that $u \in \dot{F}_{p,q}^{s,r}(\mathbb{R}^d)$ if and only if $\| \|2^{js} |\Delta_j u|\|_{l_j^q(\mathbb{Z})} \|_{L^{p,r}} < \infty$ by the equivalent characterization (2.8) of Lorentz space $L^{p,r}$. Then $\dot{F}_{p,q}^s(\mathbb{R}^d) \simeq \dot{F}_{p,q}^{s,p}(\mathbb{R}^d)$ and $L^{p,r}(\mathbb{R}^d) \simeq \dot{F}_{p,2}^{0,r}(\mathbb{R}^d)$ (see [16, Theorem 5]). For $s > 0$, the inhomogeneous Triebel-Lizorkin-Lorentz spaces $F_{p,q}^{s,r}(\mathbb{R}^d)$ are defined by

$$F_{p,q}^{s,r}(\mathbb{R}^d) = L^{p,r} \cap \dot{F}_{p,q}^{s,r}(\mathbb{R}^d) \quad (2.11)$$

with norm

$$\|u\|_{F_{p,q}^{s,r}(\mathbb{R}^d)} = \|u\|_{L^{p,r}} + \|u\|_{\dot{F}_{p,q}^{s,r}(\mathbb{R}^d)}. \quad (2.12)$$

Clearly, $F_{p,q}^{s,r}(\mathbb{R}^d)$ is a Banach space. It has been proved that there exist $1 < p_1$ and $p_2 < \infty$ such that $F_{p,q}^{s,r}(\mathbb{R}^d)$ are the interpolation spaces between $F_{p_1,q}^s(\mathbb{R}^d)$ and $F_{p_2,q}^s(\mathbb{R}^d)$ (see [16, Theorem 6]). Thus by the boundedness of Riesz transform on the Triebel-Lizorkin spaces (see Frazier et al. [17]) we have the following.

Lemma 2.3. *Riesz transform is bounded from the Triebel-Lizorkin-Lorentz space $F_{p,q}^{s,r}(\mathbb{R}^d)$ into itself. Also by $L^{p,r} \hookrightarrow L^p$ for $r \leq p$, we have the following Sobolev's embedding theorem:*

$$F_{p,q}^{s,r} \hookrightarrow F_{p,q}^{s,p} = F_{p,q}^s \hookrightarrow L^\infty, \quad \text{for } s > \frac{d}{p}. \quad (2.13)$$

The following lemma is referred as Bernstein's inequality, which describes the way derivatives act on spectrally localized functions.

Lemma 2.4 (Bernstein's inequality). *Let $1 < p, r < \infty$, and $u \in L^{p,r}(\mathbb{R}^d)$. Then*

$$\text{supp } \mathcal{F}(u) \subset \left\{ \xi \in \mathbb{R}^d : |\xi| \sim 2^j \right\} \implies \|\partial^\alpha u\|_{L^{p,r}} \sim 2^{j|\alpha|} \|u\|_{L^{p,r}}. \quad (2.14)$$

Here and thereafter, we use $A \lesssim B$ and $A \sim B$ to denote $A \leq CB$ and $cB \leq A \leq CB$ for some constants $C > c > 0$, respectively. Thus Bernstein's inequality together with the Littlewood-Paley decomposition gives that

$$\|\partial^\alpha u\|_{\dot{F}_{p,q}^{s,r}} \sim \|u\|_{\dot{F}_{p,q}^{s+|\alpha|,r}}, \quad 1 < p, r < \infty \quad (2.15)$$

for any $u \in \dot{F}_{p,q}^{s+|\alpha|,r}(\mathbb{R}^d)$.

To conclude this section, we recall three important propositions in the Triebel-Lizorkin-Lorentz spaces, which will be frequently used in the proof of Theorem 1.1. For their proofs, we refer to our recent work [14] (for the case in the Triebel-Lizorkin spaces, see also Chae [12] and Chen et al. [13]). The first one is related to the product estimates.

Proposition 2.5. *Let $1 < p, q, r < \infty$. There exists a constant $C > 0$ such that for $s > 0$*

$$\|u\theta\|_{\dot{F}_{p,q}^{s,r}} \leq C \left(\|u\|_{L^\infty} \|\theta\|_{\dot{F}_{p,q}^{s,r}} + \|\theta\|_{L^\infty} \|u\|_{\dot{F}_{p,q}^{s,r}} \right), \quad (2.16)$$

and for $s > -1$

$$\|u\theta\|_{\dot{F}_{p,q}^{s,r}} \leq C \left(\|u\|_{L^\infty} \|\theta\|_{\dot{F}_{p,q}^{s,r}} + \|\theta\|_{L^\infty} \|u\|_{\dot{F}_{p,q}^{s,r}} \right), \quad (2.17)$$

as long as the right-hand sides of (2.16) and (2.17) are finite.

The next one concerns the commutator estimates.

Proposition 2.6. *Let $s > 0, 1 < p, q, r < \infty$, and let the vector field v be divergence free. There exists a constant $C > 0$ such that*

$$\left\| \left\| 2^{js} ([v, \Delta_j] \cdot \nabla \theta) \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} \leq C \left(\|\nabla v\|_{L^\infty} \|\theta\|_{\dot{F}_{p,q}^{s,r}} + \|\nabla \theta\|_{L^\infty} \|v\|_{\dot{F}_{p,q}^{s,r}} \right), \quad (2.18)$$

$$\left\| \left\| 2^{js} ([v, \Delta_j] \cdot \nabla \theta) \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} \leq C \left(\|\nabla v\|_{L^\infty} \|\theta\|_{\dot{F}_{p,q}^{s,r}} + \|\theta\|_{L^\infty} \|\nabla v\|_{\dot{F}_{p,q}^{s,r}} \right), \quad (2.19)$$

as long as the right-hand sides of (2.18) and (2.19) are finite.

The last one is the logarithmic Triebel-Lizorkin-Lorentz inequality with features similar to the classical logarithmic Sobolev inequality [18] and to the logarithmic Triebel-Lizorkin inequality [12].

