Research Article

Normality Criteria of Meromorphic Functions That Share a Holomorphic Function

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Let \( F \) be a family of meromorphic functions defined in \( D \), let \( \psi \equiv 0 \), \( a_0, a_1, ..., a_{k-1} \) be holomorphic functions in \( D \), and let \( k \) be a positive integer. Suppose that, for every function \( f \in F \), \( f \neq 0, P(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1f' + a_0f \neq 0 \) and, for every pair functions \((f,g) \in F\), \( P(f), P(g) \) share \( \psi \), then \( F \) is normal in \( D \).

1. Introduction and Main Results

Let \( \mathbb{C} \) be complex plane. Let \( D \) be a domain in \( \mathbb{C} \). Let \( F \) be a family of meromorphic functions defined in the domain \( D \). \( F \) is said to be normal in \( D \), in the sense of Montel, if for any sequence \( \{f_n\} \subset F \), there exists a subsequence \( \{f_{n_j}\} \) such that \( f_{n_j} \) converges spherically locally uniformly in \( D \), to a meromorphic function or \( \infty \).

Let \( f(z) \) and \( g(z) \) be two meromorphic functions, let \( a \) be a finite complex number. If \( f(z) - a \) and \( g(z) - a \) have the same zeros, then we say they share \( a \) or share \( a \) IM (ignoring multiplicity) (see [1–3]).

Definition 1.1. Let \( a_i(z), (i = 1,2,\ldots, q-1) \), \( b_j(z), (j = 1,2,\ldots, n) \) be analytic in \( D \), let \( n_0, n_1, \ldots, n_k \) be nonnegative integers, set

\[
P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \cdots + a_1(z)\omega,
\]

\[
M(f,f',\ldots,f^{(k)}) = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k},
\]

\[
\gamma_M = n_0 + n_1 + \cdots + n_k,
\]

\[
\Gamma_M = n_0 + 2n_1 + \cdots + (k+1)n_k,
\]

(1.1)
where $M(f, f', \ldots, f^{(k)})$ is called a differential monomial of $f$, $\gamma_M$ the degree of $M(f, f', \ldots, f^{(k)})$, and $\Gamma_M$ the weight of $M(f, f', \ldots, f^{(k)})$.

From Definition 1.1, we give Definition 1.2.

**Definition 1.2.** Let $M_j(f, f', \ldots, f^{(k)})$, $(j = 1, 2, \ldots, n)$ be differential monomials of $f$. Set

\[
H(f, f', \ldots, f^{(k)}) = b_1(z)M_1(f, f', \ldots, f^{(k)}) + \cdots + b_n(z)M_n(f, f', \ldots, f^{(k)}),
\]

\[
\gamma_H = \max\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_n}\},
\]

\[
\Gamma_H = \max\{\Gamma_{M_1}, \Gamma_{M_2}, \ldots, \Gamma_{M_n}\},
\]

where $H(f, f', \ldots, f^{(k)})$ is called the differential polynomial of $f$, $\gamma_H$ the degree of $H(f, f', \ldots, f^{(k)})$, and $\Gamma_H$ the weight of $H(f, f', \ldots, f^{(k)})$.

\[
\frac{\Gamma}{\gamma_H} = \max\left\{\frac{\Gamma_{M_1}}{\gamma_{M_1}}, \frac{\Gamma_{M_2}}{\gamma_{M_2}}, \ldots, \frac{\Gamma_{M_n}}{\gamma_{M_n}}\right\},
\]

\[
G(f) = P(f^{(k)}) + H(f, f', \ldots, f^{(k)}).
\]

In 1979, Gu [4] proved the following result.

**Theorem A.** Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, let $k$ be a positive integer, and let $a$ be a nonzero constant. If, for each function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq a$ in $D$, then $\mathcal{F}$ is normal in $D$.

Yang [5] and Schwick [6] proved that Theorem A still holds if $a$ is replaced by a holomorphic function $\varphi(\neq 0)$ in Theorem A.

Xu [7] improved Theorem A by the ideas of shared values and obtained the following result.

