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Research Article

A Rogalski-Cornet Type Inclusion Theorem Based on Two Hausdorff Locally Convex Vector Spaces

Yingfan Liu and Youguo Wang

Department of Mathematics, College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210046, China

Correspondence should be addressed to Yingfan Liu, yingfanliu@hotmail.com

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A Rogalski-Cornet type inclusion theorem based on two Hausdorff locally convex vector spaces is proved and composed of two parts. An example is presented to show that the associated set-valued map in the first part does not need any conventional continuity conditions including upper hemicontinuous. As an application, solvability results regarding an abstract von Neumann inclusion system are obtained.

1. Introduction

The Rogalski-Cornet inclusion theorem, based on a Hausdorff locally convex vector space and as an useful tool to deal with some inclusion problems described by an upper hemicontinuous set-valued map, can be stated as follows.

Theorem 1.1 (see [1]). *Let X be a convex compact subset of a Hausdorff locally convex vector space U, and S an upper hemicontinuous set-valued map from X to U with nonempty closed convex values. If S is outward, that is, $\forall p \in U^*$, $\forall x \in X$ with $\langle p, x\rangle = \inf_{y\in X}\langle p, y\rangle$, $\langle p, x\rangle \geq \inf_{y\in Sx}\langle p, y\rangle$, then $SX = \cup_{x\in X} Sx \supseteq X$.*

Two comments are in order.

(1) Theorem 1.1 can be used to deal with those inclusion problems described by a set-valued map whose domain and range are contained in a same space. For example, if R^n_+ denotes the set of all nonnegative vectors of the n dimensional Euclidean space R^n, $c \in R^n_+$ is an expected demand of the market, $X \subset R^n_+$ some enterprise's admission output bundle set, and A or F is the enterprise's consuming map or correspondence (namely, set-valued map)

from X to R^n_+. Then a class of Leontief type input-output inclusion system, as a nonlinear extension to the classical input-output equation [2–4], is composed of

$$\begin{aligned} &\text{(a) } x \in X \quad \text{s.t. } x - Ax = c, \\ &\text{(b) } x \in X \quad \text{s.t. } x - Fx \ni c, \end{aligned} \tag{1.1}$$

and has been studied by Sandberg [5] and Fujimoto [6] with the nonlinear analysis methods. Moreover, some primary extensions to Theorem 1.1, also based on a Hausdorff locally convex vector space, have been made by Liu and Zhang [7] such that the associated correspondence S no longer needs the upper hemicontinuous condition, and the obtained results (see [7]) also have been used to deal with the solvability of (1.1)(b) (see [8]).

(2) However, if a problem is concerned with two different spaces, then Theorem 1.1 is generally useless even if this problem can be changed into an inclusion system. For example, assume that $c \in R^m_+$ is an expected demand of the market, $X \subset R^n_+$ is some enterprise's raw material bundle set, and B, A (or G, F) are the enterprise's output and input maps (or correspondences) from X to R^m_+, respectively. If the semiordering in R^m is defined by $y^1 \geq y^2$ iff $y^1 - y^2 \in R^m_+$, then the following von Neumann type input-output inequality system, composed of a single-valued inequality and a set-valued inequality,

$$\begin{aligned} &\text{(a) } x \in X \quad \text{s.t. } Bx - Ax \geq c, \\ &\text{(b) } x \in X \quad \text{s.t. } \exists y \in Gx,\ \exists z \in Fx \text{ with } y - z \geq c, \end{aligned} \tag{1.2}$$

has been studied by Liu and Zhang [9], and Liu [10, 11] with the nonlinear analysis methods including the minimax and saddle point techniques attributed to [1, 12, 13]. We claim that (1.2) also includes some economic growth problems. For example, if we restrict $\lambda \geq \lambda_0 > 0$ (where λ_0 may be viewed as the minimal growth fact of the output regarding input accepted by the enterprise) and replace A by λA, respectively, then (1.2)(a) reduces to a single-valued von Neumann economic growth model $\lambda \geq \lambda_0$, s.t. $\exists x \in X$ with $Bx \geq \lambda Ax + c$, which has been studied by Medvegyev [14], and Bidard and Hosoda [15]. Some other research regarding economic growth has also been made by Jones [16, 17], and Jones, Williams [18]. It is easy to see that (1.2)(b) yields a set-valued economic growth problem. However, up to now, no any corresponding references can be seen. This shows that to study (1.2)(b) is also useful.

Returning to (1.2), if we set $S_1 x = Ax + R^m_+$ and $S_2 x = Fx + R^m_+$ for $x \in X$, then (1.2) equals to the following inclusion system:

$$\begin{aligned} &\text{(a) } x \in X \quad \text{s.t. } Bx - S_1 x \ni c, \\ &\text{(b) } x \in X \quad \text{s.t. } Gx - S_2 x \ni c, \end{aligned} \tag{1.3}$$

which is difficult to be handled by Theorem 1.1 except for $n = m$. Moreover, the following

$$\begin{aligned} &\text{(a) } x \in X \quad \text{s.t. } Bx - Ax = c, \\ &\text{(b) } x \in X \quad \text{s.t. } Gx - Fx \ni c \end{aligned} \tag{1.4}$$

(as an inclusion system described by the preceding B, A or G, F, which is clearly more practical than (1.1) and more fine than (1.2)) can also hardly be tackled by Theorem 1.1 even along with the minimax method.