Proposition 2.7. *Let $1 < r \leq p < \infty, 1 < q < \infty$, and $s > d/p$. There exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty} \leq C \left(1 + \|u\|_{\dot{F}_{\infty,\infty}^0} \left(\log^+ \|u\|_{\dot{F}_{p,q}^{s,r}} + 1 \right) \right). \quad (2.20)$$

3. Proof of Theorem 1.1

In this section, we prove the local well-posedness of solutions to (1.1) by constructing the successive approximations and using the *a priori* estimate and establish the blow-up criterion of smooth solutions by the logarithmic Triebel-Lizorkin-Lorentz inequality. The proof is divided into five steps.

Step 1 (a priori estimate). For a given fluid velocity $u(x, t)$, we let $X(\alpha, t)$ be the particle-trajectory mapping defined by the solutions of the following ordinary differential equations:

$$\begin{aligned}\frac{d}{dt}X(\alpha, t) &= u(X(\alpha, t), t), \\ X(\alpha, 0) &= \alpha.\end{aligned}\tag{3.1}$$

Then from the Boussinesq equations (1.1) and the ODEs (3.1), we have

$$\begin{aligned}\frac{d}{dt}u(X(\alpha, t), t) &= -\nabla p(X(\alpha, t), t) + \theta(X(\alpha, t), t)e_d, \\ \frac{d}{dt}\theta(X(\alpha, t), t) &= 0, \\ u(X(\alpha, 0), 0) &= u_0(\alpha), \quad \theta(X(\alpha, 0), 0) = \theta_0(\alpha),\end{aligned}\tag{3.2}$$

from which we get

$$\begin{aligned}u(X(\alpha, t), t) &= u_0(\alpha) - \int_0^t \nabla p(X(\alpha\tau), \tau) d\tau + \int_0^t \theta(X(\alpha, \tau), \tau) e_d d\tau, \\ \theta(X(\alpha, t), t) &= \theta_0(\alpha).\end{aligned}\tag{3.3}$$

It follows from $\operatorname{div} u = 0$ that the mapping $\alpha \rightarrow X(\alpha, t)$ is a volume-preserving diffeomorphism for any given $t > 0$. Thus by Lemma 2.1 we have

$$\begin{aligned}\|u(t)\|_{L^{p,r}} + \|\theta(t)\|_{L^{p,r}} &= \|u(X(\alpha, t), t)\|_{L^{p,r}} + \|\theta(X(\alpha, t), t)\|_{L^{p,r}} \\ &\leq \|u_0\|_{L^{p,r}} + \|\theta_0\|_{L^{p,r}} + \int_0^t \|\nabla p(X(\alpha, \tau), \tau)\|_{L^{p,r}} d\tau + \int_0^t \|\theta(X(\alpha, \tau), \tau)\|_{L^{p,r}} d\tau \\ &= \|u_0\|_{L^{p,r}} + \|\theta_0\|_{L^{p,r}} + \int_0^t \|\nabla p(\tau)\|_{L^{p,r}} d\tau + \int_0^t \|\theta(\tau)\|_{L^{p,r}} d\tau.\end{aligned}\tag{3.4}$$

To deal with the pressure term, we take the divergence on both sides of (1.1)₁ and then obtain the representation of the pressure:

$$p = (-\Delta)^{-1} \partial_i u_j \partial_j u_i - (-\Delta)^{-1} \partial_d \theta = (-\Delta)^{-1} \partial_i \partial_j (u_i u_j) - (-\Delta)^{-1} \partial_d \theta,\tag{3.5}$$

where we used the Einstein convention on the summation over repeated indices and the fact that $\operatorname{div} u = 0$. Thus by the boundedness of Riesz transform on $L^{p,r}$, we have

$$\begin{aligned}\|\nabla p(\tau)\|_{L^{p,r}} &\leq \left\| \nabla (-\Delta)^{-1} \partial_i \partial_j (u_i u_j)(\tau) \right\|_{L^{p,r}} + \left\| \nabla (-\Delta)^{-1} \partial_d \theta(\tau) \right\|_{L^{p,r}} \\ &\lesssim \|u \cdot \nabla u(\tau)\|_{L^{p,r}} + \|\theta(\tau)\|_{L^{p,r}} \\ &\leq \|u(\tau)\|_{L^{p,r}} \|\nabla u(\tau)\|_{\infty} + \|\theta(\tau)\|_{L^{p,r}}.\end{aligned}\tag{3.6}$$

Substituting the last inequality into (3.4), we obtain

$$\|u(t)\|_{L^{p,r}} + \|\theta(t)\|_{L^{p,r}} \lesssim \|u_0\|_{L^{p,r}} + \|\theta_0\|_{L^{p,r}} + \int_0^t \|u(\tau)\|_{L^{p,r}} \|\nabla u(\tau)\|_{\infty} d\tau + \int_0^t \|\theta(\tau)\|_{L^{p,r}} d\tau. \quad (3.7)$$

On the other hand, we apply the frequency projection Δ_j to both sides of (1.1) to obtain

$$\begin{aligned} \partial_t \Delta_j u + u \cdot \nabla \Delta_j u &= [u, \Delta_j] \cdot \nabla u - \nabla \Delta_j p + \Delta_j \theta e_d, \\ \partial_t \Delta_j \theta + u \cdot \nabla \Delta_j \theta &= [u, \Delta_j] \cdot \nabla \theta. \end{aligned} \quad (3.8)$$

Then by (3.1), we have

$$\begin{aligned} \frac{d}{dt} \Delta_j u(X(\alpha, t), t) &= [u, \Delta_j] \cdot \nabla u(X(\alpha, t), t) - \nabla \Delta_j p(X(\alpha, t), t) + \Delta_j \theta(X(\alpha, t), t) e_d, \\ \frac{d}{dt} \Delta_j \theta(X(\alpha, t), t) &= [u, \Delta_j] \cdot \nabla \theta(X(\alpha, t), t). \end{aligned} \quad (3.9)$$