**Theorem B.** Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, let $\varphi(\neq 0)$ be a holomorphic functions and with only simple zeros in $D$, and let $k$ be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f$ has all multiple poles and $f \neq 0$. If, for every pair of functions $f$ and $g$, $f^{(k)}$ and $g^{(k)}$ share $\varphi$ in $D$, then $\mathcal{F}$ is normal in $D$.

Recently, Xu [7] did not know whether the condition $\varphi$ has only simple zero in $D$ and $f$ has all multiple poles are necessary or not in Theorem B.

In 2007, Fang and Chang considered the case $a = 0$ in Theorem A. In this note, Fang and Chang [8] proved the following result.

**Theorem C.** Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, and let $b$ be a nonzero complex number. If, for each $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least $(k + 2)/k$, then $\mathcal{F}$ is normal in $D$.

**Remark 1.3.** The number $(k + 2)/k$ is sharp, as is shown by the examples in [8].

In 2009, Xia and Xu [9] replaced the constant $1$ by a function $\varphi(z) \neq 0$ in Theorem C. They obtained the following result.
Theorem D. Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, let $\psi(\neq 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in $D$, and let $k$ be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1f' + a_0f \neq 0$ and all zeros of $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1f' + a_0f - \psi(z)$ have multiplicity at least $(k + 2)/k$. If, for $k = 1$, $\psi$ has only zeros with multiplicities at most 2 and, for $k \geq 2$, $\psi$ has only simple zeros, then $\mathcal{F}$ is normal in $D$.

It is natural to ask whether Theorem D can be improved by the ideas of shared values.

In this paper, we investigate the problem and obtain the following results.

Theorem 1.4. Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, let $\psi(\neq 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in $D$, and let $k$ be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $P(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1f' + a_0f \neq 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $P(f)$ and $P(g)$ share $\psi$, then $\mathcal{F}$ is normal in $D$.

By Theorem 1.4, we immediately deduce.

Corollary 1.5. Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, let $\psi(\neq 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in $D$, and let $k$ be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and for every pair functions $(f, g) \in \mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share $\psi$, then $\mathcal{F}$ is normal in $D$.

Remark 1.6. By the ideas of sharing values, Theorem 1.4 and Corollary 1.5 yield the number $(k + 2)/k$ can be omitted.

Remark 1.7. Obviously, Corollary 1.5 omitted the conditions $\psi$ with only simple zeros, and, for every function $f \in \mathcal{F}$, $f$ has all multiple poles in Theorem D. But the condition for every function $f \in \mathcal{F}$, $f^{(k)} \neq 0$ is additional. Hence, Corollary 1.5 improves Theorem B in some sense.

The condition $\psi \neq 0$ in Theorem 1.4 is necessary. For example, we consider the following families.

Example 1.8. $\mathcal{F} = \{f_m(z) = e^{mz}, m = 1, 2, \ldots\}$, obviously, any $f \in F$ satisfies $f \neq 0$, $f^{(k)} \neq 0$. For distinct positive integers $m, l$, $f^{(k)}_m$, and $f^{(k)}_l$ share 0 IM. However, the families $\mathcal{F}$ are not normal at $z = 0$.

Remark 1.9. Some ideas of this paper are based on [7, 9, 10].

2. Preliminary Lemmas

In order to prove our theorems, we need the following lemmas.

The well-known Zalcman’s lemma is a very important tool in the study of normal families. It has also undergone various extensions and improvements. The following is one up-to-date local version, which is due to Pang and Zalcman [11].

Lemma 2.1 (see [11, 12]). Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc $\Delta$ with the property that, for each $f \in \mathcal{F}$, all zeros are of multiplicity at least $k$. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f = 0$. If $\mathcal{F}$ is not normal in $\Delta$, then, for $0 \leq \alpha \leq k$, there exist

1. a number $r \in (0, 1)$;
2. a sequence of complex numbers $z_n, |z_n| < r$;
(3) a sequence of functions \( f_n \in \mathcal{T} \);
(4) a sequence of positive numbers \( \rho_n \to 0^+ \);

such that \( g_n(\xi) = \rho_n^{-a} f_n(z_n + \rho_n \xi) \) converge locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function \( g(\xi) \) on \( \mathbb{C} \), and, moreover, the zeros of \( g(\xi) \) are of multiplicity at least \( k \), \( g(\xi) \leq g(0) = kA + 1 \). In particular, \( g \) has order at most 2.