Now let $v \in R^m$, $Y \subset R^m$, and $S : X \subset R^n \to R^m$ be a set-valued map, then (1.2)–(1.4) (equivalently, (1.3)-(1.4)) can be viewed as the special examples of the first inclusion in the following von Neumann type inclusion system

$$\begin{aligned}
&\text{(a) } x \in X \quad \text{s.t. } Sx \ni v, \text{ that is, } v \in SX,\\
&\text{(b) } x \in X \quad \text{s.t. } Sx \cap Y \neq \emptyset, \text{ that is, } SX \cap Y \neq \emptyset,\\
&\text{(c) } \forall y \in Y,\ \exists x \in X \quad \text{s.t. } Sx \ni y, \text{ that is, } Y \subseteq SX.
\end{aligned} \tag{1.5}$$

We claim (1.5) an abstract von Neumann inclusion system if R^n and R^m are replaced by two Hausdorff locally convex vector spaces U and V, respectively.

In the sequel, we attempt to extend Theorem 1.1 to a new situation, that is, to present a Rogalski-Cornet type theorem composed of existence and continuity parts (as the main result proved in Section 3), such that the abstract von Neumann inclusion system (1.5) could be tackled, which means that the domain and range of S may probably be contained in two different Hausdorff locally convex vector spaces U and V, respectively. Since Theorem 15.1.9 in [1] and Theorem 6.4.10 in [12] are such type of results while their correspondences need the upper hemicontinuous assumption, and a new paper completed by Lignola [19] provides some Ky Fan inequalities and Nash equilibrium points without semicontinuity and compactness, we concentrate our attention to the instance such that the associated correspondence no longer needs any conventional continuity conditions including upper hemicontinuous.

The paper is arranged as follows. We introduce some necessary concepts in the next section. In Section 3, we first prove a Rogalski-Cornet type inclusion theorem based on two different Hausdorff locally convex vector spaces, followed by an example to show that the first part of this theorem does not need any conventional continuity conditions such as upper semicontinuous, lower semicontinuous, and upper hemicontinuous. Then, as an application of this theorem, we provide some solvability results for (1.5). Finally we present the conclusion in Section 4.

2. Preliminary

In the sections below, without any special explanations, we always assume U, V are two Hausdorff locally convex vector spaces, U^*, V^* their duals and $\langle \cdot, \cdot \rangle$ the duality paring on $\langle U^*, U \rangle$ or $\langle V^*, V \rangle$. We need some concepts with respect to a function f defined on U or V such as convex or concave, upper semicontinuous or lower semicontinuous (in short, u.s.c. or l.s.c.) and continuous, and concepts to a set-valued map S from a subset $X \subset U$ to V including closed, upper, and lower semicontinuous (in short, u.s.c. and l.s.c.) and continuous, whose definitions can be consulted in [1, 12, 13], so the details are omitted here.

Let $Z \subset V$ and $q \in V^*$. Denote

$$\sigma^{\flat}(Z,q) = \inf_{y\in Z}\langle q,y\rangle, \qquad \sigma^{\sharp}(Z,q) = \sup_{y\in Z}\langle q,y\rangle,$$
$$\partial(Z,q) = \left\{y \in Z : \langle q,y\rangle = \sigma^{\flat}(Z,q)\right\} \text{ (the support set of } Z \text{ at } q). \tag{2.1}$$

We also need the following.

Definition 2.1. (see [1, 12, 13]). Let U be a Hausdorff topological space and S a set-valued map from U to V. Then S is said to be upper hemicontinuous (in short, u.h.c.) if for any $q \in V^*$, $x \mapsto \sigma^{\sharp}(Sx,q) = \sup_{v\in Sx}\langle q,v\rangle$ is u.s.c. on U.

3. Main Theorem and Application to (1.5)

In this section, we always assume that

$$\begin{aligned}
&\text{(a) } X \subset U,\ Y \subset V \text{ are two nonempty convex compact subsets,}\\
&\text{(b) } S : X \to V \text{ is a set valued map with nonempty closed convex values,}\\
&\text{(c) } L \in \mathcal{L}(U,V) \text{ is a continuous linear map from } U \text{ to } V.
\end{aligned} \tag{3.1}$$

Under some additional assumptions, we first prove a Rogalski-Cornet type inclusion theorem for a set-valued map S from $X \subset U$ to V with and without u.h.c. condition, then present an application of this theorem to the abstract von Neumann inclusion system (1.5).

3.1. Main Theorem and a Counterexample

The Rogalski-Cornet type inclusion theorem we will prove is the following.

Theorem 3.1.

(A) Results without u.h.c. Condition.