Integrating the previous two equations from 0 to t gives

$$\begin{aligned} &\Delta_j u(X(\alpha, t), t) + \Delta_j \theta(X(\alpha, t), t) \\ &= \Delta_j u_0(\alpha) + \Delta_j \theta_0(\alpha) + \int_0^t ([u, \Delta_j] \cdot \nabla u(X(\alpha, \tau), \tau) + [u, \Delta_j] \cdot \nabla \theta(X(\alpha, \tau), \tau)) d\tau \\ &\quad - \int_0^t \nabla \Delta_j p(X(\alpha, \tau), \tau) d\tau + \int_0^t \theta(X(\alpha, \tau), \tau) e_d d\tau. \end{aligned} \quad (3.10)$$

We first multiply both sides by 2^{js} , take the $l^q(\mathbb{Z})$ norm, and then use Minkowski's inequality to obtain

$$\begin{aligned} &\left\| 2^{js} |\Delta_j u(X(\alpha, t), t)| \right\|_{l^q(\mathbb{Z})} + \left\| 2^{js} |\Delta_j \theta(X(\alpha, t), t)| \right\|_{l^q(\mathbb{Z})} \\ &\leq \left\| 2^{js} |\Delta_j u_0(\alpha)| \right\|_{l^q(\mathbb{Z})} + \left\| 2^{js} |\Delta_j \theta_0(\alpha)| \right\|_{l^q(\mathbb{Z})} \\ &\quad + \int_0^t \left(\left\| 2^{js} |[u, \Delta_j] \cdot \nabla u(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} + \left\| 2^{js} |[u, \Delta_j] \cdot \nabla \theta(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} \right) d\tau \\ &\quad + \int_0^t \left\| 2^{js} |\nabla \Delta_j p(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} d\tau + \int_0^t \left\| 2^{js} |\theta(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} d\tau, \end{aligned} \quad (3.11)$$

and then we take the $L^{p,r}$ norm with respect to α and use Minkowski's inequality to get

$$\begin{aligned}
& \left\| \left\| 2^{js} |\Delta_j u(X(\alpha, t), t)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} + \left\| \left\| 2^{js} |\Delta_j \theta(X(\alpha, t), t)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} \\
& \leq \|u_0\|_{\dot{F}_{p,q}^{s,r}} + \|\theta_0\|_{\dot{F}_{p,q}^{s,r}} + \int_0^t \left\| \left\| 2^{js} |[u, \Delta_j] \cdot \nabla u(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau \\
& \quad + \int_0^t \left\| \left\| 2^{js} |[u, \Delta_j] \cdot \nabla \theta(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau \\
& \quad + \int_0^t \left\| \left\| 2^{js} |\nabla \Delta_j p(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau + \int_0^t \left\| \left\| 2^{js} |\theta(X(\alpha, \tau), \tau)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau.
\end{aligned} \tag{3.12}$$

Recalling the fact that $\alpha \rightarrow X(\alpha, t)$ is a volume-preserving diffeomorphism for any given $t > 0$ and using Lemma 2.1, we have

$$\begin{aligned}
\|u(t)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(t)\|_{\dot{F}_{p,q}^{s,r}} & \lesssim \|u_0\|_{\dot{F}_{p,q}^{s,r}} + \|\theta_0\|_{\dot{F}_{p,q}^{s,r}} \\
& \quad + \int_0^t \left\| \left\| 2^{js} |[u, \Delta_j] \cdot \nabla u(\tau)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} + \left\| \left\| 2^{js} |[u, \Delta_j] \cdot \nabla \theta(\tau)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau \\
& \quad + \int_0^t \|\nabla p(\tau)\|_{\dot{F}_{p,q}^{s,r}} d\tau + \int_0^t \|\theta(\tau)\|_{\dot{F}_{p,q}^{s,r}} d\tau.
\end{aligned} \tag{3.13}$$

Thanks to the commutator estimates (2.18), the integrand in the first integral is dominated by

$$\|\nabla u(\tau)\|_{L^\infty} \|u(\tau)\|_{\dot{F}_{p,q}^{s,r}} + \|\nabla u(\tau)\|_{L^\infty} \|\theta(\tau)\|_{\dot{F}_{p,q}^{s,r}} + \|\nabla \theta(\tau)\|_{L^\infty} \|u(\tau)\|_{\dot{F}_{p,q}^{s,r}}, \tag{3.14}$$

and thus

$$\begin{aligned}
\|u(t)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(t)\|_{\dot{F}_{p,q}^{s,r}} & \lesssim \|u_0\|_{\dot{F}_{p,q}^{s,r}} + \|\theta_0\|_{\dot{F}_{p,q}^{s,r}} + \int_0^t \|\nabla p(\tau)\|_{\dot{F}_{p,q}^{s,r}} d\tau \\
& \quad + \int_0^t (1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla \theta(\tau)\|_{L^\infty}) \left(\|u(\tau)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(\tau)\|_{\dot{F}_{p,q}^{s,r}} \right) d\tau.
\end{aligned} \tag{3.15}$$

For the term related to the pressure, we use (3.5) to obtain

$$\partial_k \partial_l p = \partial_k \partial_l (-\Delta)^{-1} \partial_i u_j \partial_j u_i - \partial_k \partial_l (-\Delta)^{-1} \partial_d \theta := R_k R_l (\partial_i u_j \partial_j u_i) - R_k R_l (\partial_d \theta), \tag{3.16}$$

where R_k and R_l are Riesz transform, and then

$$\|\nabla p\|_{\dot{F}_{p,q}^{s,r}} \leq \|\partial_k \partial_l p\|_{\dot{F}_{p,q}^{s-1,r}} \leq \|\partial_i u_j \partial_j u_i\|_{\dot{F}_{p,q}^{s-1,r}} + \|\partial_d \theta\|_{\dot{F}_{p,q}^{s-1,r}} \leq \|u\|_{\dot{F}_{p,q}^{s,r}} \|\nabla u\| + \|\theta\|_{\dot{F}_{p,q}^{s,r}}, \tag{3.17}$$

where we used the boundedness of Riesz transform on $\dot{F}_{p,q}^{s,r}$ and the product estimate (2.16). Substituting the last inequality into (3.15), we have

$$\begin{aligned} & \|u(t)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(t)\|_{\dot{F}_{p,q}^{s,r}} \\ & \lesssim \|u_0\|_{\dot{F}_{p,q}^{s,r}} + \|\theta_0\|_{\dot{F}_{p,q}^{s,r}} + \int_0^t \left(1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla \theta(\tau)\|_{L^\infty}\right) \left(\|u(\tau)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(\tau)\|_{\dot{F}_{p,q}^{s,r}}\right) d\tau. \end{aligned} \quad (3.18)$$

