Research Article

Approximating Common Fixed Points of Nonspreading-Type Mappings and Nonexpansive Mappings in a Hilbert Space

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We obtain some fundamental properties for k -strictly pseudo-nonspreading mappings in a Hilbert space. We study approximation of common fixed points of k -strictly pseudo-nonspreading mappings and nonexpansive mappings in a Hilbert space by using a new iterative scheme. Furthermore, we suggest some open problems.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . Then a mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad (1.1)$$

for all $x, y \in C$.

Recently, Kohsaka and Takahashi [1] introduced a class of mappings called nonspreading mappings: Let E be a real smooth, strictly convex, and reflexive Banach space, and let J denote the duality mapping of E . Let C be a nonempty closed convex subset of E . They called a mapping $T : C \rightarrow C$ is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \quad (1.2)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in E$. They considered the class of nonspreading mappings to study the resolvents of a maximal monotone operator in the Banach space. Observe that, if E is a real Hilbert space, then J is the identity and

$$\phi(x, y) = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 = \|x - y\|^2 \quad (1.3)$$

for all $x, y \in E$. Thus, if C is a nonempty closed convex subset of a Hilbert space, then $T : C \rightarrow C$ is nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Tx - y\|^2 \quad (1.4)$$

for all $x, y \in C$. It is shown in [2] that (1.4) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle \quad (1.5)$$

for all $x, y \in C$.

Following the terminology of Browder-Petryshyn [3, (page 198)], a mapping $T : C \rightarrow H$ is *k-strictly pseudo-nonspreading* if there exists $k \in [0, 1)$ such that

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle \\ &\quad + k\|x - Tx - (y - Ty)\|^2, \end{aligned} \quad (1.6)$$

for all $x, y \in C$. Clearly, every nonspreading mapping is *k-strictly pseudo-nonspreading*.

The following is an example of nonspreading mapping which is not nonexpansive mapping.

Example 1.1 (see [2]). Let H be a Hilbert space. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $T : C \rightarrow C$ as follows:

$$Tx = \begin{cases} 0 & \text{if } x \in D, \\ P_E x & \text{if } x \in C \setminus D, \end{cases} \quad (1.7)$$

where P_E is the metric projection of H onto E . Then, T is not nonexpansive but nonspreading mapping.

The following example shows that the class of *k-strictly pseudo-nonspreading* mapping is more general than the class of nonspreading mappings.

Example 1.2 (see [4]). Let $\mathbb{R}$ denotes the real numbers with the usual norm. Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined for each $x \in \mathbb{R}$ by

$$Tx = \begin{cases} x & \text{if } x \in (-\infty, 0), \\ -2x & \text{if } x \in [0, \infty). \end{cases} \quad (1.8)$$

Then, T is *k-strictly pseudo-nonspreading* but not nonspreading mapping.

Remark 1.3 (see [4]). Let C be a nonempty closed convex subset of a real Hilbert space H , and let $S : C \rightarrow C$ be a k -strictly pseudo-nonspreading mapping. If $F(S) \neq \emptyset$, then it is closed and convex.

Iemoto and Takahashi [2] introduced the following Moudafi iterative procedure [5]:

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n\{\beta_n Sx_n + (1 - \beta_n)Tx_n\} \end{aligned} \quad (1.9)$$

for finding the approximation of common fixed points of nonspreading mapping S and nonexpansive mapping T in a Hilbert space.

In this paper, we obtain some fundamental properties for k -strictly pseudo-nonspreading mappings in a Hilbert space. We study approximation of common fixed points of k -strictly pseudo-nonspreading mappings and nonexpansive mappings in a Hilbert space by using a new iterative scheme. Furthermore, we suggest some open problems.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

Definition 2.1. Let E be a real Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at a point $p \in D(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to a point $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p , then $Tx = p$.

Lemma 2.2 (see [2]). *Let H be a real Hilbert space. Then, the following well known results hold:*

- (1) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$ for all $x, y \in H$ and for all $t \in [0, 1]$,
- (2) $2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$ for all $x, y, z, w \in H$.

Lemma 2.3 (see [6]). *Let C be a nonempty closed convex subset of H . If $T : C \rightarrow C$ is nonexpansive, then $I - T : C \rightarrow H$ is $1/2$ -inverse strongly monotone, that is,*

$$\frac{1}{2} \|(I - T)x - (I - T)y\|^2 \leq \langle x - y, (I - T)x - (I - T)y \rangle \quad (2.1)$$

for all $x, y \in C$.

The following lemma is one of the characterizations of a k -strictly pseudo-nonspreading mapping.

Lemma 2.4. *Let C be a nonempty closed convex subset of H . Then, a mapping $S : C \rightarrow C$ is k -strictly pseudo-nonspreading if and only if*

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^3 + k\|(I - S)x - (I - S)y\|^2 \quad (2.2)$$

for all $x, y \in C$.