(i) *Existence. (1) Assume that*

$$\begin{aligned}
&\text{(a) } \forall q \in V^*, \quad \left\{x \in X : \sigma^{\flat}(Sx,q) \le \sigma^{\sharp}(Y,q)\right\} \text{ is closed,}\\
&\text{(b) } \forall q \in V^*, \quad \forall x \in X \text{ with } Lx \in \partial(LX,q), \quad \sigma^{\flat}(Sx,q) \le \sigma^{\sharp}(Y,q),
\end{aligned} \tag{3.2}$$

then $Y \cap SX \neq \emptyset$, that is, there exist $y \in Y$ and $x \in X$ such that $y \in Sx$.
(2) If Y_0 is the set of all $y \in Y$ such that

$$\begin{aligned}
&\text{(a) } \forall q \in V^*, \quad \left\{x \in X : \sigma^{\flat}(Sx,q) \le \langle q,y\rangle\right\} \text{ is closed,}\\
&\text{(b) } \forall q \in V^*, \quad \forall x \in X \text{ with } Lx \in \partial(LX,q), \quad \sigma^{\flat}(Sx,q) \le \langle q,y\rangle,
\end{aligned} \tag{3.3}$$

then $Y_0 \subset SX$, that is, for each $y \in Y_0$, there exists $x \in X$ such that $y \in Sx$.

(ii) *Continuity. Assume that Y_1, the set of all $y \in Y$ such that*

$$\begin{aligned} &\text{(a) } \forall q \in V^*,\ \forall c \geq 0, \quad \left\{x \in X : \sigma^{\flat}(Sx, q) \leq \langle q, y\rangle + c\right\} \text{ is closed,} \\ &\text{(b) } (3.3)(b) \text{ holds also,} \end{aligned} \tag{3.4}$$

is nonempty, then $Y_1 \subseteq SX$. Moreover, Y_1 is compact, and the inverse of S restricting to Y_1 defined by

$$S_{Y_1}^{-1} : Y_1 \longrightarrow X : y \longmapsto S_{Y_1}^{-1}(y) = \{x \in X : y \in Sx\} \tag{3.5}$$

is a u.s.c. and u.h.c. set-valued map.

(B) Results with u.h.c. Condition.

Assume that S is u.h.c., then the following are true.

(i) *If* (3.2)*(b) holds, then $Y \cap SX$ is nonempty and compact, and the inverse of S restricting to $Y \cap SX$ defined by*

$$S_{Y\cap SX}^{-1} : Y \cap SX \longrightarrow X : y \longmapsto S_{Y\cap SX}^{-1}(y) = \{x \in X : y \in Sx\} \tag{3.6}$$

is a u.s.c. and u.h.c. set-valued map.

(ii) *If Y_1, the set of all $y \in Y$ satisfying* (3.3)*(b), is nonempty, then Y_1 is compact with $Y_1 \subset SX$, and the inverse of S restricting to Y_1 defined by* (3.5) *is also a u.s.c and u.h.c. set-valued map.*

To prove this theorem, we need some known results and state them in lemmas as follows.

Lemma 3.2 (see [1]). *Let K be a convex compact subset of U and let $\phi : K \times K \to R$ satisfy that $\forall y \in K$, $x \mapsto \phi(x, y)$ is lower semicontinuous, $\forall x \in K$, $y \mapsto \phi(x, y)$ is quasi-concave, and $\sup_{y\in K}\phi(y, y) \leq 0$. Then there exists $\overline{x} \in K$ such that $\sup_{y\in K}\phi(\overline{x}, y) \leq 0$.*

Lemma 3.3 (see [12]). *Let U be a Hausdorff topological space, V a compact Hausdorff topological space, and S a closed set-valued map from U to V. Then S is u.s.c.*

Lemma 3.4 (see [12]). *Let U be a Hausdorff topological space, V supplied with the weak topology $\sigma(V, V^*)$, and S a u.s.c. set-valued map from U to V. Then S is u.h.c.*

Remark 3.5. Since the original vector topology τ on V is stronger than the weak topology $\sigma(V, V^*)$, this Lemma is also true if V is supplied with the original topology τ.

Lemma 3.6 (see [12]). *Let U be a Hausdorff topological space, V supplied with the weak topology, and let S be a u.h.c. set-valued map from U to V with nonempty closed convex values. Then the graph of S denoted by graph S is closed, that is, S is closed.*

With these lemmas, we proceed to prove Theorem 3.1.

Proof. Proof of Part A. (i) First we prove (i).

(1) Under the assumptions (3.1) and (3.2), we will prove $Y \cap SX \neq \emptyset$ by contradiction. If $Y \cap Sx = \emptyset$ for all $x \in X$, then by (3.1), we see that for each $x \in X$, $Sx - Y$ is closed convex with $0 \overline{\in} Sx - Y$. So the Hahn-Banach separation theorem implies that

$$\forall x \in X,\ \exists q = q_x \in V^* \quad \text{s.t. } \sigma^{\flat}(Sx - Y, q) = \inf_{v \in Sx - Y} \langle q, v \rangle > \langle q, 0 \rangle = 0. \tag{3.7}$$

