Research Article

Fixed Points and Endpoints of Set-Valued Contractions in Cone Metric Spaces

Ing-Jer Lin,¹ Chi-Ming Chen,²
Mirko Jovanović,³ and Tzi-Huei Wu²

¹ Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 824, Taiwan

² Department of Applied Mathematics, National Hsinchu University of Education,
Hsinchu 30014, Taiwan

³ Faculty of Electrical Engineering, University of Belgrade, Kralja Aleksandra 73, 11120 Beograd, Serbia

Correspondence should be addressed to Chi-Ming Chen, ming@mail.nhcue.edu.tw

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The purpose of this paper is to present the fixed points and endpoints of set-valued contractions concerning with the stronger Meir-Keeler cone-type mappings in cone metric spaces. Our results generalize the recent results of Kadelburg and Radenović, 2011; Wardowski, 2009.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, $\mathbb{R}$ we denote the sets of all nonnegative real numbers and all real numbers, respectively, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D a subset of X , and $f : D \rightarrow X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that, for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \quad (1.1)$$

The well-known Banach fixed-point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed-point in X . It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, Kannan [2] and Chatterjea [3] introduced two conditions that can replace (1.1) in Banach's theorem as follows:

Kannan [2]: There exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,

$$d(fx, fy) \leq \frac{\alpha}{2} [d(x, fx) + d(y, fy)]. \quad (1.2)$$

Chatterjea [3]: There exists $\alpha \in [0, 1)$ such that for all $x, y \in X$,

$$d(fx, fy) \leq \frac{\alpha}{2} [d(x, fy) + d(y, fx)]. \quad (1.3)$$

After these three conditions, many papers have been written generalizing some of the conditions (1.1), (1.2), and (1.3). In 1969, Boyd and Wong [4] showed the following fixed-point theorem.

Theorem 1.1 (see [4]). *Let (X, d) be a complete metric space and $f : X \rightarrow X$ a map. Suppose there exists a function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\phi(0) = 0$, $\phi(t) < t$ for all $t > 0$ and ϕ is right upper semicontinuous such that*

$$d(fx, fy) \leq \phi(d(x, y)) \quad \forall x, y \in X. \quad (1.4)$$

Then f has a unique fixed-point in X .

Later, Meir-Keeler [5], using a result of Chu and Diaz [6], extended Boyd-Wong's result to mappings satisfying the following more general condition:

$$\forall \eta > 0 \quad \exists \delta > 0 \quad \text{such that } \eta \leq d(x, y) < \eta + \delta \implies d(fx, fy) < \eta, \quad (1.5)$$

and Meir-Keeler proved the following very interesting fixed-point theorem which is a generalization of the Banach contraction principle.

Theorem 1.2 (Meir-Keeler [5]). *Let (X, d) be a complete metric space and let f be a Meir-Keeler contraction; that is, for every $\eta > 0$, there exists $\delta > 0$ such that $d(x, y) < \eta + \delta$ implies $d(fx, fy) < \eta$ for all $x, y \in X$. Then f has a unique fixed-point.*

Subsequently, some authors worked on this notion of Meir-Keeler contraction (e.g., [7–10]). In this paper, we introduce a new notion of the stronger Meir-Keeler-type mapping $\psi : [0, \infty) \rightarrow [0, b)$, $b \in (0, 1]$, as follows.

Definition 1.3. A mapping $\psi : [0, \infty) \rightarrow [0, b)$, $b \in (0, 1]$ is called a stronger Meir-Keeler-type mapping in a metric space X , if for each $\eta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \leq d(x, y) < \eta + \delta$, there exists $\gamma_\eta \in [0, b)$ such that $\psi(d(x, y)) < \gamma_\eta$.

Example 1.4. Let (X, d) be a metric space, and define $\psi : [0, \infty) \rightarrow [0, b)$, $b \in (0, 1]$ by

$$\psi(d(x, y)) = \frac{bd(x, y)}{d(x, y) + 1}, \quad \text{for } x, y \in X. \quad (1.6)$$

Then ψ is a stronger Meir-Keeler-type mapping.

The existence of fixed-points for various multi valued contractive mappings had been studied by many authors under different conditions. In 1969, Nadler [11] extended the famous Banach contraction principle from single-valued mapping to multi valued mapping and proved the following fixed-point theorem for multi valued contraction. Let $N(X)$ denote the collection of all nonempty subsets of X , $B(X)$ the collection of all nonempty bounded subsets of X , $C(X)$ the collection of all nonempty closed subsets of X , $CB(X)$ the collection of all nonempty closed and bounded subsets of X , and $K(X)$ the collection of all nonempty sequentially compact subsets of X .