Proof. We have that for all $x, y \in C$,

$$\begin{aligned}
\|Sx - Sy\|^2 &\leq \|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle + k\|x - Sx - (y - Sy)\|^2 \\
\iff 2\|Sx - Sy\|^2 &\leq \|Sx - Sy\|^2 + \|x - y\|^2 + 2\langle Sx - x, x - Sx - (y - Sy) \rangle \\
&\quad + 2\|Sx - x\|^2 + k\|x - Sx - (y - Sy)\|^2 \\
&= \|Sx - Sy\|^2 + 2\langle x - Sx, Sx - Sy \rangle + \|x - Sx\|^2 \\
&\quad + \|x - y\|^2 + 2\langle Sx - x, x - y \rangle + \|Sx - x\|^2 + k\|x - Sx - (y - Sy)\|^2 \\
&= \|x - Sy\|^2 + \|Sx - y\|^2 + k\|(I - S)x - (I - S)y\|^2.
\end{aligned} \tag{2.3}$$

This completes the proof of Lemma 2.4. $\square$

Lemma 2.5. *Let C be a nonempty closed convex subset of H . Let S be a k -strictly pseudo-nonspreading mapping of C into itself and let $A = I - S$.*

Then,

$$(2 - k)\|Ax - Ay\|^2 \leq 2\langle x - y, Ax - Ay \rangle + \|Ax\|^2 + \|Ay\|^2 \tag{2.4}$$

for all $x, y \in C$.

Proof. Let $A = I - S$. We have

$$\begin{aligned}
\|Ax - Ay\|^2 &= \langle Ax - Ay, Ax - Ay \rangle \\
&= \langle x - y - (Sx - Sy), Ax - Ay \rangle \\
&= \langle x - y, Ax - Ay \rangle - \langle Sx - Sy, Ax - Ay \rangle
\end{aligned} \tag{2.5}$$

for all $x, y \in C$. From Lemma 2.2-(2) and Lemma 2.4, we get

$$\begin{aligned}
2\langle Sx - Sy, Ax - Ay \rangle &= 2\langle Sx - Sy, x - y - (Sx - Sy) \rangle \\
&= 2\langle Sx - Sy, x - y \rangle - 2\|Sx - Sy\|^2 \\
&\geq \|Sx - y\|^2 + \|Sy - x\|^2 - \|Sx - x\|^2 - \|Sy - y\|^2 \\
&\quad - \left(\|Sx - y\|^2 + \|x - Sy\|^2 + k\|Ax - Ay\|^2 \right) \\
&= -\|x - Sx\|^2 - \|y - Sy\|^2 - k\|Ax - Ay\|^2 \\
&= -\|Ax\|^2 - \|Ay\|^2 - k\|Ax - Ay\|^2.
\end{aligned} \tag{2.6}$$

So, from (2.5) and (2.6), we have

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle + \frac{1}{2}(\|Ax\|^2 + \|Ay\|^2) + \frac{k}{2}\|Ax - Ay\|^2. \quad (2.7)$$

Therefore, we get

$$(2 - k)\|Ax - Ay\|^2 \leq 2\langle x - y, Ax - Ay \rangle + \|Ax\|^2 + \|Ay\|^2 \quad (2.8)$$

for all $x, y \in C$. □

Lemma 2.6 (see [4]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $S : C \rightarrow C$ be a k -strictly pseudo-nonspreading mapping. Then $I - S$ is demiclosed at 0.*

Tan and Xu [7] proved the following; see also [6, 8].

Lemma 2.7. *Let $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.*

Lemma 2.8. *Let $\{\alpha_n\}, \{\beta_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, then $\liminf_{n \rightarrow \infty} \beta_n = 0$.*

3. Main Theorem

In this section, we prove our main theorem for finding common fixed points of k -strictly pseudo-nonspreading mapping and nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \rightarrow C$ be a k -strictly pseudo-nonspreading mapping and let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be sequences in $[0, 1]$ such that $\beta_n \in (k, 1]$. Define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Sx_n) + \alpha_n(\gamma_n x_n + (1 - \gamma_n)Tx_n) \end{aligned} \quad (3.1)$$

for all $n \in \mathbb{N}$. Then, the followings hold:

- (1) if $\liminf_{n \rightarrow \infty} \alpha_n(\beta_n - \gamma_n) > 0$, $\sum_{n=1}^{\infty} \alpha_n(1 - \gamma_n) < \infty$, and $1 + k < (2 - \alpha_n)\beta_n + \alpha_n\gamma_n$, then $\{x_n\}$ converges weakly to $q \in F(S)$,
- (2) if $\beta_n > \gamma_n$, $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, $2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n) > 0$, and $\liminf_{n \rightarrow \infty} \alpha_n(\beta_n - \gamma_n)(2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n)) > 0$, then $\{x_n\}$ converges weakly to $q \in F(T)$,
- (3) if $\liminf_{n \rightarrow \infty} \alpha_n > 0$, $\liminf_{n \rightarrow \infty} (1 - \alpha_n) > 0$, $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$, and $\liminf_{n \rightarrow \infty} \gamma_n(1 - \gamma_n) > 0$, then $\{x_n\}$ converges weakly to $q \in F(S) \cap F(T)$.