Setting $X^{\flat}(q) = \{x \in X : \sigma^{\flat}(Sx - Y, q) > 0\}$ for $q \in V^*$, from (3.7) we have $X = \bigcup_{q \in V^*} X^{\flat}(q)$. Since X is compact and $X^{\flat}(q) = \{x \in X : \sigma^{\flat}(Sx, q) - \sigma^{\sharp}(Y, q) > 0\} = X \setminus \{x \in X : \sigma^{\flat}(Sx, q) \leq \sigma^{\sharp}(Y, q)\}$, by (3.2)(a) we know that $\{X^{\flat}(q) : q \in V^*\}$ forms an open covering of X. Therefore, $X = \bigcup_{i=1}^{n} X^{\flat}(q_i)$ holds for some finite subset $\{q_1, q_2, \ldots, q_n\} \subset V^*$, and there exists a continuous partition of unit $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ associated to this finite covering of X such that

$$\begin{aligned} &\text{(a) } \alpha_i(x)\ (1 \leq i \leq n) \text{ are nonnegative continuous on } X \text{ with } \sum_{i=1}^{n} \alpha_i(x) = 1, \\ &\text{(b) } \operatorname{supp} \alpha_i \subset X^{\flat}(q_i), \text{ hence, } \sigma^{\flat}(Sx - Y, q_i) > 0 \text{ if } \alpha_i(x) > 0\ (1 \leq i \leq n), \end{aligned} \tag{3.8}$$

where $\operatorname{supp} \alpha_i = \overline{\{x \in X : \alpha_i(x) > 0\}}$ is the closure of the set $\{x \in X : \alpha_i(x) > 0\}$.

Define $\varphi(\cdot, \cdot)$ on $X \times X$ by

$$\varphi(x, y) = \left\langle \sum_{i=1}^{n} \alpha_i(x) q_i, L(x - y) \right\rangle, \quad (x, y) \in X \times X. \tag{3.9}$$

Clearly, φ satisfies the assumptions of Lemma 3.2. Indeed, it is easy to see that for any $y \in X$, $x \mapsto \varphi(x, y)$ is continuous, for any $x \in X$, $y \mapsto \varphi(x, y)$ is affine, and $\sup_{y \in X} \varphi(y, y) = 0$. So there exists $\overline{x} \in X$ such that $\sup_{y \in X} \varphi(\overline{x}, y) \leq 0$. Take $\overline{q} = \sum_{i=1}^{n} \alpha_i(\overline{x}) q_i$ and $I = \{i \in \{1, 2, \ldots, n\} : \alpha_i(\overline{x}) > 0\}$, then from (3.8) and (3.9) we know that $I \neq \emptyset$ and $0 \leq \langle \overline{q}, L\overline{x} \rangle - \inf_{y \in X} \langle \overline{q}, Ly \rangle = \sup_{y \in X} \langle \overline{q}, L(\overline{x} - y) \rangle = \sup_{y \in Y} \varphi(\overline{x}, y) \leq 0$. This further implies

$$\begin{aligned} &\text{(a) } \langle \overline{q}, L\overline{x} \rangle = \inf_{y \in X} \langle \overline{q}, Ly \rangle = \sigma^{\flat}(LX, \overline{q}), \quad \text{that is, } L\overline{x} \in \partial(LX, \overline{q}), \\ &\text{(b) } \forall i \in I, \quad \alpha_i(\overline{x}) > 0, \quad \sigma^{\flat}(S\overline{x} - Y, q_i) > 0, \quad \overline{q} = \sum_{i \in I} \alpha_i(\overline{x}) q_i. \end{aligned} \tag{3.10}$$

Associating this with (3.2)(b), we obtain that

$$\begin{aligned} 0 &\geq \sigma^{\flat}(S\overline{x}, \overline{q}) - \sigma^{\sharp}(Y, \overline{q}) = \sigma^{\flat}\left(S\overline{x} - Y, \sum_{i \in I} \alpha_i(\overline{x}) q_i \right) \\ &\geq \sum_{i \in I} \alpha_i(\overline{x}) \sigma^{\flat}(S\overline{x} - Y, q_i) > 0. \end{aligned} \tag{3.11}$$

This is a contradiction. Therefore, $Y \cap SX \neq \emptyset$.

(2) Statement (2) immediately follows from (1) because for each $y \in Y_0$, $\{y\}$ is compact, and (3.2) holds for $Y = \{y\}$ because of the assumption (3.3).

(ii) Then we prove (ii).

By (3.4) and statement (i)(2), it is easy to see that $Y_1 \subseteq SX$, so the left is to show that Y_1 is compact and $S_{Y_1}^{-1}$ defined by (3.5) is u.s.c. and u.h.c.

(1) Since Y is compact and $Y_1 \subseteq Y$, it is sufficient to verify that Y_1 is closed, that is, to verify that if $\{y_\lambda : \lambda \in \Lambda\} \subset Y_1$ is a generalized sequence such that $y_\lambda \to y_0 \in Y$, then $y_0 \in Y_1$. To this end, for each $q \in V^*$, $y \in Y$ and each $c \geq 0$, we define

$$\begin{aligned} &\text{(a) } X^\flat(S,q,y,c) = \left\{x \in X : \sigma^\flat(Sx,q) \leq \langle q,y\rangle + c\right\}, \\ &\text{(b) } X^\flat(S,q,y) = X^\flat(S,q,y,0), \end{aligned} \tag{3.12}$$

and will prove that

$$\begin{aligned} &\text{(a) } \forall q \in V^*,\ \forall c \geq 0, \quad X^\flat(S,q,y_0,c) \text{ is closed}, \\ &\text{(b) } \forall q \in V^*,\ \forall x \in X \text{ with } Lx \in \partial(LX,q), \quad x \in X^\flat(S,q,y_0), \end{aligned} \tag{3.13}$$

which implies by (3.4) and (3.12) that $y_0 \in Y_1$, and thus Y_1 is closed.