Here, as usual, \( g(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2) \) is the spherical derivative.

**Lemma 2.2** (see [1]). Let \( f(z) \) be a transcendental meromorphic function in \( \mathbb{C} \), let \( k(\geq 1) \) be an integer, and let \( b \) be a non-zero finite value, then \( f \) or \( f^{(k)} - b \) has infinite zeros.

**Lemma 2.3** (see [7]). Let \( f(z) \) be a nonconstant rational function. Let \( k \geq 1 \) be an integer, and let \( b \) be a non-zero finite value. If \( f \neq 0 \), then \( f^{(k)}(z) - b \) has at least two distinct zeros in the plane.

**Lemma 2.4.** Let \( f(z) \) be a nonconstant rational function. Let \( k \geq 1 \) be an integer, and let \( l \) be a positive integer. If \( f \neq 0 \), \( f^{(k)}(z) \neq 0 \), then \( f^{(k)}(z) - z^l \) has at least two distinct zeros in the plane.

**Proof.** Since \( f \neq 0 \) and \( f^{(k)} \neq 0 \), then \( f \) is a nonpolynomial rational function and has the form

\[
f(z) = \frac{A}{(z-z_1)^{m_1}(z-z_2)^{m_2} \cdots (z-z_t)^{m_t}},
\]

where \( A \neq 0 \) is a constant, and \( m_1, m_2, \ldots, m_t \) are positive integers. Set \( m = m_1 + m_2 + \cdots + m_t \). Then,

\[
f'(z) = \frac{-A(mz^{l-1} + b_{l-2}z^{l-2} + \cdots + b_0)}{(z-z_1)^{m_1+1}(z-z_2)^{m_2+1} \cdots (z-z_t)^{m_t+1}},
\]

where \( b_{l-2}, \ldots, b_0 \) are constants. For \( k \geq 2 \), by mathematical induction, we have

\[
f^{(k)}(z) = \frac{Bz^{kt-k} + c_{kt-k-1}z^{kt-k-1} + \cdots + c_0}{(z-z_1)^{m_1+k}(z-z_2)^{m_2+k} \cdots (z-z_t)^{m_t+k}},
\]

where \( B = (-1)^k m(m+1)(m+2) \cdots (m+k-1)A \neq 0 \), \( c_{kt-k-1}, \ldots, c_0 \) are constants. Since \( f^{(k)} \neq 0 \), we deduce that \( t = 1 \), and thus

\[
f(z) = \frac{A}{(z-z_1)^{m_1}},
\]

\[
f^{(k)}(z) = \frac{B}{(z-z_1)^{m_1+k}}.
\]

**Case 1** (if \( f^{(k)} - z^l \) has exactly one zero \( z_0 \)). From (2.5), we set

\[
f^{(k)}(z) - z^l = \frac{B}{(z-z_1)^{m_1+k}} - z^l = \frac{B'(z-z_0)^{m_1+k+l}}{(z-z_1)^{m_1+k}}.
\]

Obviously, \( B' \) is a nonzero constant and \( l \geq 1 \).
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From (2.6), we obtain

\[ f^{(k+1)}(z) = \frac{(z-z_0)^{m_1+k-1}P_i(z)}{(z-z_1)^{m_1+k+1}}, \quad (2.7) \]

where \( P_i(z) \neq 0 \). By (2.4), we deduce

\[ f^{(k+1)}(z) = \frac{A'}{(z-z_1)^{m_1+k+1}}, \quad (2.8) \]

where \( A' \) is nonzero constant.

Comparing (2.7) and (2.8), we obtain that \( \deg A' = 0 \geq m_1 + k - 1 \) is impossible.

Case 2 (if \( f^{(k)}(z) - z_i \neq 0 \)). By (2.5), clearly Case 2 is impossible.

Lemma 2.4 is proved.