This together with Gronwall's inequality gives the *a priori* estimates in the homogeneous Triebel-Lizorkin-Lorentz spaces:

$$\|u(t)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(t)\|_{\dot{F}_{p,q}^{s,r}} \leq C \left(\|u_0\|_{\dot{F}_{p,q}^{s,r}} + \|\theta_0\|_{\dot{F}_{p,q}^{s,r}} \right) \exp \left(C \int_0^t (1 + \|\nabla v(\tau)\|_{L^\infty} + \|\nabla \theta(\tau)\|_{L^\infty}) d\tau \right). \quad (3.19)$$

Similarly, by (3.7) and (3.18), we get

$$\begin{aligned} & \|u(t)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(t)\|_{\dot{F}_{p,q}^{s,r}} \\ & \lesssim \|u_0\|_{\dot{F}_{p,q}^{s,r}} + \|\theta_0\|_{\dot{F}_{p,q}^{s,r}} + \int_0^t \left(1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla \theta(\tau)\|_{L^\infty}\right) \left(\|u(\tau)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(\tau)\|_{\dot{F}_{p,q}^{s,r}}\right) d\tau, \end{aligned} \quad (3.20)$$

and thus Gronwall's inequality yields the *a priori* estimates in the inhomogeneous spaces:

$$\|u(t)\|_{\dot{F}_{p,q}^{s,r}} + \|\theta(t)\|_{\dot{F}_{p,q}^{s,r}} \leq C \left(\|u_0\|_{\dot{F}_{p,q}^{s,r}} + \|\theta_0\|_{\dot{F}_{p,q}^{s,r}} \right) \exp \left(C \int_0^t (1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla \theta(\tau)\|_{L^\infty}) d\tau \right). \quad (3.21)$$

Step 2 (approximate solutions and uniform estimates). In order to construct the approximate solutions of (1.1), we first set $(u^{(0)}, \theta^{(0)}) = (0, 0)$ and then define $\{(u^{(n)}, \theta^{(n)})\}_{n \in \mathbb{N}}$ as the solutions of the linear equations:

$$\begin{aligned} \partial_t u^{(n+1)} + u^{(n)} \cdot \nabla u^{(n+1)} + \nabla p^{(n+1)} &= \theta^{(n+1)} e_d, \quad \text{in } \mathbb{R}^d \times (0, \infty), \\ \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} &= 0, \quad \text{in } \mathbb{R}^d \times (0, \infty), \\ \operatorname{div} u^{(n)} &= 0, \quad \text{in } \mathbb{R}^d \times (0, \infty), \end{aligned} \quad (3.22)$$

with initial data $u^{(n+1)}(x, 0) = u_0^{(n+1)}(x) := S_{n+2}u_0(x)$, $\theta^{(n+1)}(x, 0) = \theta_0^{(n+1)}(x) := S_{n+2}\theta_0(x)$.

For each $n \in \mathbb{N}$, similar to Step 1, we let the particle-trajectory mapping $X^{(n)}(\alpha, t)$ be the solution of the following ordinary differential equation:

$$\begin{aligned} \frac{d}{dt} X^{(n)}(\alpha, t) &= u^{(n)}(X^{(n)}(\alpha, t), t), \\ X^{(n)}(\alpha, 0) &= \alpha. \end{aligned} \quad (3.23)$$

Then following the same procedure of estimate leading to (3.20), we obtain

$$\begin{aligned}
\|u^{(n+1)}(t)\|_{F_{p,q}^{s,r}} + \|\theta^{(n+1)}(t)\|_{F_{p,q}^{s,r}} &\lesssim \|S_{n+2}u_0\|_{F_{p,q}^{s,r}} + \|S_{n+2}\theta_0\|_{F_{p,q}^{s,r}} \\
&\quad + \int_0^t \left(1 + \|\nabla u^{(n)}(\tau)\|_{L^\infty}\right) \\
&\quad \times \left(\|u^{(n+1)}(\tau)\|_{F_{p,q}^{s,r}} + \|\theta^{(n+1)}(\tau)\|_{F_{p,q}^{s,r}}\right) d\tau d\tau. \\
&\quad + \int_0^t \left(\|\nabla u^{(n+1)}(\tau)\|_{L^\infty} + \|\nabla \theta^{(n+1)}(\tau)\|_{L^\infty}\right) \|u^{(n)}(\tau)\|_{F_{p,q}^{s,r}} d\tau.
\end{aligned} \tag{3.24}$$

Note that

$$\|S_{n+2}u_0\|_{F_{p,q}^{s,r}} \leq \|u_0\|_{F_{p,q}^{s,r}}, \quad \|S_{n+2}\theta_0\|_{F_{p,q}^{s,r}} \leq \|\theta_0\|_{F_{p,q}^{s,r}}. \tag{3.25}$$

This together with Sobolev embedding theorem $F_{p,q}^{s-1,r} \hookrightarrow L^\infty$ for $s-1 > d/p$ yields that

$$\begin{aligned}
\|u^{(n+1)}(t)\|_{F_{p,q}^{s,r}} + \|\theta^{(n+1)}(t)\|_{F_{p,q}^{s,r}} &\lesssim \|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \\
&\quad + \int_0^t \left(1 + \|u^{(n)}(\tau)\|_{F_{p,q}^{s,r}}\right) \left(\|u^{(n+1)}(\tau)\|_{F_{p,q}^{s,r}} + \|\theta^{(n+1)}(\tau)\|_{F_{p,q}^{s,r}}\right) d\tau.
\end{aligned} \tag{3.26}$$