(a) First we prove (3.13)(a). Suppose that $q \in V^*$, $c \geq 0$ are fixed, and $\{x_\alpha : \alpha \in I\} \subset X^\flat(S,q,y_0,c)$ is a generalized sequence such that $x_\alpha \to x_0$. By (3.12)(a) we have

$$\sigma^\flat(Sx_\alpha,q) \leq \langle q,y_0\rangle + c, \quad \forall \alpha \in I. \tag{3.14}$$

Since $y_\lambda \to y_0$, for each $\varepsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that

$$\left|\langle q, y_{\lambda_0} - y_0\rangle\right| < \frac{\varepsilon}{2}. \tag{3.15}$$

Associating this with (3.14), we obtain

$$\begin{aligned} \sigma^\flat(Sx_\alpha,q) - \langle q,y_{\lambda_0}\rangle - c - \frac{\varepsilon}{2} &\leq \sigma^\flat(Sx_\alpha,q) - \langle q,y_0\rangle - c - \left(\frac{\varepsilon}{2} - \left|\langle q,y_0 - y_{\lambda_0}\rangle\right|\right) \\ &\leq 0 \quad \text{for any } \alpha \in I. \end{aligned} \tag{3.16}$$

As $y_{\lambda_0} \in Y_1$, from (3.4)(a), (3.12)(a), and (3.16), we conclude that $\{x_\alpha\} \subseteq X^\flat(S,q,y_{\lambda_0},c+\varepsilon/2)$ and $x_0 \in X^\flat(S,q,y_{\lambda_0},c+\varepsilon/2)$ because $X^\flat(S,q,y_{\lambda_0},c+\varepsilon/2)$ is closed. Hence, $\sigma^\flat(Sx_0,q) \leq \langle q,y_{\lambda_0}\rangle + c + \varepsilon/2$, which together with (3.15) yields

$$\sigma^\flat(Sx_0,q) - \langle q,y_0\rangle - c - \varepsilon \leq \sigma^\flat(Sx_0,q) - \langle q,y_{\lambda_0}\rangle - c - \frac{\varepsilon}{2} - \left(\frac{\varepsilon}{2} - \left|\langle q,y_{\lambda_0} - y_0\rangle\right|\right) \leq 0. \tag{3.17}$$

By taking $\varepsilon \to 0$, from (3.17) we obtain that $\sigma^\flat(Sx_0, q) \le \langle q, y_0\rangle + c$. Hence, $x_0 \in X^\flat(S, q, z_0, c)$ and (3.13)(a) follows.

(b) Then we prove (3.13)(b). Suppose that $q \in V^*$, $x \in X$ satisfy $Lx \in \partial(LX, q)$, and (3.15) holds for this q and some $\lambda_0 \in \Lambda$. Since $y_{\lambda_0} \in Y_1$, by (3.4)(b), we have $\sigma^\flat(Sx, q) \le \langle q, y_{\lambda_0}\rangle$. In view of (3.15), we get

$$\sigma^\flat(Sx, q) - \langle q, y_0\rangle \le \sigma^\flat(Sx, q) - \langle q, y_{\lambda_0}\rangle + \left|\langle q, y_{\lambda_0} - y_0\rangle\right| \le \frac{\varepsilon}{2}. \tag{3.18}$$

Thus, $\sigma^\flat(Sx, q) \le \langle q, y_0\rangle$ because ε is arbitrary. This shows that $x \in X^\flat(S, q, y_0)$ and (3.13)(b) is also true. Therefore, $y_0 \in Y_1$.

(2) To prove the continuity of $S_{Y_1}^{-1}$, by Lemmas 3.3 and 3.4 and Remark 3.5, it is sufficient to verify that $S_{Y_1}^{-1} : Y_1 \to X(\subset U)$ is closed because X is compact. Assume that $\{(y_\alpha, x_\alpha) : \alpha \in I\} \subset \text{graph } S_{Y_1}^{-1}$ satisfy $(y_\alpha, x_\alpha) \to (y_0, x_0) \in Y_1 \times X$, then $y_\alpha \in Sx_\alpha$ $(\alpha \in I)$. This implies that

$$\forall q \in V^*,\ \forall \alpha \in \Lambda, \quad \sigma^\flat(Sx_\alpha, q) \le \langle q, y_\alpha\rangle. \tag{3.19}$$

Let $q \in V^*$ be fixed. Since $y_\alpha \to y_0 \in Y_1$, for any $\varepsilon > 0$, there exists $\alpha_0 \in I$ such that $|\langle q, y_\alpha - y_0\rangle| < \varepsilon$ for $\alpha \succ \alpha_0$, which together with (3.19) yields

$$\sigma^\flat(Sx_\alpha, q) - \langle q, y_0\rangle - \varepsilon \le \sigma^\flat(Sx_\alpha, q) - \langle q, y_\alpha\rangle - \left(\varepsilon - \left|\langle q, y_\alpha - y_0\rangle\right|\right) \le 0 \quad (\alpha \succ \alpha_0). \tag{3.20}$$

Therefore, $\{x_\alpha : \alpha \succ \alpha_0\} \subset X^\flat(S, q, y_0, \varepsilon)$. This implies $x_0 \in X^\flat(S, q, y_0, \varepsilon)$ because $y_0 \in Y_1$ and thus $X^\flat(S, q, y_0, \varepsilon)$ is closed. Hence, $\sigma^\flat(Sx_0, q) \le \langle q, y_0\rangle + \varepsilon$. By letting $\varepsilon \to 0$, it follows that $\sigma^\flat(Sx_0, q) \le \langle q, y_0\rangle$ holds for any fixed $q \in V^*$. Since Sx_0 is closed convex by assumption (3.1), we have $y_0 \in Sx_0$ (also thanks to the Hahn-Banach separation theorem). Combing this with the fact $y_0 \in Y_1$, we conclude that $(y_0, x_0) \in \text{graph } S_{Y_1}^{-1}$. Therefore, $S_{Y_1}^{-1}$ defined by (3.5) is closed, and statement (ii) follows.