**Lemma 2.5** (see [7]). Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \), let \( k \) be a positive integer, and let \( \varphi(\neq 0) \) be a holomorphic function in \( D \). If, for any \( f \in \mathcal{F} \) satisfying \( f \neq 0 \) and if \( f^{(k)}, g^{(k)} \) share \( \varphi \) IM for every pair of functions \( f, g \in \mathcal{F} \), then \( \mathcal{F} \) is normal in \( D \).

In this paper, by the same method of [7], we consider the differential polynomial in Lemma 2.5 and prove a more general result.

**Lemma 2.6.** Let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \), let \( k \) be a positive integer, and let \( \varphi(\neq 0) \) be a holomorphic function in \( D \). If, for any \( f \in \mathcal{F} \) satisfying \( f \neq 0 \) and if \( G(f), G(g) \) share \( \varphi \) IM for every pair of functions \( f, g \in \mathcal{F} \), where \( G(f) \) is a differential polynomial of \( f \) as the definition 1 satisfying \( q \geq \gamma_H \), and \( \Gamma / |\gamma_H| < k + 1 \), then \( \mathcal{F} \) is normal in \( D \), where \( q, \Gamma / |\gamma_H| \) are as in Definitions 1.1 and 1.2.

**Proof.** We may assume that \( D = \Delta = \{ |z| < 1 \} \). Suppose that \( \mathcal{F} \) is not normal in \( D \). Without loss of generality, we assume that \( \mathcal{F} \) is not normal at \( z_0 = 0 \). Then, by Lemma 2.1, there exists a number \( r \in (0, 1) \); a sequence of complex numbers \( z_j, z_j \to 0 \) \( (j \to \infty) \); a sequence of functions \( f_j \in \mathcal{F} \); a sequence of positive numbers \( \rho_j \to 0^+ \) such that \( g_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi) \) converges uniformly with respect to the spherical metric to a nonconstant meromorphic functions \( \overline{g}(\xi) \) in \( C \). Moreover, \( \overline{g}(\xi) \) is of order at most 2. Hurwitz’s theorem implies that \( \overline{g}(\xi) \neq 0 \).

We have

\[ G(f_j)(z_j + \rho_j \xi) = P_{f_j}(f_j^{(k)}(z_j + \rho_j \xi)) + H(f_j, f_j^{(k)})(z_j + \rho_j \xi), \]

\[ H(f_j, f_j^{(k)})(z_j + \rho_j \xi) = \sum_{i=1}^{n} b_i(z_j + \rho_j \xi) p_{i}^{(k+1)}(\Gamma, M_i) M_i(g_j, g_j^{(k)}). \quad (2.9) \]
Considering $b_i(z)$ is analytic on $D(i = 1, 2, \ldots, n)$, we have

$$|b_i(z_j + \rho_j \xi)| \leq M \left( \frac{1 + r}{2}, b_i(z) \right) < \infty, \quad (i = 1, 2, \ldots, n)$$  \hspace{1cm} (2.10)

for sufficiently large $j$.

Hence, we deduce from $\Gamma/\gamma |H < k + 1$ that

$$\sum_{i=1}^{n} b_i(z_j + \rho_j \xi)\rho_j^{(k+1)\gamma M_i - \Gamma M_i} M_i \left( g_{i}, g'_{i}, \ldots, g^{(k)}_{i} \right)(\xi)$$  \hspace{1cm} (2.11)

converges uniformly to 0 on every compact subset of $C$ which contains no poles of $\overline{g}(\xi)$.

Thus, we have

$$G(f_j)(z_j + \rho_j \xi) \rightarrow P\left( \overline{g}^{(k)}(\xi) \right),$$

$$G(f_j)(z_j + \rho_j \xi) - \psi(z_j + \rho_j \xi) \rightarrow P\left( \overline{g}^{(k)}(\xi) - \psi(z_0) \right)$$

on every compact subset of $C$ which contains no poles of $\overline{g}(\xi)$.

