Research Article

Periodic Solutions of Some Impulsive Hamiltonian Systems with Convexity Potentials

Dezhu Chen and Binxiang Dai

School of Mathematical Sciences and Computing Technology, Central South University, Hunan, Changsha 410083, China

Correspondence should be addressed to Binxiang Dai, binxiangdai@126.com

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We study the existence of periodic solutions of some second-order Hamiltonian systems with impulses. We obtain some new existence theorems by variational methods.

1. Introduction

Consider the following systems:

$$\begin{aligned} \ddot{u}(t) &= f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ \Delta \dot{u}(t_k) &= g_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \\ u(T) - u(0) &= \dot{u}(T) - \dot{u}(0) = 0, \end{aligned} \quad (1.1)$$

where $k \in \mathbb{Z}$, $u \in \mathbb{R}^n$, $\Delta \dot{u}(t_k) = \dot{u}(t_k^+) - \dot{u}(t_k^-)$ with $\dot{u}(t_k^\pm) = \lim_{t \rightarrow t_k^\pm} \dot{u}(t)$, $g_k(u) = \text{grad}_u G_k(u)$, $G_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for each $k \in \mathbb{Z}$, there exists an $m \in \mathbb{Z}$ such that $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, and we suppose that $f(t, u) = \text{grad}_u F(t, u)$ satisfies the following assumption.

(A) $F(t, x)$ is measurable in t for $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| + |f(t, x)| \leq a(|x|)b(t), \quad (1.2)$$

for all $x \in \mathbb{R}^n$ and $t \in [0, T]$.

Many solvability conditions for problem (1.1) without impulsive effect are obtained, such as, the coercivity condition, the convexity conditions (see [1–4] and their references), the sublinear nonlinearity conditions, and the superlinear potential conditions. Recently, by using variational methods, many authors studied the existence of solutions of some second-order differential equations with impulses. More precisely, Nieto in [5, 6] considers linear conditions, [7–10] the sublinear conditions, and [11–16] the sublinear conditions and the other conditions. But to the best of our knowledge, except [7] there is no result about convexity conditions with impulsive effects. By using different techniques, we obtain different results from [7].

We recall some basic facts which will be used in the proofs of our main results. Let

$$H_T^1 = \left\{ u : [0, T] \rightarrow \mathbb{R}^n \text{ absolutely continuous; } u(0) = u(T), \dot{u}(t) \in L^2(0, T; \mathbb{R}^n) \right\}, \quad (1.3)$$

with the inner product

$$\langle u, v \rangle = \int_0^T (u(t), v(t)) dt + \int_0^T (\dot{u}(t), \dot{v}(t)) dt, \quad \forall u, v \in H_T^1, \quad (1.4)$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$. The corresponding norm is defined by

$$\|u\| = \left(\int_0^T (u(t), u(t)) dt + \int_0^T (\dot{u}(t), \dot{u}(t)) dt \right)^{1/2}, \quad \forall u \in H_T^1. \quad (1.5)$$

The space H_T^1 has some important properties. For $u \in H_T^1$, let $\bar{u} = (1/2T) \int_0^T u(t) dt$, and $\tilde{u} = u(t) - \bar{u}$. Then one has Sobolev's inequality (see Proposition 1.3 in [1]):

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt. \quad (1.6)$$

Consider the corresponding functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t)) dt + \sum_{k=1}^m G_k(u(t_k)). \quad (1.7)$$

It follows from assumption (A) and the continuity of g_k one has that φ is continuously differentiable and weakly lower semicontinuous on H_T^1 . Moreover, we have

$$\langle \varphi'(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t)) dt + \int_0^T (f(t, u(t)), v(t)) dt + \sum_{k=1}^m (g_k(u(t_k)), v(t_k)), \quad (1.8)$$

for $u, v \in H_T^1$ and φ' is weakly continuous and the weak solutions of problem (1.1) correspond to the critical points of φ (see [8]).

Theorem 1.1 ([2, Theorem 1.1]). Suppose that V and W are reflexive Banach spaces, $\varphi \in C^1(V \times W, R)$, $\varphi(v, \cdot)$ is weakly upper semi-continuous for all $v \in V$, and $\varphi(\cdot, w) : V \rightarrow R$ is convex for all $w \in W$ and φ is weakly continuous. Assume that

$$\varphi(0, w) \longrightarrow -\infty \quad (1.9)$$

as $\|w\| \rightarrow \infty$ and for every $M > 0$,

$$\varphi(v, w) \longrightarrow +\infty, \quad (1.10)$$

as $\|v\| \rightarrow \infty$ uniformly for $\|w\| \leq M$. Then φ has at least one critical point.