Proof of Part B. Since S is u.h.c., by Definition 2.1 we know that for each $q \in V^*$, $x \mapsto \sigma^\flat(Sx, q) = -\sigma^\sharp(Sx, -q)$ is lower semicontinuous. Hence the lower sections $\{x \in X : \sigma^\flat(Sx, q) \le \sigma^\sharp(Y, q)\}$ and $\{x \in X : \sigma^\flat(Sx, q) \le \langle q, y\rangle + c\}$ of the function $x \mapsto \sigma^\flat(Sx, q)$ are closed in X for all $q \in V^*$, $y \in Y$ and all $c \in R^1$. This implies that all the conditions (a) from (3.2) to (3.4) are satisfied, and thus all the statements of part (A) are true. So the remaining is to show that both $Y \cap SX$ and graph $S_{Y\cap SX}^{-1}$ (where $S_{Y\cap SX}^{-1}$ is defined by (3.6)) are closed also thanks to Lemmas 3.3 and 3.4 and Remark 3.5.

(1) Assume that $\{y_\alpha : \alpha \in I\}$ is a generalized sequence of $Y \cap SX$ with $y_\alpha \to y_0$, then $y_0 \in Y$ (because Y is compact), and for each $\alpha \in I$ there exists $x_\alpha \in X$ such that $y_\alpha \in Sx_\alpha$. As X is compact, choosing a generalized subsequence if necessary, we may assume $x_\alpha \to x_0 \in X$. On the other hand, it is easy to see that in Definition 2.1, if τ denotes the original vector topology on V, then V can be supplied with any compatible topologies $\mathbb{T}$ of τ including the weak topology $\sigma(V, V^*)$ because $(V, \mathbb{T})^* = (V, \sigma(V, V^*))^* = V^*$. Associating this with (3.1)(b) and using Hahn-Banach's separation theorem, we conclude that $S : X \subset U \to V$ is also a u.h.c. correspondence with nonempty closed convex values when V is supplied with the weak topology $\sigma(V, V^*)$, which implies by Lemma 3.6 that S is closed. Therefore, $y_0 \in Sx_0$ and $Y \cap SX$ is closed.

(2) Assume that $\{(y_\alpha, x_\alpha) : \alpha \in \Lambda\} \subset \text{graph } S^{-1}_{Y\cap SX}$ is a generalized sequence such that $(y_\alpha, x_\alpha) \to (y_0, x_0)$, then $y_0 \in Y \cap SX$ (because $Y \cap SX$ is closed), $x_0 \in X$ and $y_\alpha \in Sx_\alpha$ $(\alpha \in \Lambda)$. Thus, $y_0 \in Sx_0$ by the closeness of S, and $S^{-1}_{Y\cap SX}$ is closed. This completes the proof. □

Remark 3.7. We claim that the part (A) of Theorem 3.1 does not need any conventional continuity conditions such as u.s.c., l.s.c. and u.h.c. See the following counterexample.

Example 3.8. Let $U = R, V = R^n$ $(n \geq 2)$, $e_i = (\overbrace{0,\ldots,0,1}^{i},0,\ldots,0)$ $(i = 1,2,\ldots,n)$, $X = [0,1]$, $Y = [0,1]^n$, and let Q_0 be the set of all rational numbers of $[0,1]$, $P_0 = [0,1] \setminus Q_0$. Assume that $k \in \{1,2,\ldots,n\}$ is a fixed number, L a single-valued map from R^1 to R^n, and S a set-valued map from X to R^n defined by

$$Lx = xe_k, \quad \text{for } x \in R^1, \qquad Sx = \begin{cases} [x-3, x+4]^n & \text{if } x \in Q_0, \\ [x-4, x+3]^n & \text{if } x \in P_0. \end{cases} \tag{3.21}$$

Then we have the following (i) It is easy to see that $L \in \mathcal{L}(R^1, R^n)$ (i.e., L is a continuous linear map from R^1 to R^n), Sx is a nonempty convex compact subset of R^n for each $x \in X$, and S is not u.s.c. or l.s.c. at any point of X. It can also be shown that S is not u.h.c. on X. Indeed, if $q = (q_1, q_2, \ldots, q_n) \in V^* = R^n$ with $q_i > 0$ $(i = 1,2,\ldots,n)$, then $x \mapsto \sigma^\#(Sx, q)$ is not u.s.c. at any points of P_0 because

$$\sigma^\#(Sx, q) = \sup_{(y_1,y_2,\ldots,y_n)\in Sx} \sum_{i=1}^n \langle q_i, y_i \rangle = \begin{cases} \sum_{i=1}^n q_i(x+4) & \text{if } x \in Q_0, \\ \sum_{i=1}^n q_i(x+3) & \text{if } x \in P_0. \end{cases} \tag{3.22}$$

(ii) Now we verify that (3.2)–(3.4) hold for this example.