It follows from Gronwall's inequality that

$$\|u^{(n+1)}(t)\|_{F_{p,q}^{s,r}} + \|\theta^{(n+1)}(t)\|_{F_{p,q}^{s,r}} \leq C \left(\|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \right) \exp \left(C \int_0^t \left(1 + \|u^{(n)}(\tau)\|_{F_{p,q}^{s,r}}\right) d\tau \right) \tag{3.27}$$

for some $C > 0$ independent of n . Choosing $T_0 := T_0(\|u_0\|_{F_{p,q}^{s,r}}, \|\theta_0\|_{F_{p,q}^{s,r}}) > 0$ such that

$$C \left(1 + 2C\|u_0\|_{F_{p,q}^{s,r}} + 2C\|\theta_0\|_{F_{p,q}^{s,r}}\right) T_0 \leq \log 2, \tag{3.28}$$

and using the standard induction arguments, we obtain for any $n \in \mathbb{N}_0$

$$\sup_{0 \leq t \leq T_0} \left(\|u^{(n+1)}(t)\|_{F_{p,q}^{s,r}} + \|\theta^{(n+1)}(t)\|_{F_{p,q}^{s,r}} \right) \leq 2C \left(\|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \right). \tag{3.29}$$

Step 3 (existence). To prove the local existence, we show that there exists $T_1 \in (0, T_0]$ independent of n such that $(u^{(n)}, \theta^{(n)})$ is a Cauchy sequence in $(C([0, T_1]; F_{p,q}^{s-1,r}(\mathbb{R}^d)))^{d+1}$. For this purpose, we denote

$$\delta u^{(n+1)} := u^{(n+1)} - u^{(n)}, \quad \delta \theta^{(n+1)} := \theta^{(n+1)} - \theta^{(n)}, \quad \delta p^{(n+1)} := p^{(n+1)} - p^{(n)}. \quad (3.30)$$

It follows that $(\delta u^{(n+1)}, \delta \theta^{(n+1)}, \delta p^{(n+1)})$ satisfies the following:

$$\begin{aligned} \partial_t \delta u^{(n+1)} + u^{(n)} \cdot \nabla \delta u^{(n+1)} + \nabla \delta p^{(n+1)} &= -\delta u^{(n)} \cdot \nabla u^{(n)} + \delta \theta^{(n+1)} e_d, \quad \text{in } \mathbb{R}^d \times (0, \infty), \\ \partial_t \delta \theta^{(n+1)} + u^{(n)} \cdot \nabla \delta \theta^{(n+1)} &= -\delta u^{(n)} \cdot \nabla \theta^{(n)}, \quad \text{in } \mathbb{R}^d \times (0, \infty), \\ \operatorname{div} u^{(n)} &= 0, \quad \text{in } \mathbb{R}^d \times (0, \infty), \\ \delta u^{(n+1)}(x, 0) &= \Delta_{n+1} u_0(x), \quad \delta \theta^{(n+1)}(x, 0) = \Delta_{n+1} \theta_0(x), \quad \text{in } \mathbb{R}^d. \end{aligned} \quad (3.31)$$

Since $X^{(n)}(\alpha, t)$ is a solution of the ordinary differential equations (3.23), we have

$$\begin{aligned} \frac{d}{dt} \delta u^{(n+1)}(X^{(n)}(\alpha, t), t) &= -\delta u^{(n)} \cdot \nabla u^{(n)}(X^{(n)}(\alpha, t), t) - \nabla \delta p^{(n+1)}(X^{(n)}(\alpha, t), t) \\ &\quad + \delta \theta^{(n+1)}(X^{(n)}(\alpha, t), t) e_d, \\ \frac{d}{dt} \delta \theta^{(n+1)}(X^{(n)}(\alpha, t), t) &= -\delta u^{(n)} \cdot \nabla \theta^{(n)}(X^{(n)}(\alpha, t), t). \end{aligned} \quad (3.32)$$

Integrating the previous equations on $[0, t]$, we have

$$\begin{aligned} &\delta u^{(n+1)}(X^{(n)}(\alpha, t), t) + \delta \theta^{(n+1)}(X^{(n)}(\alpha, t), t) \\ &= \Delta_{n+1} u_0(\alpha) + \Delta_{n+1} \theta_0(\alpha) - \int_0^t \delta u^{(n)} \cdot \nabla (u^{(n)} + \theta^{(n)})(X^{(n)}(\alpha, \tau), \tau) d\tau \\ &\quad - \int_0^t \nabla \delta p^{(n+1)}(X^{(n)}(\alpha, \tau), \tau) d\tau + \int_0^t \delta \theta^{(n+1)}(X^{(n)}(\alpha, \tau), \tau) e_d d\tau. \end{aligned} \quad (3.33)$$

Similar to Step 1, we take the $L^{p,r}$ norm on both sides, use Minkowski's inequality and Hölder's inequality in Lorentz spaces, and then obtain

$$\begin{aligned} &\|\delta u^{(n+1)}(t)\|_{L^{p,r}} + \|\delta \theta^{(n+1)}(t)\|_{L^{p,r}} \\ &\lesssim \|\Delta_{n+1} u_0\|_{L^{p,r}} + \|\Delta_{n+1} \theta_0\|_{L^{p,r}} + \int_0^t \|\delta u^{(n)} \cdot \nabla (u^{(n)} + \theta^{(n)})(\tau)\|_{L^{p,r}} d\tau \\ &\quad + \int_0^t \|\nabla \delta p^{(n+1)}\|_{L^{p,r}} d\tau + \int_0^t \|\delta \theta^{(n+1)}(\tau)\|_{L^{p,r}} d\tau \end{aligned}$$