Next, we will prove that $G(f_j)(\xi) - \psi(z_0)$ has just a unique zero. By way of contradiction, let $\xi_0$ and $\xi_0^*$ be two distinct solutions of $G(f_j)(\xi) - \psi(z_0)$, and choose $\delta(0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \emptyset$ where $D(\xi_0, \delta) = \{ \xi : |\xi - \xi_0| < \delta \}$ and $D(\xi_0^*, \delta) = \{ \xi : |\xi - \xi_0^*| < \delta \}$. By Hurwitz’s theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that, for sufficiently large $j$,

$$G(f_j)(z_j + \rho_j \xi_j) - \psi(z_0) = 0,$$

$$G(f_j)(z_j + \rho_j \xi_j^*) - \psi(z_0) = 0.$$  \hspace{1cm} (2.13)

By the hypothesis that for each pair of functions $f$ and $g$ in $\mathcal{F}$, $G(f)$ and $G(g)$ share $\psi(z_0)$ in $D$, we know that, for any positive integer $m$,

$$G(f_m)(z_j + \rho_j \xi_j) - \psi(z_0) = 0,$$

$$G(f_m)(z_j + \rho_j \xi_j^*) - \psi(z_0) = 0.$$  \hspace{1cm} (2.14)

Fix $m$, take $j \rightarrow \infty$, and note $z_j + \rho_j \xi_j \rightarrow 0$, $z_j + \rho_j \xi_j^* \rightarrow 0$, then

$$G(f_m)(0) - \psi(z_0) = 0.$$  \hspace{1cm} (2.15)

Since the zeros of $G(f_m)(0) - \psi(z_0) = 0$ have no accumulation point, so $z_j + \rho_j \xi_j = 0$, $z_j + \rho_j \xi_j^* = 0$.

Hence,

$$\xi_j = - \frac{z_j}{\rho_j}, \quad \xi_j^* = - \frac{z_j}{\rho_j}.$$  \hspace{1cm} (2.16)
This contradicts with $\zeta_j \in D(\zeta_0, \delta)$, $\zeta_j^* \in D(\zeta_0^*, \delta)$, and $D(\zeta_0, \delta) \cap D(\zeta_0^*, \delta) = \emptyset$. So $G(f_j) - \varphi(z_0)$ has just a unique zero. By Hurwitz’s theorem, we know $P(\bar{G}(\zeta_j)) - \varphi(z_0)$ has just a unique zero.

By Lemmas 2.2 and 2.3, we know $G_n^{(k)}(\zeta) - \varphi(z_0)$ has at least two distinct zeros. From the definition of $P(w)$, we deduce that $P(\bar{G}(\zeta_j)) - \varphi(z_0)$ has more than two distinct zeros, a contradiction.

So $\mathcal{F}$ is normal in $D$. Lemma 2.6 is proved.

By Lemma 2.6, we immediately deduce the following lemma.

**Lemma 2.7.** Let $\mathcal{F}$ be a family of meromorphic functions defined in $D$, let $\varphi(\not=0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in $D$, and let $k$ be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \not= 0$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1f + a_0f \not= 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1f' + a_0f, g^{(k)} + a_{k-1}g^{(k-1)} + \cdots + a_1g' + a_0g$ share $\varphi$, then $\mathcal{F}$ is normal in $D$.

**Lemma 2.8** (see [1]). Let $f(z)$ be a meromorphic function. Let $k$ be a positive integer. If $f(z) \not= 0$, then $f^{(k)}(z) \not= 1$, then $f$ is a constant.

**Lemma 2.9** (see [13, 14]). Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$, and let $P(\not=0)$ be a polynomial. Let $k$ be a positive integer. If all zeros (except at most finite zeros) of $f(z)$ have the multiplicity at least 3, then $f^{(k)}(z) - P(z)$ has infinite zeros.

### 3. Proof of Theorem 1.4

**Proof.** Since normality is a local property, without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$, and

$$\varphi(z) = z^l \varphi(z) \quad (z \in \Delta),$$

where $l$ is a positive integer, $\varphi(0) = 1$, $\varphi(z) \not= 0$ on $\Delta' = \{z : 0 < |z| < 1\}$. By Lemma 2.6, we only need to prove that $\mathcal{F}$ is normal at $z = 0$.

If $f \in \mathcal{F}$, $P(f)(0) \not= \varphi(0)$, then there exists $\delta > 0$ such that $P(f)(z) \not= \varphi(z)$ on $\Delta_\delta$. By condition of Theorem, for every $g \in \mathcal{F}$, we know $P(g)(z) \not= \varphi(z)$ on $\Delta_\delta$. By theorem D, $\mathcal{F}$ is normal on $\Delta_\delta$, so $\mathcal{F}$ is normal on $z = 0$.