Proof. Putting $U_n = \beta_n I + (1 - \beta_n)S$ and $V_n = \gamma_n I + (1 - \gamma_n)T$. We first show that the sequence $\{x_n\}$ is bounded. Indeed, from Lemma 2.2-(1) and S as a k -strictly pseudo-nonspreading mapping, we have

$$\begin{aligned}
\|U_n x - U_n y\|^2 &= \|\beta_n(x - y) + (1 - \beta_n)(Sx - Sy)\|^2 \\
&= \beta_n\|x - y\|^2 + (1 - \beta_n)\|Sx - Sy\|^2 \\
&\quad - \beta_n(1 - \beta_n)\|x - Sx - (y - Sy)\|^2 \\
&\leq \beta_n\|x - y\|^2 + (1 - \beta_n) \\
&\quad \times \left(\|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle + k\|x - Sx - (y - Sy)\|^2 \right) \\
&\quad - \beta_n(1 - \beta_n)\|x - Sx - (y - Sy)\|^2 \\
&= \|x - y\|^2 + 2(1 - \beta_n)\langle x - Sx, y - Sy \rangle \\
&\quad - (1 - \beta_n)(\beta_n - k)\|x - Sx - (y - Sy)\|^2 \\
&\leq \|x - y\|^2 + 2(1 - \beta_n)\langle x - Sx, y - Sy \rangle \\
&= \|x - y\|^2 + \frac{2}{1 - \beta_n}\langle x - U_n x, y - U_n y \rangle,
\end{aligned} \tag{3.2}$$

for all $x, y \in C$. Let $p \in F(S) \cap F(T)$, then

$$U_n p = \beta_n p + (1 - \beta_n)Sp = p. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\|U_n x_n - p\| = \|U_n x_n - U_n p\| \leq \|x_n - p\|. \tag{3.4}$$

Since T is a nonexpansive mapping and $F(T) \neq \emptyset$, we get

$$\begin{aligned}
\|V_n x_n - p\| &= \|\gamma_n x_n + (1 - \gamma_n)Tx_n - p\| \\
&\leq \gamma_n\|x_n - p\| + (1 - \gamma_n)\|Tx_n - p\| \\
&\leq \|x_n - p\|
\end{aligned} \tag{3.5}$$

for all $p \in F(S) \cap F(T)$. From (3.4) and (3.5), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)U_n x_n + \alpha_n V_n x_n - p\|^2 \\
&\leq (1 - \alpha_n)\|U_n x_n - p\|^2 + \alpha_n\|V_n x_n - p\|^2 \\
&\leq \|x_n - p\|^2
\end{aligned} \tag{3.6}$$

for all $n \in \mathbb{N}$. Therefore, there exists $\lim_{n \rightarrow \infty} \|x_n - p\|$ and hence $\{x_n\}$ is bounded.

Let

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (3.7)$$

To prove (1), let

$$z_{n+1} = (1 - \alpha_n)U_n x_n + \alpha_n(\gamma_n x_n + (1 - \gamma_n)Sx_n) \quad (3.8)$$

and $A = I - S$. Then, we have

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &= \|(1 - \alpha_n)U_n x_n + \alpha_n V_n x_n \\ &\quad - (1 - \alpha_n)U_n x_n - \alpha_n(\gamma_n x_n + (1 - \gamma_n)Sx_n)\| \\ &= \alpha_n \|\gamma_n x_n + (1 - \gamma_n)Tx_n - \gamma_n x_n - (1 - \gamma_n)Sx_n\| \\ &= \alpha_n(1 - \gamma_n)\|Tx_n - Sx_n\|. \end{aligned} \quad (3.9)$$

Since $\sum_{n=1}^{\infty} \alpha_n(1 - \gamma_n) < \infty$, we have $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$. From the continuity of $\|\cdot\|$, we get

$$\lim_{n \rightarrow \infty} \|z_n - p\| = \lim_{n \rightarrow \infty} \|x_n - p\| = c. \quad (3.10)$$

Since $U_n = \beta_n I + (1 - \beta_n)S$, we get

$$\begin{aligned} U_n x_n - Sx_n &= \beta_n x_n + (1 - \beta_n)Sx_n - Sx_n \\ &= \beta_n(x_n - Sx_n) = \beta_n(I - S)x_n \\ &= \beta_n Ax_n, \\ U_n x_n - x_n &= \beta_n x_n + (1 - \beta_n)Sx_n - x_n \\ &= -(1 - \beta_n)(x_n - Sx_n) \\ &= -(1 - \beta_n)Ax_n. \end{aligned} \quad (3.11)$$