$$\begin{aligned}
&\lesssim \|\Delta_{n+1}u_0\|_{L^{p,r}} + \|\Delta_{n+1}\theta_0\|_{L^{p,r}} + \int_0^t \|\delta u^{(n)}\|_{L^{p,r}} \left(\|\nabla u^{(n)}(\tau)\|_{L^\infty} + \|\nabla \theta^{(n)}(\tau)\|_{L^\infty} \right) d\tau \\
&\quad + \int_0^t \left(1 + \|\nabla u^{(n)}(\tau)\|_{L^\infty} \right) \left(\|\delta u^{(n+1)}(\tau)\|_{L^{p,r}} + \|\delta \theta^{(n+1)}(\tau)\|_{L^{p,r}} \right) d\tau \\
&\lesssim \|\Delta_{n+1}u_0\|_{L^{p,r}} + \|\Delta_{n+1}\theta_0\|_{L^{p,r}} + \int_0^t \|\delta u^{(n)}\|_{L^{p,r}} \left(\|u^{(n)}(\tau)\|_{F_{p,q}^{s,r}} + \|\theta^{(n)}(\tau)\|_{F_{p,q}^{s,r}} \right) d\tau \\
&\quad + \int_0^t \left(1 + \|u^{(n)}(\tau)\|_{F_{p,q}^{s,r}} \right) \left(\|\delta u^{(n+1)}(\tau)\|_{L^{p,r}} + \|\delta \theta^{(n+1)}(\tau)\|_{L^{p,r}} \right) d\tau,
\end{aligned} \tag{3.34}$$

where we used $F_{p,q}^{s-1,r} \hookrightarrow L^\infty$ for $s-1 > d/p$ in the last inequality.

Next, we apply Δ_j to the first equation of (3.31) to obtain

$$\begin{aligned}
\partial_t \Delta_j \delta u^{(n+1)} + u^{(n)} \cdot \nabla \Delta_j \delta u^{(n+1)} &= [u^{(n)}, \Delta_j] \cdot \nabla \delta u^{(n+1)} - \Delta_j (\delta u^{(n)} \cdot \nabla u^{(n)}) \\
&\quad - \nabla \Delta_j \delta p^{(n+1)} + \Delta_j \delta \theta^{(n+1)} e_d, \\
\partial_t \Delta_j \delta \theta^{(n+1)} + u^{(n)} \cdot \nabla \Delta_j \delta \theta^{(n+1)} &= [u^{(n)}, \Delta_j] \cdot \nabla \delta \theta^{(n+1)} - \Delta_j (\delta u^{(n)} \cdot \nabla \theta^{(n)}).
\end{aligned} \tag{3.35}$$

It follows from the definition of $X^{(n)}(\alpha, t)$ that

$$\begin{aligned}
\frac{d}{dt} \Delta_j \delta u^{(n+1)}(X^{(n)}(\alpha, t), t) &= [u^{(n)}, \Delta_j] \cdot \nabla \delta u^{(n+1)}(X^{(n)}(\alpha, t), t) \\
&\quad - \Delta_j (\delta u^{(n)} \cdot \nabla u^{(n)})(X^{(n)}(\alpha, t), t) \\
&\quad - \nabla \Delta_j \delta p^{(n+1)}(X^{(n)}(\alpha, t), t) + \Delta_j \delta \theta^{(n+1)}(X^{(n)}(\alpha, t), t) e_d, \\
\frac{d}{dt} \Delta_j \delta \theta^{(n+1)}(X^{(n)}(\alpha, t), t) &= [u^{(n)}, \Delta_j] \cdot \nabla \delta \theta^{(n+1)}(X^{(n)}(\alpha, t), t) \\
&\quad - \Delta_j (\delta u^{(n)} \cdot \nabla \theta^{(n)})(X^{(n)}(\alpha, t), t).
\end{aligned} \tag{3.36}$$

Integrating the previous equations on $[0, t]$ yields

$$\begin{aligned}
&\Delta_j \delta u^{(n+1)}(X^{(n)}(\alpha, t), t) + \Delta_j \delta \theta^{(n+1)}(X^{(n)}(\alpha, t), t) \\
&= \Delta_j \Delta_{n+1} u_0(\alpha) + \Delta_j \Delta_{n+1} \theta_0(\alpha) + \int_0^t [u^{(n)}, \Delta_j] \cdot \nabla \delta u^{(n+1)}(X^{(n)}(\alpha, \tau), \tau) d\tau \\
&\quad + \int_0^t [u^{(n)}, \Delta_j] \cdot \nabla \delta \theta^{(n+1)}(X^{(n)}(\alpha, \tau), \tau) d\tau - \int_0^t \Delta_j (\delta u^{(n)} \cdot \nabla u^{(n)})(X^{(n)}(\alpha, \tau), \tau) d\tau \\
&\quad - \int_0^t \Delta_j (\delta u^{(n)} \cdot \nabla \theta^{(n)})(X^{(n)}(\alpha, \tau), \tau) d\tau - \int_0^t \nabla \Delta_j \delta p^{(n+1)}(X^{(n)}(\alpha, \tau), \tau) d\tau \\
&\quad + \int_0^t \Delta_j \delta \theta^{(n+1)}(X^{(n)}(\alpha, \tau), \tau) e_d d\tau.
\end{aligned} \tag{3.37}$$

Then following the similar procedure of estimate leading to (3.20), we get

$$\begin{aligned}
& \left\| \delta u^{(n+1)}(t) \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(t) \right\|_{\dot{F}_{p,q}^{s-1,r}} \\
& \lesssim \left\| \Delta_{n+1} u_0 \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \Delta_{n+1} \theta_0 \right\|_{\dot{F}_{p,q}^{s-1,r}} + \int_0^t \left\| \left\| 2^{j(s-1)} \left[u^{(n)}, \Delta_j \right] \cdot \nabla \delta u^{(n+1)}(\tau) \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau \\
& \quad + \int_0^t \left\| \left\| 2^{j(s-1)} \left[u^{(n)}, \Delta_j \right] \cdot \nabla \delta \theta^{(n+1)}(\tau) \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau + \int_0^t \left\| \delta u^{(n)} \cdot \nabla u^{(n)} \right\|_{\dot{F}_{p,q}^{s-1,r}} d\tau \\
& \quad + \int_0^t \left\| \delta u^{(n)} \cdot \nabla \theta^{(n)} \right\|_{\dot{F}_{p,q}^{s-1,r}} d\tau + \int_0^t \left\| \nabla \delta p^{(n+1)} \right\|_{\dot{F}_{p,q}^{s-1,r}} d\tau + \int_0^t \left\| \delta \theta^{(n+1)} \right\|_{\dot{F}_{p,q}^{s-1,r}} d\tau \\
& \lesssim \left\| \Delta_{n+1} u_0 \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \Delta_{n+1} \theta_0 \right\|_{\dot{F}_{p,q}^{s-1,r}} \\
& \quad + \int_0^t \left(1 + \left\| \nabla u^{(n)}(\tau) \right\|_{\infty} \right) \left(\left\| \delta u^{(n+1)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} \right) d\tau \\
& \quad + \int_0^t \left(\left\| \delta u^{(n+1)}(\tau) \right\|_{\infty} + \left\| \delta \theta^{(n+1)}(\tau) \right\|_{\infty} \right) \left\| \nabla u^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} d\tau \\
& \quad + \int_0^t \left\| \delta u^{(n)}(\tau) \right\|_{\infty} \left(\left\| \nabla u^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \nabla \theta^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} \right) d\tau \\
& \quad + \int_0^t \left\| \delta u^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} \left(\left\| \nabla u^{(n)}(\tau) \right\|_{\infty} + \left\| \nabla \theta^{(n)}(\tau) \right\|_{\infty} \right) d\tau,
\end{aligned} \tag{3.38}$$

