Research Article

Fixed Point of Strong Duality Pseudocontractive Mappings and Applications

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Let $E$ be a smooth Banach space with the dual $E^*$, an operator $T : E \to E^*$ is said to be $\alpha$-strong duality pseudocontractive if

$$
\langle x - y, Tx - Ty \rangle \leq \langle x - y, Jx - Jy \rangle - \alpha \|Jx - Jy - (Tx - Ty)\|^2,
$$

for all $x, y \in E$, where $\alpha$ is a nonnegative constant. An element $x \in E$ is called a duality fixed point of $T$ if $Tx = Jx$. The purpose of this paper is to introduce the definition of $\alpha$-strong duality pseudocontractive mappings and to study its fixed point problem and applications for operator equation and variational inequality problems.

1. Introduction and Preliminaries

Let $E$ be a real Banach space with the dual $E^*$: let $T$ be an operator from $E$ into $E^*$. We consider the first operator equation problem of finding an element $x^* \in E$ such that

$$
\langle Tx^*, x^* \rangle = \|Tx^*\|^2 = \|x^*\|^2.
$$

(1.1)

We also consider the second variational inequality problem of finding an element $x^* \in E$ such that

$$
\langle Tx^*, x^* - x \rangle \geq 0, \quad \forall \|x\| \leq \|x^*\|.
$$

(1.2)

Let $E$ be a real Banach space with the dual $E^*$. Let $p$ be a given real number with $p > 1$. The generalized duality mapping $J_p$ from $E$ into $2^{E^*}$ is defined by

$$
J_p(x) = \left\{ f \in E^* : \langle x, f \rangle = \|f\|^p, \|f\| = \|x\|^{p-1} \right\}, \quad \forall x \in E,
$$

(1.3)
where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In particular, $J = J_2$ is called the normalized duality mapping and $J_p(x) = \|x\|^{p-2} J(x)$ for all $x \neq 0$. If $E$ is a Hilbert space, then $J = I$, where $I$ is the identity mapping. The duality mapping $J$ has the following properties:

(i) if $E$ is smooth, then $J$ is single valued;

(ii) if $E$ is strictly convex, then $J$ is one to one;

(iii) if $E$ is reflexive, then $J$ is a mapping of $E$ onto $^*E$;

(iv) if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$;

(v) if $E^*$ is uniformly convex, then $J$ is uniformly continuous on each bounded subsets of $E$ and $J$ is single valued and also one to one.

For more details, see [1, 2].

Let $E$ be a smooth Banach space with the dual $E^*$. Let $T : E \rightarrow E^*$ be an operator; an element $x^* \in E$ is called a duality fixed point of $T$, if $Tx^* = Jx^*$.

We also consider the third variational inequality problem of finding an element $x^* \in E$ such that

$$
\langle Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C,
$$

where $C$ is a closed convex subset of $E$. The set of solutions of the variational inequality problem (1.4) is denoted by $VI(C, T)$.

We also consider the fourth variational inequality problem of finding an element $x^* \in E$ such that

$$
\langle Jx^* - Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C,
$$

where $C$ is a closed convex subset of $E$. The set of solutions of the variational inequality problem (1.5) is denoted by $VI(C, J, T)$.

**Conclusion 1.** If $x^*$ is a duality fixed point of $T$, then $x^*$ must be a solution of problem (1.1).

**Proof.** If $x^*$ is a normalized fixed point of $T$, then $Tx^* = Jx^*$, so that

$$
\langle Tx^*, x^* \rangle = \langle Jx^*, x^* \rangle = \|Jx^*\|^2 = \|Tx^*\|^2 = \|x^*\|^2.
$$

This completes the proof.

**Conclusion 2.** If $x^*$ is a duality fixed point of $T$, then $x^*$ must be a solution of variational inequality problem (1.2).

**Proof.** Suppose $x^*$ is a duality fixed point of $T$; then

$$
\langle Tx^*, x^* \rangle = \|Tx^*\|^2 = \|x^*\|^2.
$$

(1.7)
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Obverse that

\[
(Tx^*, x^* - x) = (Tx^*, x^*) - (Tx^*, x) \\
\geq \|Tx^*\|^2 - \|Tx^*\|\|x\| \\
= \|Tx^*\|\|x\| - \|Tx^*\|\|x\| \\
= \|Tx^*\|\|x\| - \|Tx^*\|\|x\| \geq 0,
\]

for all \(\|x\| \leq \|x^*\|\). This completes the proof.