Now, we consider $P(f)(0) = \varphi(0)$. Suppose $P(f)(z) \not= \varphi(z)$ on the neighborhood $|z| < \delta$ (where $\delta$ is a small positive number) (otherwise, $P(f)(z) \equiv \varphi(z)$ on the neighborhood $|z| < \delta$, by condition of theorem, for every $g \in \mathcal{F}$, we also obtain $P(g)(z) \equiv \varphi(z)$). So $P(g)(z) \not= \varphi(z) + 1$. By Theorem D, $\mathcal{F}$ is normal at $z = 0$. So Theorem 1.4 is proved). there exists $\delta > 0$ such that $P(f)(z) \not= \varphi(z)$ on $(z \in \Delta'_\delta)$. So, for every $g \in \mathcal{F}$, we obtain

$$P(g)(z) \not= \varphi(z) \quad (z \in \Delta'_\delta).$$

By Theorem D, $\mathcal{F}$ is normal on $\Delta'$.

Next, we will prove $\mathcal{F}$ is normal at $z = 0$. Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z = 0 \in \Delta$, then there exists a sequence functions (we also denote $\mathcal{F}$) that has no any normal subsequence on $z = 0$. 

Consider the family \( \mathcal{H} = \{g(z) = (f(z)/\psi(z)) : f \in \mathcal{F}, z \in \Delta \} \). Since \( f \neq 0 \) for \( f \in \mathcal{F} \), we have that \( g(0) = \infty \) for each \( g \in \mathcal{H} \).

We first prove that \( \mathcal{H} \) is normal in \( \Delta \). Suppose, on the contrary, that \( \mathcal{H} \) is not normal at \( z_0 \in \Delta \). By Lemma 2.1, there exist a sequence of functions \( g_n \in \mathcal{H} \), a sequence of complex numbers \( z_n \to z_0 \), and a sequence of positive numbers \( \rho_n \to 0 \), such that

\[
G_n(\xi) = \frac{g_n(z_n + \rho_n \xi)}{\rho_n^k} \to G(\xi)
\]

(3.3)

converges spherically uniformly on compact subsets of \( \mathbb{C} \) where \( G(\xi) \) is a nonconstant meromorphic function on \( \mathbb{C} \), and \( G(\xi) \neq 0 \).

We distinguish two cases.

Case 1 \((z_n/\rho_n \to \infty)\). By a simple calculation, for \( 0 \leq i \leq k \), we have

\[
\frac{f_n^{(i)}(z)}{\psi(z)} - \frac{i}{\rho_n} \sum_{j=1}^{i} C_i^j \delta_n^{(i-j)}(z) \sum_{l=0}^{i} A_{jl} 1 \frac{\psi^{(l)}(z)}{\psi(z)}
\]

(3.4)

where \( A_{jl} = l(l - 1) \cdots (l - j + t + 1)C_j^l \) if \( l < j \), for \( t = 0, 1, \ldots, j - 1 \) and \( A_{jj} = 1 \).

Thus, from (3.4), we have

\[
\frac{\rho_n^k - g_n^{(i)}(\xi)}{\rho_n^k} = \frac{g_n^{(i)}(z_n + \rho_n \xi)}{\rho_n^k}
\]

\[
= \frac{f_n^{(i)}(z_n + \rho_n \xi)}{\psi(z_n + \rho_n \xi)} - \frac{i}{\rho_n} \sum_{j=1}^{i} C_i^j \delta_n^{(i-j)}(z_n + \rho_n \xi) \sum_{l=0}^{i} A_{jl} \frac{1}{(z_n + \rho_n \xi)^{l-j}} \frac{\psi^{(l)}(z_n + \rho_n \xi)}{\psi(z_n + \rho_n \xi)}
\]

(3.5)

On the other hand, we have

\[
\lim_{n \to \infty} \frac{1}{(z_n/\rho_n) + \xi} = 0,
\]