where we used the commutator estimates (2.19) and the product estimates (2.16) in last inequality. This together with (3.34), Sobolev embedding theorem, and the boundedness of Riesz transform gives that

$$\begin{aligned}
& \left\| \delta u^{(n+1)}(t) \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(t) \right\|_{\dot{F}_{p,q}^{s-1,r}} \\
& \lesssim \left\| \Delta_{n+1} u_0 \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \Delta_{n+1} \theta_0 \right\|_{\dot{F}_{p,q}^{s-1,r}} \\
& \quad + \int_0^t \left(1 + \left\| u^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s,r}} \right) \left(\left\| \delta u^{(n+1)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} \right) d\tau \\
& \quad + \int_0^t \left\| \delta u^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s-1,r}} \left(\left\| u^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s,r}} + \left\| \theta^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s,r}} \right) d\tau \\
& \lesssim 2^{-(n+1)} \left(\left\| u_0 \right\|_{\dot{F}_{p,q}^{s,r}} + \left\| \theta_0 \right\|_{\dot{F}_{p,q}^{s,r}} \right) \\
& \quad + T_1 \left(1 + \sup_{t \in [0, T_1]} \left\| u^{(n)}(t) \right\|_{\dot{F}_{p,q}^{s,r}} \right) \sup_{t \in [0, T_1]} \left(\left\| \delta u^{(n+1)}(t) \right\|_{\dot{F}_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(t) \right\|_{\dot{F}_{p,q}^{s-1,r}} \right) \\
& \quad + T_1 \sup_{t \in [0, T_1]} \left(\left\| u^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s,r}} + \left\| \theta^{(n)}(\tau) \right\|_{\dot{F}_{p,q}^{s,r}} \right) \sup_{t \in [0, T_1]} \left\| \delta u^{(n)}(t) \right\|_{\dot{F}_{p,q}^{s-1,r}}.
\end{aligned} \tag{3.39}$$

It follows from the uniform bounds (3.29) that

$$\begin{aligned}
& \sup_{t \in [0, T_1]} \left(\left\| \delta u^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} \right) \\
& \leq C 2^{-(n+1)} \left(\|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \right) + C T_1 \sup_{t \in [0, T_1]} \left(\left\| \delta u^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} \right) \\
& \quad + C T_1 \sup_{t \in [0, T_1]} \left(\left\| \delta u^{(n)}(t) \right\|_{F_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n)}(t) \right\|_{F_{p,q}^{s-1,r}} \right).
\end{aligned} \tag{3.40}$$

If we choose T_1 small enough such that $C T_1 \leq 1/2$, then

$$\begin{aligned}
& \sup_{t \in [0, T_1]} \left(\left\| \delta u^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} \right) \\
& \leq C 2^{-(n+1)} \left(\|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \right) + 2 C T_1 \sup_{t \in [0, T_1]} \left(\left\| \delta u^{(n)}(t) \right\|_{F_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n)}(t) \right\|_{F_{p,q}^{s-1,r}} \right).
\end{aligned} \tag{3.41}$$

By the standard induction arguments, we obtain

$$\sup_{t \in [0, T_1]} \left(\left\| \delta u^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} + \left\| \delta \theta^{(n+1)}(t) \right\|_{F_{p,q}^{s-1,r}} \right) \leq 2 C 2^{-n} \left(\|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \right), \tag{3.42}$$

which implies that $(u^{(n)}(x, t), \theta^{(n)}(x, t))$ is a Cauchy sequence in $(C([0, T_1]; F_{p,q}^{s-1,r}))^{d+1}$ with $T_1 \leq \min\{T_0, 1/2C\}$. Thus $u^{(n)}(x, t) \rightarrow u(x, t)$ and $\theta^{(n)}(x, t) \rightarrow \theta(x, t)$ in $(C([0, T_1]; F_{p,q}^{s,r}))^{d+1}$ as $n \rightarrow \infty$. From (3.22) and the uniform estimates (3.29), we have actually $(u(x, t), \theta(x, t)) \in (C([0, T_1]; F_{p,q}^{s,r}))^{d+1}$ solving the Boussinesq equation (1.1).