Let \(U = \{x \in E : \|x\| = 1\}\). A Banach space \(E\) is said to be strictly convex if for any \(x, y \in U, x \neq y\) implies \(\|(x + y)/2\| < 1\). It is also said to be uniformly convex if for each \(\varepsilon \in (0, 2]\), there exists \(\delta > 0\) such that for any \(x, y \in U, \|x - y\| \geq \varepsilon\) implies \(\|(x + y)/2\| < 1 - \delta\). It is known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function \(\delta : [0, 2] \to [0, 1]\) called the modulus of convexity of \(E\) as follows:

\[
\delta(\varepsilon) = \left\{ 1 - \frac{x + y}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}. \tag{1.9}
\]

It is known that \(E\) is uniformly convex if and only if \(\delta(\varepsilon) > 0\) for all \(\varepsilon \in (0, 2]\). Let \(p\) be a fixed real number with \(p \geq 2\). Then \(E\) is said to be \(p\)-uniformly convex if there exists a constant \(c > 0\) such that \(\delta(\varepsilon) \geq ce^p\) for all \(\varepsilon \in [0, 2]\). For example, see [3, 4] for more details. The constant \(1/c\) is said to be uniformly convexity constant of \(E\).

A Banach space \(E\) is said to be smooth if the limit

\[
\lim_{t \to 0} \frac{\|tx + ty\| - \|x\|}{t}
\]

exists for all \(x, y \in U\). It is also said to be uniformly smooth if the above limit is attained uniformly for \(x, y \in U\). One should note that no Banach space is \(p\)-uniformly convex for \(1 < p < 2\); see [5] for more details. It is well known that the Hilbert and the Lebesgue \(L^q(1 < q \leq 2)\) spaces are 2-uniformly convex and uniformly smooth. Let \(X\) be a Banach space, and let \(L^q(X) = \{\Omega, \Sigma, \mu, X\}, 1 < q \leq \infty\) be the Lebesgue-Bochner space on an arbitrary measure space \((\Omega, \Sigma, \mu)\). Let \(2 \leq p < \infty\), and let \(1 < q \leq p\). Then \(L^q(X)\) is \(p\)-uniformly convex if and only if \(X\) is \(p\)-uniformly convex; see [4].

In this paper, we first propose the definition of generalized \(\alpha\)-strongly pseudocontractive mappings from a smooth Banach \(E\) into its dual \(E^*\) as follows. We also discuss the problem of fixed point for generalized \(\alpha\)-strongly pseudocontractive mappings and its applications.

Let \(E\) be a smooth Banach space and \(E^*\) denote the dual of \(E\). An operator \(A : E \to E^*\) is said to be

1. \(\alpha\)-inverse-strongly monotone if there exists nonnegative real number \(\alpha\) such that

\[
\langle x - y, Ax - Ay \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in E. \tag{1.11}
\]
(2) $\alpha$-strong duality pseudocontractive mapping, if there exists a nonnegative real number $\alpha$ such that

$$
\langle x - y, Ax - Ay \rangle \leq \langle x - y, Jx - Jy \rangle - \alpha \| Jx - Jy - (Ax - Ay) \|^2
$$

(1.12)

for all $x, y \in E$.

It is easy to show that $A$ is $\alpha$-strong duality pseudocontractive if and only if $(J - A)$ is $\alpha$-inverse-strongly monotone.

Let $E$ be a smooth Banach space and $E^*$ denote the dual of $E$. Let $A : E \to E^*$ be an operator. The set of zero points of $A$ is defined by $A^{-1}0 = \{ x \in E : Ax = 0 \}$. The set of duality fixed points of $A$ is defined by $F(A) = \{ x \in E : Ax = Jx \}$. It is also easy to show that, an element $u \in E$ is a zero point of an $\alpha$-inverse-strongly monotone operator $A$ if and only if $u$ is a duality fixed point of the $\alpha$-strong duality pseudocontractive mapping $(J - A)$.

## 2. Main Results and Applications

Recently, Zegeye and Shahzad [6] proved the following result.