(3.6)

\[
\lim_{n \to \infty} \frac{\rho_n^k \psi^{(i)}(z_n + \rho_n \xi)}{\psi(z_n + \rho_n \xi)} = 0,
\]

for \( t \geq 1 \). Noting that \( g_n^{(i-j)}(z_n + \rho_n \xi)/\rho_n^j \) is locally bounded on \( \mathbb{C} \) minus the set of poles of \( G(\xi) \) since \( g_n(z_n + \rho_n \xi)/\rho_n^k \to G(\xi) \). Therefore, on every subset of \( \mathbb{C} \) which contains no poles
of $G(\xi)$, we have
\[
\frac{f_n^{(k)}(z_n + \rho_n \xi)}{q(z_n + \rho_n \xi)} \to G^{(k)}(\xi),
\]
(3.7)
for $i = 0, 1, \ldots, k - 1$, and thus
\[
\frac{f_n^{(k)}(z_n + \rho_n \xi) + \sum_{i=0}^{k-1} a_i(z_n + \rho_n \xi) f_n^{(i)}(z_n + \rho_n \xi)}{q(z_n + \rho_n \xi)} \to G^{(k)}(\xi),
\]
(3.8)
since $a_0, \ldots, a_{k-1}$ are analytic in $D$.

By $G(\xi) \neq 0$, we know $G^{(k)}(\xi) \neq 1$. In fact, if $G^{(k)}(\xi_0) = 1$, by Hurwitz’s theorem, then there exists $\xi_n \to \xi_0$, for $n$ sufficiently large,
\[
P(f)(z_n + \rho_n \xi_n) = q(z_n + \rho_n \xi_n).
\]
(3.9)
By the condition of the theorem, for every positive number $m$, we obtain $P(f_n)(z_n + \rho_n \xi_n) = q(z_n + \rho_n \xi_n)$. We know $z_n + \rho_n \xi_n \to z_0 \in \Delta_\delta$, and, for sufficiently large $n$, $z_n + \rho_n \xi_n \in \Delta_\delta$. However, $z_n + \rho_n \xi_n \neq 0$ (otherwise, $z_n + \rho_n \xi_n = 0$, so $\xi_n = -(z_n/\rho_n) \to \infty$, a contradiction), so for sufficiently large $n$, $z_n + \rho_n \xi_n \in \Delta_\delta$. This contradicts with (3.2).

So $G(\xi) \neq 0$ and $G^{(k)}(\xi) \neq 1$, by Lemma 2.8, we obtain $G$ is a constant, a contradiction.

**Case 2.** $z_n/\rho_n \to \alpha$ is a finite complex number. Then,
\[
\frac{g_n (\rho_n \xi)}{\rho_n^k} = \frac{g_n(z_n + \rho_n (\xi - (z_n/\rho_n)))}{\rho_n^k} = G_n \left( \xi - \frac{z_n}{\rho_n} \right) \to G(\xi - \alpha) = \mathbb{G}(\xi).
\]
(3.10)
Obviously, $G(\xi) \neq 0$, and $\xi = 0$ is a pole of $\mathbb{G}$ with order at least $l$.

Set
\[
H_n(\xi) = \frac{f_n(\rho_n \xi)}{\rho_n^{k+l}}.
\]
(3.11)
Then,
\[
H_n(\xi) = \frac{q(\rho_n \xi)}{\rho_n^k} \frac{f_n(\rho_n \xi)}{\rho_n^l q(\rho_n \xi)} = \frac{q(\rho_n \xi)}{\rho_n^l} \frac{g_n(\rho_n \xi)}{\rho_n^k}.
\]
(3.12)
Noting that $\psi(\rho_n \xi) / \rho_n^i \to \xi^i$, thus
\[
H_n(\xi) \to \xi^i G(\xi) = H(\xi),
\] (3.13)
uniformly on compact subsets of $\mathbb{C}$. Since $G$ has a pole of order at least at $\xi = 0$, we have $H(0) \neq 0$, so that $H(\xi) \neq 0$.