Step 4 (uniqueness). To get the uniqueness we closely follow the arguments in last step. Assume that (u, θ) and $(\tilde{u}, \tilde{\theta})$ are two solutions of the Boussinesq equations (1.1) with the same initial data (u_0, θ_0) in the class $L^\infty([0, T_1]; F_{p,q}^{s,r}(\mathbb{R}^d))$. Denote

$$\delta u := \tilde{u} - u, \quad \delta \theta := \tilde{\theta} - \theta, \quad \delta p := \tilde{p} - p. \tag{3.43}$$

Then it follows that $\delta \theta$ satisfies the following:

$$\begin{aligned}
& \partial_t \delta u + \tilde{u} \cdot \nabla \delta u + \nabla \delta p = -\delta u \cdot \nabla u + \delta \theta e_d, \quad \text{in } \mathbb{R}^d \times (0, T_1), \\
& \partial_t \delta \theta + \tilde{u} \cdot \nabla \delta \theta = -\delta u \cdot \nabla \theta, \quad \text{in } \mathbb{R}^d \times (0, T_1), \\
& \operatorname{div} \tilde{u} = 0, \quad \text{in } \mathbb{R}^d \times (0, T_1), \\
& \delta u(x, 0) = 0, \quad \delta \theta(x, 0) = 0, \quad \text{in } \mathbb{R}^d.
\end{aligned} \tag{3.44}$$

We follow the strategy used to derive (3.40) to obtain

$$\sup_{t \in [0, T]} \left(\|\delta u(t)\|_{F_{p,q}^{s-1,r}} + \|\delta \theta(t)\|_{F_{p,q}^{s-1,r}} \right) \leq CT \sup_{t \in [0, T]} \left(\|\delta u(t)\|_{F_{p,q}^{s-1,r}} + \|\delta \theta(t)\|_{F_{p,q}^{s-1,r}} \right) \quad (3.45)$$

for any $T \leq T_1$. Whenever T is small enough such that $CT < 1$, we have $\delta u(x, t) \equiv \delta \theta(x, t) \equiv 0$ for any $t \leq T$, that is, $\tilde{u}(x, t) \equiv u(x, t)$ and $\tilde{\theta}(x, t) \equiv \theta(x, t)$. By using the standard continuity argument, we see $\tilde{u}(x, t) \equiv u(x, t)$ and $\tilde{\theta}(x, t) \equiv \theta(x, t)$ for any $t \leq T_1$.

Step 5 (blow-up criterion). By the *a priori* estimate (3.21), we only need to dominate $\|\nabla u\|_\infty$ and $\|\nabla \theta\|_\infty$. Indeed, it follows from the logarithmic Triebel-Lizorkin-Lorentz inequality (2.20) with $s - 1 > d/p$ that

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\lesssim 1 + \|\nabla u\|_{\dot{F}_{\infty,\infty}^0} \left(\log^+ \|\nabla u\|_{F_{p,q}^{s-1,r}} + 1 \right) \lesssim 1 + \|u\|_{\dot{F}_{\infty,\infty}^1} \left(\log^+ \|u\|_{F_{p,q}^{s,r}} + 1 \right), \\ \|\nabla \theta\|_{L^\infty} &\lesssim 1 + \|\nabla \theta\|_{\dot{F}_{\infty,\infty}^0} \left(\log^+ \|\nabla \theta\|_{F_{p,q}^{s-1,r}} + 1 \right) \lesssim 1 + \|\theta\|_{\dot{F}_{\infty,\infty}^1} \left(\log^+ \|\theta\|_{F_{p,q}^{s,r}} + 1 \right). \end{aligned} \quad (3.46)$$

Thus, the *a priori* estimate (3.21) gives that

$$\begin{aligned} \|u(t)\|_{F_{p,q}^{s,r}} + \|\theta(t)\|_{F_{p,q}^{s,r}} &\leq C \left(\|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \right) \exp \left(C \int_0^t \left(\|u(\tau)\|_{\dot{F}_{\infty,\infty}^1} + \|\theta(\tau)\|_{\dot{F}_{\infty,\infty}^1} \right) \right. \\ &\quad \left. \times \left(\log^+ (\|u(\tau)\|_{F_{p,q}^{s,r}} + \|\theta(\tau)\|_{F_{p,q}^{s,r}}) + 1 \right) d\tau \right). \end{aligned} \quad (3.47)$$

By using Gronwall's inequality, we have

$$\|u(t)\|_{F_{p,q}^{s,r}} + \|\theta(t)\|_{F_{p,q}^{s,r}} \leq C \left(\|u_0\|_{F_{p,q}^{s,r}} + \|\theta_0\|_{F_{p,q}^{s,r}} \right) \exp \left(C \exp \left(C \int_0^t \left(\|u(\tau)\|_{\dot{F}_{\infty,\infty}^1} + \|\theta(\tau)\|_{\dot{F}_{\infty,\infty}^1} \right) d\tau \right) \right). \quad (3.48)$$

Thus if $\limsup_{t \rightarrow T^*} (\|u(t)\|_{F_{p,q}^{s,r}} + \|\theta(t)\|_{F_{p,q}^{s,r}}) = \infty$, then $\int_0^{T^*} (\|u(t)\|_{\dot{F}_{\infty,\infty}^1} + \|\theta(t)\|_{\dot{F}_{\infty,\infty}^1}) dt = \infty$.

On the other hand, it follows from Sobolev embedding $F_{p,q}^{s,r} \hookrightarrow W^{1,\infty} \hookrightarrow \dot{F}_{\infty,\infty}^1$ for $s - 1 > d/p$ that

$$\begin{aligned} \int_0^{T^*} \left(\|u(t)\|_{\dot{F}_{\infty,\infty}^1} + \|\theta(t)\|_{\dot{F}_{\infty,\infty}^1} \right) dt &\leq T^* \sup_{0 \leq \tau \leq T^*} \left(\|u(\tau)\|_{\dot{F}_{\infty,\infty}^1} + \|\theta(\tau)\|_{\dot{F}_{\infty,\infty}^1} \right) \\ &\lesssim T^* \sup_{0 \leq \tau \leq T^*} (\|\nabla u(\tau)\|_{L^\infty} + \|\nabla \theta(\tau)\|_{L^\infty}) \\ &\lesssim T^* \sup_{0 \leq \tau \leq T^*} \left(\|u(\tau)\|_{F_{p,q}^{s,r}} + \|\theta(\tau)\|_{F_{p,q}^{s,r}} \right). \end{aligned} \quad (3.49)$$

Thus $\int_0^{T^*} (\|u(t)\|_{\dot{F}_{\infty,\infty}^1} + \|\theta(t)\|_{\dot{F}_{\infty,\infty}^1}) dt = \infty$ implies $\limsup_{t \rightarrow T^*} (\|u(t)\|_{F_{p,q}^{s,r}} + \|\theta(t)\|_{F_{p,q}^{s,r}}) = \infty$.

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