**Theorem 2.1** (see, [6]). Let $E$ be a uniformly smooth and 2-uniformly convex real Banach space with the dual $E^*$: let $A : E \to E^*$ be a $\gamma$-inverse-strongly monotone mapping and $T : E \to E$ a relatively weak nonexpansive mapping with $A^{-1}0 \cap F(T) \neq \emptyset$. Assume that $0 < \alpha \leq \lambda_n \leq \gamma c^2 / 2$, where $1/c$ is the uniformly convexity constant. Define a sequence $\{ x_n \}$ in $E$ by the following algorithm:

$$
x_0 \in E \text{ chosen arbitrarily},
$$

$$
z_n = J^{-1}(Jx_n - \lambda_n Ax_n),
$$

$$
y_n = Tz_n,
$$

$$
C_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, z_n) \leq \phi(z, z_n) \leq \phi(z, x_n) \right\},
$$

$$
C_0 = \left\{ z \in E : \phi(z, y_0) \leq \phi(z, z_0) \leq \phi(z, x_0) \right\},
$$

$$
Q_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \right\},
$$

$$
Q_0 = E,
$$

$$
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
$$

where $J$ is the duality mapping on $E$. Then $\{ x_n \}$ converges strongly to $\Pi_{A^{-1}0 \cap F(T)}x_0$, where $\Pi_{A^{-1}0 \cap F(T)}$ is the generalized projection from $E$ onto $A^{-1}0 \cap F(T)$.

If taking $T = I$, then Theorem 2.1 reduces to the following result.

**Theorem 2.2.** Let $E$ be a uniformly smooth and 2-uniformly convex real Banach space with the dual $E^*$, let $A : E \to E^*$ be a $\gamma$-inverse strongly monotone mapping with $A^{-1}0 \neq \emptyset$. Assume that
0 < \alpha \leq \lambda_n \leq \gamma c^2/2$, where $1/c$ is the uniformly convexity constant. Define a sequence $\{x_n\}$ in $E$ by the following algorithm:

\[
x_0 \in E \text{ chosen arbitrarily,}
\]
\[
y_n = J^{-1}(Jx_n - \lambda_n Ax_n),
\]
\[
C_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n) \right\},
\]
\[
C_0 = \left\{ z \in E : \phi(z, y_0) \leq \phi(z, x_0) \right\},
\]
\[
Q_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : (x_n - z, Jx_0 - Jx_n) \geq 0 \right\},
\]
\[
Q_0 = E,
\]
\[
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\]

where $J$ is the duality mapping on $E$. Then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x_0$, where $\Pi_{A^{-1}0}$ is the generalized projection from $E$ onto $A^{-1}0$.

**Theorem 2.3.** Let $E$ be a uniformly smooth and 2-uniformly convex real Banach space; let $A : E \to E^*$ be an $\alpha$-strong duality pseudocontractive mapping with nonempty set of duality fixed points $F(A)$. Let $T : E \to E$ be a relatively weak nonexpansive mapping and $F(A) \cap F(T) = \emptyset$. Assume $0 < \alpha \leq \lambda_n \leq \gamma c^2/2L$. Define a sequence $\{x_n\}$ in $E$ by the following algorithm:

\[
x_0 \in E \text{ chosen arbitrarily,}
\]
\[
z_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_n Ax_n),
\]
\[
y_n = Tz_n,
\]
\[
C_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, z_n) \leq \phi(z, x_n) \right\},
\]
\[
C_0 = \left\{ z \in E : \phi(z, y_0) \leq \phi(z, z_0) \leq \phi(z, x_0) \right\},
\]
\[
Q_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : (x_n - z, Jx_0 - Jx_n) \geq 0 \right\},
\]
\[
Q_0 = E,
\]
\[
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0),
\]

where $J$ is the duality mapping on $E$. Then $\{x_n\}$ converges strongly to a common element $x^* \in F(A) \cap F(T)$. This element is also a common solution of operator equation (1.1) and variational inequality (1.2).

**Proof.** Let $B = J - A$, then $B : E \to E^*$ is $\alpha/L$-inverse-strongly monotone and $\alpha$-strongly monotone, so that $B^{-1}0 = F(A)$ has only one element. On the other hand, we have

\[
z_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_n Ax_n) = J^{-1}(Jx_n - \lambda_n Bx_n).
\]

By using Theorem 2.1 and Conclusions 1 and 2, we obtain the conclusion of Theorem 2.3.