(1) It is easy to see that $\forall p \in R^1$, $\forall x \in X = [0,1]$, and $\forall y \in [0,1]$, we have

$$\begin{aligned} \sigma^\flat([x-3, x+4] - [0,1], p) &= \begin{cases} p(x-4) < 0, & \text{if } p > 0, \\ p(x+4) < 0, & \text{if } p < 0, \end{cases} \\ \sigma^\flat([x-3, x+4] - y, p) &= \begin{cases} p(x-3-y) < 0, & \text{if } p > 0, \\ p(x+4-y) < 0, & \text{if } p < 0, \end{cases} \\ \sigma^\flat([x-4, x+3] - [0,1], p) &= \begin{cases} p(x-5) < 0, & \text{if } p > 0, \\ p(x+3) < 0, & \text{if } p < 0, \end{cases} \\ \sigma^\flat([x-4, x+3] - y, p) &= \begin{cases} p(x-4-y) < 0, & \text{if } p > 0, \\ p(x+3-y) < 0, & \text{if } p < 0. \end{cases} \end{aligned} \tag{3.23}$$

Since $\prod_{i=1}^{n} A_i - \prod_{i=1}^{n} B_i = \prod_{i=1}^{n}(A_i - B_i)$ and $\sigma^{\flat}(\prod_{i=1}^{n} F_i, q) = \sum_{i=1}^{n} \sigma^{\flat}(F_i, q_i)$ for $A_i, B_i, F_i \subset R^1$ $(i = 1, 2, \ldots, n)$ and $q = (q_1, q_2, \ldots, q_n) \in R^n$, from (3.23) and using the fact that $\sigma^{\flat}(F_0, 0) = 0$ for any $F_0 \subseteq R^1$, we obtain that $\forall x \in X = [0,1]$, $\forall q = (q_1, q_2, \ldots, q_n) \in R^n$ and $\forall y = (y_1, y_2, \ldots, y_n) \in Y = [0,1]^n$,

$$\begin{aligned} \sigma^{\flat}(Sx - Y, q) &= \sum_{i=1}^{n} \sigma^{\flat}([x-3, x+4] - [0,1], q_i) \le 0, \\ \sigma^{\flat}(Sx - y, q) &= \sum_{i=1}^{n} \sigma^{\flat}([x-3, x+4] - y_i, q_i) \le 0, \end{aligned} \quad \text{if } x \in Q_0, \tag{3.24}$$

$$\begin{aligned} \sigma^{\flat}(Sx - Y, q) &= \sum_{i=1}^{n} \sigma^{\flat}([x-4, x+3] - [0,1], q_i) \le 0, \\ \sigma^{\flat}(Sx - y, q) &= \sum_{i=1}^{n} \sigma^{\flat}([x-4, x+3] - y_i, q_i) \le 0, \end{aligned} \quad \text{if } x \in P_0. \tag{3.25}$$

Both (3.24) and (3.25) imply that

$$\forall q \in R^n,\ \forall y \in Y,\ \forall c \ge 0, \quad \begin{aligned} &\left\{x \in X : \sigma^{\flat}(Sx, q) \le \sigma^{\sharp}(Y, q)\right\} = [0,1], \\ &\left\{x \in X : \sigma^{\flat}(Sx, q) \le \langle q, y\rangle\right\} = [0,1], \\ &\left\{x \in X : \sigma^{\flat}(Sx, q) \le \langle q, y\rangle + c\right\} = [0,1]. \end{aligned} \quad \text{are closed,} \tag{3.26}$$

Therefore, all (a) of (3.2)–(3.4) are satisfied.

(2) As $Lx = (\overbrace{0, \ldots, 0, x}^{k}, 0, \ldots, 0)$ $(x \in X = [0,1])$, $LX = \prod_{i=1}^{n} X_i$, where $X_k = X$ and $X_i = \{0\}$ for $i \ne k$, then for any $q = (q_1, q_2, \ldots, q_n) \in R^n$ and any $x \in X$, we see that

$$\langle q, Lx\rangle = \sigma^{\flat}(LX, q) \Longleftrightarrow q_k x = \sigma^{\flat}(X, q_k) = \begin{cases} 0, & q_k \ge 0 \\ q_k, & q_k < 0 \end{cases} \Longleftrightarrow \begin{cases} x = 0, & \text{if } q_k > 0, \\ x = 1, & \text{if } q_k < 0, \\ x \in X, & \text{if } q_k = 0. \end{cases} \tag{3.27}$$

From (3.26) and (3.27), it follows that $\forall q = (q_1, q_2, \ldots, q_n) \in R^n$ and $\forall x \in X = [0,1]$ with $Lx \in \partial(LX, q)$,

$$\sigma^{\flat}(Sx - Y, q) = \begin{cases} \sigma^{\flat}(S0 - Y, q) \le 0, & \text{if } q_k > 0, \\ \sigma^{\flat}(S1 - Y, q) \le 0, & \text{if } q_k < 0, \\ \sigma^{\flat}(Sx - Y, q) \le 0, & \text{if } q_k = 0,\ (x \in X), \end{cases} \tag{3.28}$$

$$\forall y \in Y, \quad \sigma^{\flat}(Sx - y, q) = \begin{cases} \sigma^{\flat}(S0 - y, q) \le 0, & \text{if } q_k > 0, \\ \sigma^{\flat}(S1 - y, q) \le 0, & \text{if } q_k < 0, \\ \sigma^{\flat}(Sx - y, q) \le 0, & \text{if } q_k = 0, \ (x \in X). \end{cases} \tag{3.29}$$