2. Main Results

Theorem 2.1. Assume that assumption (A) holds. If further

(H₁) $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$, and

(H₂) there exist $\eta, \theta > 0$ such that $G_k(x) \geq \eta|x| + \theta$, for all $x \in \mathbb{R}^n$, then (1.1) possesses at least one solution in H_T^1 .

Remark 2.2. (H₁) implies there exists a point $\bar{x}$ for which

$$\int_0^T \nabla F(t, \bar{x}) dt = 0. \quad (2.1)$$

Proof of Theorem 2.1. It follows Remark 2.2, (1.6), and (H₂) that

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T (F(t, u(t)) - F(t, \bar{x})) dt + \int_0^T F(t, \bar{x}) dt + \sum_{k=1}^m G_k(u(t_k)) \\ &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), u(t) - \bar{x}) dt + \sum_{k=1}^m G_k(u(t_k)) \\ &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), \tilde{u}) dt + \sum_{k=1}^m \eta |\tilde{u} + \bar{u}| + m\theta \\ &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \left(\int_0^T |f(t, \bar{x})| dt \right) \|\tilde{u}\|_\infty + m\eta |\bar{u}| - m\eta \|\tilde{u}\|_\infty + m\theta \\ &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - C_0 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{1/2} + m\eta |\bar{u}| + m\theta, \end{aligned} \quad (2.2)$$

for all $u \in H_T^1$ and some positive constant C_0 . As $\|u\| \rightarrow \infty$ if and only if $(|u|^2 + \|\dot{u}\|_2^2)^{1/2} \rightarrow \infty$, we have $\varphi(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$. By Theorem 1.1 and Corollary 1.1 in [1], φ has a minimum point in H_T^1 , which is a critical point of φ . Hence, problem (1.1) has at least one weak solution. $\square$

Theorem 2.3. Assume that assumption (A) and (H_1) hold. If further

(H_3) there exist $\eta, \theta > 0$ and $\alpha \in (0, 2)$ such that $G_k(x) \leq \eta|x|^\alpha + \theta$ for all $x \in \mathbb{R}^n$ and

(H_4) there exist some $\beta > \alpha$ and $\gamma > 0$ such that

$$|x|^{-\beta} \int_0^T F(t, x) dt \leq -\gamma, \quad (2.3)$$

for $|x| \geq M$ and $t \in [0, T]$, where M is a constant, then (1.1) possesses at least one solution in H_T^1 .

Remark 2.4. We can find that our condition (H_4) is very different from condition (vii) in [7] since we prove this by the saddle point theorem substituted for the least action principle.

Proof of Theorem 2.3. We prove φ satisfies the (PS) condition at first. Suppose $\{u_n\}$ is such a sequence that $\{\varphi(u_n)\}$ is bounded and $\lim_{n \rightarrow \infty} \varphi'(u_n) = 0$. We will prove it has a convergent subsequence. By (H_3) and (1.6), we have

$$\begin{aligned} \sum_{k=1}^m G_k(u(t_k)) &\leq \sum_{k=1}^m \eta |\tilde{u}(t_k) + \bar{u}|^\alpha + m\theta \\ &\leq 4m\eta (|\tilde{u}(t_k)|^\alpha + |\bar{u}|^\alpha) + m\theta \\ &\leq C_1 \|\dot{u}\|_2^\alpha + C_2 |\bar{u}|^\alpha + C_3, \end{aligned} \quad (2.4)$$

for some positive constants C_1, C_2, C_3 . By Remark 2.2, (1.6), and (2.4), we have

$$\begin{aligned} \varphi(u_n) &= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T (F(t, u_n(t)) - F(t, \bar{x})) dt + \int_0^T F(t, \bar{x}) dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), u_n(t) - \bar{x}) dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T F(t, \bar{x}) dt + \int_0^T (f(t, \bar{x}), \tilde{u}_n) dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &\geq \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt - \left(\int_0^T |f(t, \bar{x})| dt \right) \|\tilde{u}_n\|_\infty - C_1 \|\dot{u}_n\|_2^\alpha - C_2 |\bar{u}_n|^\alpha - C_4 \\ &\geq \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt - C_5 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{1/2} - C_1 \|\dot{u}_n\|_2^\alpha - C_2 |\bar{u}_n|^\alpha - C_4, \end{aligned} \quad (2.5)$$