So, we have from Lemma 2.5, Lemma 2.6 ($Ap = 0$), as well as (3.4), (3.11) that

$$\begin{aligned} \|z_{n+1} - p\|^2 &= \|(1 - \alpha_n)U_n x_n + \alpha_n(\gamma_n x_n + (1 - \gamma_n)Sx_n) - p\|^2 \\ &= \|U_n x_n - p - \alpha_n(U_n x_n - Sx_n) + \alpha_n \gamma_n(x_n - Sx_n)\|^2 \\ &= \|U_n x_n - p - \alpha_n \beta_n Ax_n + \alpha_n \gamma_n Ax_n\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|U_n x_n - p - \alpha_n(\beta_n - \gamma_n)Ax_n\|^2 \\
&\leq \|U_n x_n - p\|^2 - 2\alpha_n(\beta_n - \gamma_n)\langle U_n x_n - p, Ax_n \rangle + \alpha_n^2(\beta_n - \gamma_n)^2 \|Ax_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\alpha_n(\beta_n - \gamma_n)\langle U_n x_n - p, Ax_n - Ap \rangle + \alpha_n^2(\beta_n - \gamma_n)^2 \|Ax_n\|^2 \\
&= \|x_n - p\|^2 - 2\alpha_n(\beta_n - \gamma_n)\langle U_n x_n - x_n, Ax_n - Ap \rangle \\
&\quad - 2\alpha_n(\beta_n - \gamma_n)\langle x_n - p, Ax_n - Ap \rangle + \alpha_n^2(\beta_n - \gamma_n)^2 \|Ax_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\alpha_n(1 - \beta_n)(\beta_n - \gamma_n)\langle Ax_n, Ax_n - Ap \rangle \\
&\quad + \alpha_n(\beta_n - \gamma_n)\left(\|Ax_n\|^2 + \|Ap\|^2 - (2 - k)\|Ax_n - Ap\|^2\right) \\
&\quad + \alpha_n^2(\beta_n - \gamma_n)^2 \|Ax_n\|^2 \\
&= \|x_n - p\|^2 + 2\alpha_n(1 - \beta_n)(\beta_n - \gamma_n)\|Ax_n\|^2 \\
&\quad - (1 - k)\alpha_n(\beta_n - \gamma_n)\|Ax_n\|^2 + \alpha_n^2(\beta_n - \gamma_n)^2 \|Ax_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n(\beta_n - \gamma_n)(1 - k - 2(1 - \beta_n) - \alpha_n(\beta_n - \gamma_n))\|Ax_n\|^2.
\end{aligned} \tag{3.12}$$

Hence

$$\alpha_n(\beta_n - \gamma_n)(1 - k - 2(1 - \beta_n) - \alpha_n(\beta_n - \gamma_n))\|Ax_n\|^2 \leq \|x_n - p\|^2 - \|z_{n+1} - p\|^2. \tag{3.13}$$

Since $\liminf_{n \rightarrow \infty} \alpha_n(\beta_n - \gamma_n) > 0$, we get

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = \lim_{n \rightarrow \infty} \|Ax_n\| = 0. \tag{3.14}$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to q . From Lemma 2.6, we obtain $q \in F(S)$. To show our conclusion, it is sufficient to show that for another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $v \in F(S)$, $q = v$. Before proving this, we show that for any $z \in F(S)$, $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists. Since

$$U_n z = \beta_n z + (1 - \beta_n)Sz = z \tag{3.15}$$

and, from (3.2), we get

$$\|U_n x_n - z\|^2 = \|U_n x_n - U_n z\|^2 \leq \|x_n - z\|^2 \tag{3.16}$$

for all $z \in F(S)$. Hence, we have

$$\begin{aligned}
\|z_{n+1} - z\| &= \|(1 - \alpha_n)U_n x_n + \alpha_n(\gamma_n x_n + (1 - \gamma_n)Sx_n) - z\| \\
&= \|(1 - \alpha_n)(U_n x_n - z) + \alpha_n(\gamma_n(x_n - z) + (1 - \gamma_n)(Sx_n - z))\| \\
&\leq (1 - \alpha_n)\|U_n x_n - z\| + \alpha_n\gamma_n\|x_n - z\| + \alpha_n(1 - \gamma_n)\|Sx_n - z\| \\
&\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\|x_n - z\| + \alpha_n(1 - \gamma_n)\|Sx_n - z\| \\
&= \|x_n - z\| + \alpha_n(1 - \gamma_n)\|Sx_n - z\| \\
&\leq \|z_n - z\| + \|x_n - z_n\| + \alpha_n(1 - \gamma_n)\|Sx_n - z\|
\end{aligned} \tag{3.17}$$

for all $z \in F(S)$. From Lemma 2.7 and (3.9), $\lim_{n \rightarrow \infty} \|z_n - z\|$ exists. So, there exists $\lim_{n \rightarrow \infty} \|x_n - z\|$ for all $z \in F(S)$ because $\lim_{n \rightarrow \infty} (x_n - z_n) = 0$. Suppose that $q \neq v$. We have from Opial's theorem [9] that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|x_n - q\| &= \lim_{j \rightarrow \infty} \|x_{n_j} - q\| < \lim_{j \rightarrow \infty} \|x_{n_j} - v\| = \lim_{n \rightarrow \infty} \|x_n - v\| \\
&= \lim_{j \rightarrow \infty} \|x_{n_j} - v\| \\
&< \lim_{j \rightarrow \infty} \|x_{n_j} - q\| = \lim_{n \rightarrow \infty} \|x_n - q\|.
\end{aligned} \tag{3.18}$$

This is a contradiction. So, $\{x_n\}$ converges weakly to $q \in F(S)$.