Both (3.28) and (3.29) show that $\forall q \in R^n$, $\forall x \in X$ with $Lx \in \partial(LX, q)$, we have

$$\sigma^{\flat}(Sx, q) \le \sigma^{\sharp}(Y, q), \quad \sigma^{\flat}(S, q) \le \langle q, y \rangle \quad (y \in Y). \tag{3.30}$$

Hence, all (b) of (3.2)-(3.4) are satisfied. Therefore, Theorem 3.1(A) holds for this example. Indeed, we have $Y_0 = Y_1 = Y \subset SX$.

Remark 3.9. In case $U = V$ and $L = I$ (the identity map from U to itself), then part (A) of Theorem 3.1 is just the main results in [7] and part (B) returns to the Theorem 1.1.

Remark 3.10. From the proof of Theorem 3.1(A), we see that the Ky-Fan inequality is very important to the existence part. Regarding this inequality, some extensions have been made by Lignola [19], Lin and Simons [20], Alzer [21], as well as S. J. Li and X. B. Li [22], we think that probably in the future, these new Ky-Fan inequalities could also be used to obtain some new useful Rogalski-Cornet type theorems.

3.2. *Application to* (1.5)

From Theorem 3.1, we can obtain a solvability theorem to (1.5) as follows.

Theorem 3.11. (i) *If $v \in V$, and* (3.2) *holds for $Y = \{v\}$, then $v \in SX$, that is,* (1.5)*(a) is solvable.*

(ii) *Assume that*

$$\forall q \in V^*, \ \forall x \in X \text{ with } Lx \in \partial(LX, q), \quad \sigma^{\flat}(Sx, q) \le \langle q, Lx \rangle. \tag{3.31}$$

Then (1) $LX \subset SX$ if

$$\forall q \in V^*, \ \forall y \in LX, \quad \left\{ x \in X : \sigma^{\flat}(Sx, q) \le \langle q, y \rangle \right\} \text{ is closed.} \tag{3.32}$$

In particular,

(a) *Equation* (1.5)*(a) is solvable (i.e., $v \in SX$) when $v \in LX$,*

(b) *Equation* (1.5)*(b) has a solution (i.e., $SX \cap Y \neq \emptyset$) when $Y \cap LX \neq \emptyset$,*

(c) *Equation* (1.5)*(c) is solvable (i.e., $Y \subset SX$) when $Y \subset LX$.*

(2) The inverse of S restricting to LX defined by $S_{LX}^{-1} : LX \to X : y \mapsto S_{LX}^{-1}(y) = \{x \in X : y \in Sx\}$ is u.s.c. and u.h.c. if

$$\forall y \in LX, \ \forall q \in V^*, \ \forall c \ge 0, \quad \left\{ x \in X : \sigma^{\flat}(Sx, q) \le \langle q, y \rangle + c \right\} \text{ is closed.} \tag{3.33}$$

(iii) *If S is u.h.c., then the assumptions* (3.2)*(a),* (3.32)*, and* (3.33) *in statements (i) and (ii) can be removed.*

Proof. (i) Applying Theorem 3.1(A)(i)(1) to $Y = \{v\}$, we see that the statement (i) is true.

(ii) Set $Y = LX$. Since X is a convex compact subset of U, and L a continuous linear map from U to V, we see that Y is a convex compact subset of V. Moreover, for each $y \in LX$, as $Lx \in \partial(LX, q)$ equals to $\langle q, Lx\rangle = \sigma^{\flat}(LX, q)$, from (3.31), we obtain that

$$\forall q \in V^*,\ \forall x \in X \text{ with } Lx \in \partial(LX, q), \quad \sigma^{\flat}(Sx, q) \le \langle q, Lx\rangle = \sigma^{\flat}(LX, q) \le \langle q, y\rangle. \tag{3.34}$$

This means that the assumption (3.3)(b) holds for any $y \in LX$.

(1) Associating (3.32) with (3.34) we know that (3.3) holds for all $y \in Y$, which implies by Theorem 3.1(A)(i)(2), for $Y = LX$, that $LX = Y_0 \subset SX$, and thus all the statements (a), (b), and (c) of (ii)(1) are true.

(2) From (3.33) and (3.34), we see that Y_1 defined by (3.4) is precisely equal to LX. Hence by Theorem 3.1(A)(ii), the statement (ii)(2) follows.

(iii) The conclusion of (iii) is clear. This completes the proof. □

4. Conclusion

In this paper, we have proved a Rogalski-Cornet type theorem (namely, Theorem 3.1) based on two Hausdorff locally convex vector spaces, and presented a counterexample (i.e., Example 3.8) to show that the first part of this theorem does not need any conventional continuity conditions such as upper semicontinuous, lower semicontinuous and upper hemicontinuous conditions. Applying this theorem, by Theorem 3.11 we have also provided the solvability results for a class of abstract von Neumann input-output inclusion system (namely, (1.5)).

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