for some positive constants C_4, C_5 , which implies that

$$C |\bar{u}_n|^{\alpha/2} \geq \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{1/2} - C_6, \quad (2.6)$$

for some positive constants C, C_6 . By (1.6), the above inequality implies that

$$\|\tilde{u}_n\|_\infty \leq C_7 \left(|\bar{u}_n|^{\alpha/2} + 1 \right), \quad (2.7)$$

for the positive constant C_7 . The one has

$$|u_n(t)| \geq |\bar{u}_n| - |\tilde{u}_n| \geq |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geq |\bar{u}_n| - C_7 \left(|\bar{u}_n|^{\alpha/2} + 1 \right), \quad \forall t \in [0, T]. \quad (2.8)$$

If $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\bar{u}_n| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \quad (2.9)$$

By (2.8) and (2.9), we have

$$|u_n(t)| \geq \frac{1}{2} |\bar{u}_n|, \quad (2.10)$$

for all large n and every $t \in [0, T]$. By (2.10) and (H_4) , one has $|u_n(t)| \geq M$ for all large n . It follows from (H_4) , (2.4), (2.6), (2.7), and above inequality that

$$\begin{aligned} \varphi(u_n) &\leq \left(C |\bar{u}_n|^{\alpha/2} + C_6 \right)^2 - \int_0^T \gamma |u_n(t)|^\beta dt + C_2 \|\tilde{u}\|_\infty^\alpha + C_2 |\bar{u}|^\alpha + C_3 \\ &\leq \left(C |\bar{u}_n|^{\alpha/2} + C_6 \right)^2 - 2^{-\beta} |\bar{u}_n|^\beta T \gamma + C_8 \left(|\bar{u}_n|^{\alpha/2} + 1 \right)^\alpha + C_2 |\bar{u}|^\alpha + C_3, \end{aligned} \quad (2.11)$$

for large n and the positive constant C_8 , which contradicts the boundedness of $\varphi(u_n)$ since $\beta > \alpha$. Hence $(|\bar{u}_n|)$ is bounded. Furthermore, (u_n) is bounded by (2.6). A similar calculation to Lemma 3.1 in [9] shows that φ satisfies the (PS) condition. We now prove that φ satisfies the other conditions of the saddle point theorem. Assume that $\widetilde{H}_T^1 = \{u \in H_T^1 : \bar{u} = 0\}$, then $H_T^1 = \widetilde{H}_T^1 \oplus \mathbb{R}^n$. From above calculation, one has

$$\varphi(u) \geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - C_5 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{1/2} - C_1 \|\dot{u}\|_2^\alpha - C_4, \quad (2.12)$$

for all $u \in \widetilde{H}_T^1$, which implies that

$$\varphi(u) \longrightarrow +\infty, \quad (2.13)$$

as $\|u\| \rightarrow \infty$ in $\widetilde{H}_T^1$. Moreover, by (H_3) and (H_4) we have

$$\begin{aligned}\varphi(x) &= \int_0^T F(t, x) dt + \sum_{k=1}^m G_k(x) \\ &\leq -T\gamma|x|^\beta + m\eta|x|^\alpha + m\theta,\end{aligned}\tag{2.14}$$

for $|x| > M$, which implies that

$$\varphi(x) \longrightarrow -\infty,\tag{2.15}$$

as $|x| \rightarrow \infty$ in $\mathbb{R}^n$ since $\beta > \alpha$. Now Theorem 2.3 is proved by (2.13), (2.15), and the saddle point theorem. $\square$

Theorem 2.5. *Assume that assumption (A) holds. Suppose that $F(t, \cdot), G_k(x)$ are concave and satisfy*

(H_5) $G_k(x) \leq -\eta|x| + \theta$ for some positive constant $\eta, \theta > 0$, then (1.1) possesses at least one solution in H_T^1 .