To prove (2), let

$$z_{n+1} = (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n) + \alpha_n V_n x_n \tag{3.19}$$

and $B = I - T$. It follows that

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\| &= \|(1 - \alpha_n)U_n x_n + \alpha_n V_n x_n \\
&\quad - (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n) - \alpha_n V_n x_n\| \\
&= (1 - \alpha_n)\|\beta_n x_n + (1 - \beta_n)Sx_n - \beta_n x_n - (1 - \beta_n)Tx_n\| \\
&\leq (1 - \beta_n)\|Sx_n - Tx_n\|.
\end{aligned} \tag{3.20}$$

So, from the boundedness of $\{x_n\}$, $\{z_n\}$ is also bounded. Since T is a nonexpansive, by Lemma 2.3, B is $1/2$ -inverse strongly monotone and $Bp = 0$, we have

$$\begin{aligned}
\|z_{n+1} - p\|^2 &= \|(1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n) + \alpha_n V_n x_n - p\|^2 \\
&= \|\beta_n(x_n - p) + (1 - \beta_n)(Tx_n - p) - \alpha_n(\beta_n - \gamma_n)Bx_n\|^2 \\
&= \|\beta_n(x_n - p) + (1 - \beta_n)(Tx_n - p)\|^2 \\
&\quad - 2\alpha_n(\beta_n - \gamma_n)\langle \beta_n(x_n - p) + (1 - \beta_n)(Tx_n - p), Bx_n \rangle \\
&\quad + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&\leq (\beta_n\|x_n - p\| + (1 - \beta_n)\|Tx_n - p\|)^2 \\
&\quad - 2\alpha_n\beta_n(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp \rangle \\
&\quad - 2\alpha_n(1 - \beta_n)(\beta_n - \gamma_n)\langle Tx_n - p, Bx_n - Bp \rangle \\
&\quad + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\alpha_n\beta_n(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp \rangle \\
&\quad - 2\alpha_n(1 - \beta_n)(\beta_n - \gamma_n)\langle Tx_n - x_n, Bx_n - Bp \rangle \\
&\quad - 2\alpha_n(1 - \beta_n)(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp \rangle + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&= \|x_n - p\|^2 + 2\alpha_n(1 - \beta_n)(\beta_n - \gamma_n)\langle Bx_n, Bx_n \rangle \\
&\quad - 2\alpha_n(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp \rangle + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\alpha_n(1 - \beta_n)(\beta_n - \gamma_n)\|Bx_n\|^2 \\
&\quad - \alpha_n(\beta_n - \gamma_n)\|Bx_n - Bp\|^2 + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n(\beta_n - \gamma_n)(1 - 2(1 - \beta_n) - \alpha_n(\beta_n - \gamma_n))\|Bx_n\|^2
\end{aligned} \tag{3.21}$$

and hence

$$\alpha_n(\beta_n - \gamma_n)(1 - 2(1 - \beta_n) - \alpha_n(\beta_n - \gamma_n))\|Bx_n\|^2 \leq \|x_n - p\|^2 - \|z_{n+1} - p\|^2 \tag{3.22}$$

for $p \in F(S) \cap F(T)$. Summing from $n = 1$ to N , from (3.20), we have

$$\begin{aligned}
&\sum_{n=1}^N \alpha_n(\beta_n - \gamma_n)(2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n))\|Bx_n\|^2 \\
&\leq \|x_1 - p\|^2 + \sum_{n=1}^{N-1} (\|x_{n+1} - p\|^2 - \|z_{n+1} - p\|^2) - \|z_{N+1} - p\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_1 - p\|^2 + \sum_{n=1}^{N-1} (\|x_{n+1} - p\| + \|z_{n+1} - p\|) \|x_{n+1} - z_{n+1}\| \\
&\leq \|x_1 - p\|^2 + \sum_{n=1}^{N-1} (1 - \beta_n) (\|x_{n+1} - p\| + \|z_{n+1} - p\|) \|Sx_n - Tx_n\| \\
&\leq \|x_1 - p\|^2 + M \sum_{n=1}^{N-1} (1 - \beta_n),
\end{aligned} \tag{3.23}$$

where $M = \sup_{n \in \mathbb{N}} \{(\|x_{n+1} - p\| + \|z_{n+1} - p\|) \|Sx_n - Tx_n\|\}$. Letting $N \rightarrow \infty$, from $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \alpha_n (\beta_n - \gamma_n) (2\beta_n - 1 - \alpha_n (\beta_n - \gamma_n)) \|Bx_n\|^2 \\
&\leq \|x_1 - p\|^2 + M \sum_{n=1}^{\infty} (1 - \beta_n) \\
&< \infty.
\end{aligned} \tag{3.24}$$