From (3.11), we get
\[
H_n^{(i)}(\rho_n \xi) \to H^{(i)}(\xi),
\] (3.14)
spherically uniformly on compact subsets of $\mathbb{C}$ minus the set of poles of $G(\xi)$. As the above, on every compact subset of $\mathbb{C}$ minus the set of poles of $G(\xi)$, we have
\[
\frac{f_n^{(k)}(\rho_n \xi) + \sum_{i=0}^{k-1} a_i(\rho_n \xi) f_n^{(i)}(\rho_n \xi)}{\rho_n^i} \to H^{(k)}(\xi),
\] (3.15)
and
\[
\frac{f_n^{(k)}(\rho_n \xi) + \sum_{i=0}^{k-1} a_i(\rho_n \xi) f_n^{(i)}(\rho_n \xi) - \psi(\rho_n \xi)}{\rho_n^i} \to H^{(k)}(\xi) - \xi^i,
\] (3.16)
locally uniformly on $\mathbb{C}$.

By the assumption of Theorem and (3.16), Hurwitz’s theorem implies $H^{(k)}(\xi) \neq 0$.

Next, we proof that if $\xi \in \mathbb{C} \setminus \{0\}$, then $H^{(k)}(\xi) \neq \xi^i$.

First, $H^{(k)}(\xi) \neq \xi^i$, otherwise $H^{(k)}(\xi) \equiv \xi^i$, which contradicts with $H(\xi) \neq 0$. If there exists a $\xi_0 \neq 0$ such that $H^{(k)}(\xi_0) \equiv \xi_0^i$, by Hurwitz’s theorem and (3.16), there exists $\xi_n \to \xi_0$ such that $f_n^{(k)}(\rho_n \xi_n) + \sum_{i=0}^{k-1} a_i(\rho_n \xi_n) f_n^{(i)}(\rho_n \xi_n) = \psi(\rho_n \xi_n)$. By the assumption of Theorem 1.4, for every positive $m$ such that $P(\rho_m) (\rho_n \xi_n) = \psi(\rho_n \xi_n)$. However, for $n$ sufficiently large, $\rho_n \xi_n \in \Delta_0^i$, all of these contradict with (3.2). So if $\xi \in \mathbb{C} \setminus \{0\}$, then $H^{(k)}(\xi) \neq \xi^i$.

Noting $H(\xi) \neq 0$, By Lemma 2.9, we know $H$ must be a rational function. If $H$ is not a constant, By Lemma 2.4, we know $H^{(k)}(\xi) - \xi^i$ has at least two distinct zeros, a contradiction. So $H$ must be a nonzero constant, also contradicts with $H^{(k)}(\xi) \neq 0$. Now, we have proved the $\mathcal{H}$ is normal on $\Delta$. It remains to show that $\mathcal{G}$ is normal at $z = 0$. Since $\mathcal{J}$ is normal in $\Delta$, then the family $\mathcal{J}$ is equicontinuous on $\Delta$ with respect to the spherical distance. On the other hand, $\mathcal{G}(0) = \infty$ for each $\mathcal{G} \in \mathcal{J}$, so there exists $\delta > 0$ such that $|g(z)| \geq 1$ for all $g \in \mathcal{J}$ and each $g \in \Delta_\delta = \{z : |z| < \delta\}$. Suppose that $\mathcal{G}$ is not normal at $z = 0$. Since $\mathcal{G}$ is normal in $0 < |z| < 1$, the family $\mathcal{G}_1 = \{1/f : f \in \mathcal{J}\}$ is normal in $\Delta = \{z : 0 < |z| < 1\}$, but it is not normal at $z = 0$. Then, there exists a sequence $\{1/f_n\} \subset \mathcal{G}_1$ which converges locally uniformly in $\Delta'$, but not in $\Delta$. Noting that $f_n \neq 0$ in $\Delta$, $1/f_n$ is holomorphic in $\Delta$ for each $n$. The maximum modulus principle implies that $1/f_n \to \infty$ in $\Delta'$. Thus, $f_n \to 0$ converges locally uniformly in $\Delta'$, and hence so does $\{g_n\} \subset \mathcal{L}$, where $g_n = f_n / \psi$. But $|g_n(z)| \geq 1$ for each $z \in \Delta_\delta$, a contradiction.

This finally completes the proof of Theorem 1.4. \qed
Abstract and Applied Analysis

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