Taking $T = I$ in Theorem 2.3, we get the following result. \hfill \Box
Theorem 2.4. Let $E$ be a uniformly smooth and 2-uniformly convex real Banach space; let $A : E \to E^*$ be a $L$-Lipschitz and $\alpha$-strongly duality pseudocontractive mapping with nonempty set of duality fixed points $F(A)$. Assume $0 < a \leq \lambda_n \leq ac^2/2L$. Define a sequence $\{x_n\}$ in $E$ by the following algorithm:

$$x_0 \in E \text{ chosen arbitrarily,}$$
$$y_n = J^{-1}((1 - \lambda_n)Jx_n + \lambda_nAx_n),$$
$$C_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n) \right\},$$
$$C_0 = \left\{ z \in E : \phi(z, y_0) \leq \phi(z, x_0) \right\},$$
$$Q_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \right\},$$
$$Q_0 = E,$$
$$x_{n+1} = \Pi_{C \cap Q_n}(x_0),$$

where $J$ is the duality mapping on $E$. Then $\{x_n\}$ converges strongly to a duality fixed point $x^* \in F(A)$. This element $x^*$ is also a common solution of operator equation (1.1) and variational inequality (1.2).

Iiduka and Takahashi [7] introduce an iterative scheme for finding a solution of the variational inequality problem for an operator $A$ that satisfies the following conditions (i)–(iii) in a 2-uniformly convex and uniformly smooth Banach space $E$:

(i) $A$ is $\alpha$-inverse-strongly monotone;
(ii) $\text{VI}(C, A) \neq \emptyset$;
(iii) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in E$ and $u \in \text{VI}(C, A)$.

They proved the following convergence theorem.

Theorem 2.5 (see, [7]). Let $E$ be a 2-uniformly convex and uniformly smooth Banach space, whose duality mapping $J$ is weakly sequentially continuous, and $C$ a nonempty, closed convex subset of $E$. Assume that $A$ is an operator of $C$ into $E^*$, that satisfies the conditions (i)–(iii). Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_nAx_n),$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2\alpha^2/2$, then the sequence $\{x_n\}$ converges weakly to some element $z \in \text{VI}(C, A)$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{\text{VI}(C, A)}x_n$.

In this paper, we introduce an iterative scheme for finding a solution of the variational inequality problem for an operator $T$ that satisfies the following conditions (iv)–(vi) in a 2-uniformly convex and uniformly smooth Banach space $E$:

(iv) $T$ is $\alpha$-strong duality pseudocontractive,
(v) $\text{VI}(C, J, T) \neq \emptyset$,
(vi) $\|Jy - Ty\| \leq \|(J - T)y - (J - T)u\|$ for all $y \in E$ and $u \in \text{VI}(C, J, T)$.

By using Theorem 2.5, we prove the following convergence theorem.
Theorem 2.6. Let $E$ be a 2-uniformly convex and uniformly smooth Banach space, whose duality mapping $J$ is weakly sequentially continuous, and $C$ a nonempty, closed convex subset of $E$. Assume that $T$ is an operator of $C$ into $E^*$ that satisfies the conditions (iv)–(vi). Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \Pi_C J^{-1}((1 - \lambda_n) J x_n + \lambda_n T x_n),$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive numbers. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some $a, b$ with $0 < a < b < c^2 \alpha/2$, then the sequence $\{x_n\}$ converges weakly to some element $z \in VI(C, J, T)$, where $1/c$ is the 2-uniformly convexity constant of $E$. Further $z = \lim_{n \to \infty} \Pi_{VI(C, J, T)} x_n$.

Proof. Let $A = J - T$, then $B : E \to E^*$ is $\alpha$-inverse-strongly monotone, so that $B^{-1}0 = F(A)$. On the other hand, we have

$$x_{n+1} = \Pi_C J^{-1}((1 - \lambda_n) J x_n + \lambda_n T x_n) = J^{-1}(J x_n - \lambda_n Ax_n).$$

By using Theorem 2.5, we obtain the conclusion of Theorem 2.6.

In fact, from condition (vi), we have $F(T) = VI(C, J, T)$, so that under the conditions of Theorem 2.6, the $\{x_n\}$ converges strongly to a duality fixed point $z \in F(T)$. This element $z$ is also a common solution of operator equation (1.1) and variational inequality (1.2), where $\{x_n\}$ is defined by Algorithm (2.7).

References