Since $\sum_{n=1}^{\infty} \alpha_n (\beta_n - \gamma_n) (2\beta_n - 1 - \alpha_n (\beta_n - \gamma_n)) = \infty$, from Lemma 2.8, we get

$$\liminf_{n \rightarrow \infty} \|x_n - Tx_n\| = \liminf_{n \rightarrow \infty} \|Bx_n\| = 0. \tag{3.25}$$

Since T is a nonexpansive mapping, from (3.1), we get

$$\begin{aligned}
\|Tx_{n+1} - x_{n+1}\| &= \|Tx_{n+1} - (1 - \alpha_n)U_n x_n - \alpha_n V_n x_n\| \\
&\leq \|Tx_{n+1} - Tx_n\| + (1 - \alpha_n) \|Tx_n - U_n x_n\| + \alpha_n \|Tx_n - V_n x_n\| \\
&\leq \|x_{n+1} - x_n\| + (1 - \alpha_n) \|Tx_n - U_n x_n\| + \alpha_n \|Tx_n - V_n x_n\| \\
&\leq (1 - \alpha_n) \|U_n x_n - x_n\| + \alpha_n \|V_n x_n - x_n\| \\
&\quad + (1 - \alpha_n) \|Tx_n - \beta_n x_n - (1 - \beta_n)Sx_n\| \\
&\quad + \alpha_n \|Tx_n - \gamma_n x_n - (1 - \gamma_n)Tx_n\| \\
&= (1 - \alpha_n) \|\beta_n x_n + (1 - \beta_n)Sx_n - x_n\| + \alpha_n \|\gamma_n x_n + (1 - \gamma_n)Tx_n - x_n\| \\
&\quad + (1 - \alpha_n) \|\beta_n (Tx_n - x_n) + (1 - \beta_n)(Tx_n - Sx_n)\| + \alpha_n \|\gamma_n (Tx_n - x_n)\| \\
&\leq (1 - \alpha_n) (1 - \beta_n) \|Sx_n - x_n\| + \alpha_n (1 - \gamma_n) \|Tx_n - x_n\| \\
&\quad + (1 - \alpha_n) \|Tx_n - x_n\| + (1 - \alpha_n) (1 - \beta_n) \|Tx_n - Sx_n\| + \alpha_n \gamma_n \|Tx_n - x_n\| \\
&\leq \|Tx_n - x_n\| + (1 - \beta_n) (\|Sx_n - x_n\| + \|Tx_n - Sx_n\|).
\end{aligned} \tag{3.26}$$

Since $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, from Lemma 2.7, there exists the limit of $\{\|Tx_n - x_n\|\}$. Therefore, from (3.25), we get

$$\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0. \quad (3.27)$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to q . Since a nonexpansive mapping T is demiclosed, we have $q \in F(T)$. As in the proof of (1), $\{x_n\}$ converges weakly to $q \in F(T)$.

(3) From (3.6) and (3.7), we have that, for any $p \in F(S) \cap F(T)$,

$$0 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \longrightarrow c^2 - c^2 = 0 \quad (3.28)$$

as $n \rightarrow \infty$. We first show that $\{x_n\}$ converges weakly to some point in $F(S)$. Actually, from (3.4) and (3.5), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n) \|U_n x_n - p\|^2 + \alpha_n \|V_n x_n - p\|^2 \\ &\leq (1 - \alpha_n) \|U_n x_n - p\|^2 + \alpha_n \|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 \end{aligned} \quad (3.29)$$

for $p \in F(S) \cap F(T)$. Hence, we get

$$\begin{aligned} 0 &\leq \|x_n - p\|^2 - (1 - \alpha_n) \|U_n x_n - p\|^2 - \alpha_n \|x_n - p\|^2 \\ &= (1 - \alpha_n) (\|x_n - p\|^2 - \|\beta_n x_n + (1 - \beta_n) Sx_n - p\|^2) \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \end{aligned} \quad (3.30)$$

Since $\liminf_{n \rightarrow \infty} (1 - \alpha_n) > 0$, it follows from (3.28) and (3.30) that

$$\lim_{n \rightarrow \infty} (\|x_n - p\|^2 - \|\beta_n x_n + (1 - \beta_n) Sx_n - p\|^2) = 0. \quad (3.31)$$

From Lemma 2.2-(1), we have

$$\begin{aligned} &\|\beta_n x_n + (1 - \beta_n) Sx_n - p\|^2 \\ &= \|\beta_n (x_n - p) + (1 - \beta_n) (Sx_n - p)\|^2 \\ &= \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|Sx_n - p\|^2 - \beta_n (1 - \beta_n) \|x_n - Sx_n\|^2. \end{aligned} \quad (3.32)$$