Proof of Theorem 2.5. Consider the corresponding functional φ on $\mathbb{R}^n \times \widetilde{H}_T^1$ given by

$$\varphi(u) = -\frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, u(t)) dt - \sum_{k=1}^m G_k(u(t_k)),\tag{2.16}$$

which is continuously differentiable, bounded, and weakly upper semi-continuous on H_T^1 . Similar to the proof of Lemma 3.1 in [2], one has that $\varphi(x + w)$ is convex in $x \in \mathbb{R}^n$ for every $w \in \widetilde{H}_T^1$. By the condition, we have $-G_k(x + w) \geq -2G_k((1/2)x) + G_k(-w)$. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\varphi(x + w) &= -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - \int_0^T F(t, x + w) dt - \sum_{k=1}^m G_k(x + w) \\ &\geq -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - \left(\int_0^T |f(t, \bar{x})| dt \right) \|w\|_\infty - \sum_{k=1}^m G_k(x + w) + C_9 \\ &\geq -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - C_0 \left(\int_0^T |\dot{w}|^2 dt \right)^{1/2} - 2G_k\left(\frac{1}{2}x\right) + G_k(-w) + C_9,\end{aligned}\tag{2.17}$$

which means $\varphi(x+w) \rightarrow +\infty$ as $|x| \rightarrow \infty$, uniformly for $w \in \widetilde{H}_T^1$ with $\|w\| \leq M$ by (H_5) and (1.6). On the other hand,

$$\begin{aligned}\varphi(w) &= -\frac{1}{2} \int_0^T |\dot{w}|^2 dt - \int_0^T F(t, w) dt - \sum_{k=1}^m G_k(w) \\ &\leq -\frac{1}{2} \int_0^T |\dot{w}|^2 dt + C_0 \left(\int_0^T |\dot{w}|^2 dt \right)^{1/2} + m\eta \|w\|_\infty + C_9,\end{aligned}\tag{2.18}$$

which implies that $\varphi(w) \rightarrow -\infty$ as $\|w\| \rightarrow \infty \in \widetilde{H}_T^1$ by (H_5) and (1.6). We complete our proof by Theorem 1.1. $\square$

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References

- [1] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74, Springer, Berlin, Germany, 1989.
- [2] C.-L. Tang and X.-P. Wu, "Some critical point theorems and their applications to periodic solution for second order Hamiltonian systems," *Journal of Differential Equations*, vol. 248, no. 4, pp. 660–692, 2010.
- [3] J. Mawhin, "Semicoercive monotone variational problems," *Académie Royale de Belgique, Bulletin de la Classe des Sciences*, vol. 73, no. 3-4, pp. 118–130, 1987.
- [4] C.-L. Tang, "An existence theorem of solutions of semilinear equations in reflexive Banach spaces and its applications," *Académie Royale de Belgique, Bulletin de la Classe des Sciences*, vol. 4, no. 7–12, pp. 317–330, 1996.
- [5] J. J. Nieto, "Variational formulation of a damped Dirichlet impulsive problem," *Applied Mathematics Letters*, vol. 23, no. 8, pp. 940–942, 2010.
- [6] J. J. Nieto and D. O'Regan, "Variational approach to impulsive differential equations," *Nonlinear Analysis. Real World Applications*, vol. 10, no. 2, pp. 680–690, 2009.
- [7] J. Zhou and Y. Li, "Existence of solutions for a class of second-order Hamiltonian systems with impulsive effects," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1594–1603, 2010.
- [8] H. Zhang and Z. Li, "Periodic solutions of second-order nonautonomous impulsive differential equations," *International Journal of Qualitative Theory of Differential Equations and Applications*, vol. 2, no. 1, pp. 112–124, 2008.
- [9] J. Zhou and Y. Li, "Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2856–2865, 2009.
- [10] W. Ding and D. Qian, "Periodic solutions for sublinear systems via variational approach," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2603–2609, 2010.
- [11] H. Zhang and Z. Li, "Periodic and homoclinic solutions generated by impulses," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 39–51, 2011.
- [12] Z. Zhang and R. Yuan, "An application of variational methods to Dirichlet boundary value problem with impulses," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 155–162, 2010.
- [13] D. Zhang and B. Dai, "Infinitely many solutions for a class of nonlinear impulsive differential equations with periodic boundary conditions," *Computers & Mathematics with Applications*, vol. 61, no. 10, pp. 3153–3160, 2011.
- [14] L. Yang and H. Chen, "Existence and multiplicity of periodic solutions generated by impulses," *Abstract and Applied Analysis*, vol. 2011, Article ID 310957, 15 pages, 2011.

- [15] J. Sun, H. Chen, and J. J. Nieto, "Infinitely many solutions for second-order Hamiltonian system with impulsive effects," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 544–555, 2011.
- [16] J. Sun, H. Chen, J. J. Nieto, and M. Otero-Novoa, "The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 12, pp. 4575–4586, 2010.