Since S is a k -strictly pseudo-nonspreading mapping, from (1.6), we get

$$\begin{aligned}
 & \beta_n(1 - \beta_n)\|x_n - Sx_n\|^2 \\
 &= \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|Sx_n - p\|^2 - \|\beta_n x_n + (1 - \beta_n)Sx_n - p\|^2 \\
 &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\left(\|x_n - p\|^2 + k\|x_n - Sx_n\|^2\right) \\
 &\quad - \|\beta_n x_n + (1 - \beta_n)Sx_n - p\|^2 \\
 &= \|x_n - p\|^2 + k(1 - \beta_n)\|x_n - Sx_n\|^2 - \|\beta_n x_n + (1 - \beta_n)Sx_n - p\|^2
 \end{aligned} \tag{3.33}$$

and hence

$$(1 - \beta_n)(\beta_n - k)\|x_n - p\|^2 \leq \|x_n - p\|^2 - \|\beta_n x_n + (1 - \beta_n)Sx_n - p\|^2 \tag{3.34}$$

for $p \in F(S) \cap F(T)$. Since $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$, from (3.31), we get

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0. \tag{3.35}$$

As in the proof of (1), from Lemma 2.6, we obtain that if $\{x_{n_i}\}$ converges weakly to v , then $v \in F(S)$. We also show that such v is in $F(T)$. In fact, from (3.4) and (3.5), we get

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n)\|U_n x_n - p\|^2 + \alpha_n\|V_n x_n - p\|^2 \\
 &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|\gamma_n x_n + (1 - \gamma_n)Tx_n - p\|^2 \\
 &\leq \|x_n - p\|^2
 \end{aligned} \tag{3.36}$$

for $p \in F(S) \cap F(T)$. Hence, we have

$$\begin{aligned}
 0 &\leq \|x_n - p\|^2 - (1 - \alpha_n)\|x_n - p\|^2 - \alpha_n\|\gamma_n x_n + (1 - \gamma_n)Tx_n - p\|^2 \\
 &= \alpha_n\left(\|x_n - p\|^2 - \|\gamma_n x_n + (1 - \gamma_n)Tx_n - p\|^2\right) \\
 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
 \end{aligned} \tag{3.37}$$

Since $\liminf_{n \rightarrow \infty} \alpha_n > 0$, it follows from (3.28) that

$$\lim_{n \rightarrow \infty} \left(\|x_n - p\|^2 - \|\gamma_n x_n + (1 - \gamma_n)Tx_n - p\|^2\right) = 0. \tag{3.38}$$

Since T is a nonexpansive mapping, we have

$$\begin{aligned} & \|\gamma_n x_n + (1 - \gamma_n)Tx_n - p\|^2 \\ &= \gamma_n \|x_n - p\|^2 + (1 - \gamma_n) \|Tx_n - p\|^2 - \gamma_n(1 - \gamma_n) \|x_n - Tx_n\|^2 \\ &\leq \|x_n - p\|^2 - \gamma_n(1 - \gamma_n) \|x_n - Tx_n\|^2, \end{aligned} \quad (3.39)$$

and hence

$$\gamma_n(1 - \gamma_n) \|x_n - Tx_n\|^2 \leq \|x_n - p\|^2 - \|\gamma_n x_n + (1 - \gamma_n)Tx_n - p\|^2. \quad (3.40)$$

Since $\liminf_{n \rightarrow \infty} \gamma_n(1 - \gamma_n) > 0$, from (3.38), we get

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (3.41)$$

Since $\{x_{n_i}\}$ converges weakly to q , we have $q \in F(T)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to v . Then, we have $q = v$. In fact, if $q \neq v$, then we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - q\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - q\| \\ &< \lim_{i \rightarrow \infty} \|x_{n_i} - v\| = \lim_{n \rightarrow \infty} \|x_n - v\| = \lim_{j \rightarrow \infty} \|x_{n_j} - v\| \\ &< \lim_{j \rightarrow \infty} \|x_{n_j} - q\| = \lim_{n \rightarrow \infty} \|x_n - q\|. \end{aligned} \quad (3.42)$$

This is a contradiction. So, we have $q = v$. Therefore, we conclude that $\{x_n\}$ converges weakly to $q \in F(S) \cap F(T)$. $\square$

For nonspreading mapping S , we have $k = 0$ reduces the followings.

Corollary 3.2 (cf. [2]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $S : C \rightarrow C$ be a nonspreading mapping and let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Sx_n) + \alpha_n(\gamma_n x_n + (1 - \gamma_n)Tx_n) \end{aligned} \quad (3.43)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$. Then, the followings hold:

- (1) if $\liminf_{n \rightarrow \infty} \alpha_n(\beta_n - \gamma_n) > 0$, $\sum_{n=1}^{\infty} \alpha_n(1 - \gamma_n) < \infty$ and $1 < (2 - \alpha_n)\beta_n + \alpha_n\gamma_n$, then $\{x_n\}$ converges weakly to $q \in F(S)$,
- (2) if $\beta_n > \gamma_n$, $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, $2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n) > 0$ and $\liminf_{n \rightarrow \infty} \alpha_n(\beta_n - \gamma_n)(2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n)) > 0$, then $\{x_n\}$ converges weakly to $q \in F(T)$,

(3) if $\liminf_{n \rightarrow \infty} \alpha_n > 0$, $\liminf_{n \rightarrow \infty} (1 - \alpha_n) > 0$, $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n (1 - \gamma_n) > 0$, then $\{x_n\}$ converges weakly to $q \in F(S) \cap F(T)$.

Corollary 3.3 (cf. [2, 10]). Let C be a nonempty closed convex subset of a Hilbert space H and let $S : C \rightarrow C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) S x_n \end{aligned} \tag{3.44}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$. If $\liminf_{n \rightarrow \infty} \alpha_n > 0$, then $\{x_n\}$ converges weakly to $q \in F(S)$.

Proof. Putting $\beta_n = 0$, $\gamma_n = 1$ for $n \in \mathbb{N}$ in Theorem 3.1, we get the conclusion. $\square$

Corollary 3.4. Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T x_n \end{aligned} \tag{3.45}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$. If $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ converges weakly to $q \in F(T)$.

Proof. Putting $\beta_n = 1$, $\gamma_n = 0$ for $n \in \mathbb{N}$ in Theorem 3.1, we get the conclusion. $\square$

4. Some Open Problems

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real-valued functional on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st), \quad (r(s)f)(t) = f(ts) \tag{4.1}$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let X be a subspace of $l^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. It is well known that μ is mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s) \tag{4.2}$$

for each $f \in X$.

Let X be a translation invariant subspace of $l^\infty(S)$ (i.e., $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then, a mean μ on X is said to be *left invariant* (resp. *right invariant*) if

$$\mu(l(s)f) = \mu(f) \quad (\text{resp.} \quad \mu(r(s)f) = \mu(f)) \tag{4.3}$$

for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [11–14]. X is said to be *left* (resp., *right*) *amenable* if X has a left (resp. right) invariant mean. X is amenable if X is left and right amenable. Moreover, $l^\infty(S)$ is amenable when S is a commutative semigroup or a solvable group. However, the free group or semigroup of two generators is not left or right amenable (see [15–17]). A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically left* (resp., *right*) *invariant* if

$$\lim_\alpha (\mu_\alpha(l(s)f) - \mu_\alpha(f)) = 0 \quad \left(\text{resp. } \lim_\alpha (\mu_\alpha(r(s)f) - \mu_\alpha(f)) = 0 \right) \quad (4.4)$$

for each $f \in X$ and $s \in S$, and it is said to be left (resp., right) *strongly asymptotically invariant* (or *strong regular*) if

$$\lim_\alpha \|l^*(s)\mu_\alpha - \mu_\alpha\| = 0 \quad \left(\text{resp. } \lim_\alpha \|r^*(s)\mu_\alpha - \mu_\alpha\| = 0 \right) \quad (4.5)$$

for each $s \in S$, where $l^*(s)$ and $r^*(s)$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day in [15] where they were called *weak* invariant* and *norm invariant*, respectively.

It is easy to see that if a semigroup S is left (resp. right) amenable, then the semigroup $S' = S \cup \{e\}$ where $es' = s'e = s'$ for all $s' \in S$ is also left (resp. right) amenable and conversely.

From now on S denotes a semigroup with an identity e . S is called *left reversible* if any two right ideals of S have nonvoid intersection, that is, $aS \cap bS \neq \emptyset$ for $a, b \in S$. In this case, $(S, \leq)$ is a directed system when the binary relation “ $\leq$ ” on S is defined by $a \leq b$ if and only if $\{a\} \cup aS \supseteq \{b\} \cup bS$ for $a, b \in S$. It is easy to see that $t \leq ts$ for all $t, s \in S$. Further, if $t \leq s$, then $pt \leq ps$ for all $p \in S$. The class of left reversible semigroup includes all groups and commutative semigroups. If a semigroup S is left amenable, then S is left reversible. But the converse is not true ([18–23]).

Let S be a semigroup and let C be a closed and convex subset of E . Let $F(T)$ denote the fixed point set of T . Then, $\mathfrak{S} = \{T(s) : s \in S\}$ is called a *representation of S as nonexpansive mappings on C* if $T(s)$ is nonexpansive with $T(e) = I$ and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathfrak{S})$ the set of common fixed points of $\{T(s) : s \in S\}$, that is,

$$F(\mathfrak{S}) = \bigcap_{s \in S} F(T(s)) = \bigcap_{s \in S} \{x \in C : T(s)x = x\}. \quad (4.6)$$

We know that if μ is a mean on X and if for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^* \rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that $\mu \langle T(\cdot)x, x^* \rangle = \langle x_0, x^* \rangle$ for each $x^* \in E^*$. We denote such a point x_0 by $T_\mu x$. Note that $T_\mu x$ is contained in the closure of the convex hull of $\{T(s)x : s \in S\}$ for each $x \in C$. Note that $T_\mu z = z$ for each $z \in F(\mathfrak{S})$; see [12–14, 17, 20, 22].

Open Problems.

- (1) Does Theorem 3.1 hold for a Banach space? (cf. [24, 25])
- (2) Does Theorem 3.1 hold a commutative or amenable or reversible semigroup of nonexpansive mappings in place of T using asymptotically invariant nets?

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