Theorem 1.5 (see [11]). *Let (X, d) be a complete metric space, and, T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \leq cd(x, y) \quad \forall x, y \in X. \quad (1.7)$$

Then T has a fixed-point in X .

In 1989, Mizoguchi-Takahashi [12] proved the following fixed-point theorem.

Theorem 1.6 (see [12]). *Let (X, d) be a complete metric space and T a map from X into $CB(X)$. Assume that*

$$\mathcal{H}(Tx, Ty) \leq \xi(d(x, y)) \cdot d(x, y) \quad (1.8)$$

for all $x, y \in X$, where $\xi : [0, \infty) \rightarrow [0, 1)$ satisfies $\limsup_{s \rightarrow t^+} \xi(s) < 1$ for all $t \in [0, \infty)$. Then T has a fixed-point in X .

Remark 1.7. It is clear that if the function $\xi : [0, \infty) \rightarrow [0, b)$, $b \in (0, 1]$ satisfies

$$\limsup_{s \rightarrow t^+} \xi(s) < b, \quad \forall t \in [0, \infty), \quad (1.9)$$

then ξ is also a stronger Meir-Keeler-type function.

In 2003, Rus [13] introduced the notion of endpoints (strict fixed-points) and proved some results on strict fixed-point theory on multi valued operator. An element $x \in X$ is said to be a fixed-point of $T : X \rightarrow N(X)$, if $x \in Tx$. If $\{x\} = Tx$, then x is called an endpoint of T . We will denote the sets of all fixed-points and endpoints of T by $\text{Fix}(T)$ and $\text{End}(T)$, respectively. The following theorems were the main results of Rus [13].

Theorem 1.8 (see [13]). Let (X, d) be a complete metric space and $T : X \rightarrow B(X)$ a multi valued operator. Suppose that

- (i) $x \in Tx$ for all $x \in X$
- (ii) there exist a comparison function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ (see [14]) and a Picard sequence $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ such that

$$\delta(Tx_{n+1}) \leq \varphi(\delta(Tx_n)), \quad n \in \mathbb{N}, \quad (1.10)$$

where $\delta(A) = \sup\{d(x, y) : x, y \in A\}$.

Then $x_n \rightarrow x^*$ as $n \rightarrow \infty$ and x^* is a unique strict fixed-point of T .

Recently, Wardowski [15] proved the following theorems concerning fixed-points and endpoints of set-valued contractions in complete cone metric spaces.

Definition 1.9. A function $I : X \rightarrow \mathbb{R}$ is called lower semicontinuous, if, for any sequence $\{x_n\} \subset X$ and $x \in X$,

$$x_n \rightarrow x \text{ implies } f(x) \leq \liminf_{n \rightarrow \infty} f(x_n). \quad (1.11)$$

If (X, d) is a cone metric space and $T : X \rightarrow C(X)$, then, for $x \in X$, we denote

$$\begin{aligned} \mathfrak{D}(x, Tx) &= \{d(x, z) : z \in Tx\}, \\ \mathcal{S}(x, Tx) &= \{y \in \mathfrak{D}(x, Tx) : \|y\| = \inf\{\|z\| : z \in \mathfrak{D}(x, Tx)\}\}. \end{aligned} \quad (1.12)$$

Theorem 1.10 (see [15]). Let (X, d) be a complete cone metric space, let P be a normal cone with normal constant K , and let $T : X \rightarrow C(X)$. Assume that a function $I : X \rightarrow \mathbb{R}$ defined by $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$ is lower semicontinuous. If there exist $\lambda \in [0, 1)$, $b \in (\lambda, 1]$ such that

$$\begin{aligned} \forall x \in X \quad \exists y \in Tx \quad \exists v \in \mathfrak{D}(y, Ty) \quad \forall u \in \mathfrak{D}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \lambda d(x, y)], \end{aligned} \quad (1.13)$$

then $\text{Fix}(T) \neq \emptyset$.

Theorem 1.11 (see [15]). Let (X, d) be a complete cone metric space, let P be a normal cone with normal constant K , and let $T : X \rightarrow K(X)$. Assume that a function $I : X \rightarrow \mathbb{R}$ defined by $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$ is lower semicontinuous. The following hold

- (a) If there exist $\lambda \in [0, 1)$, $b \in (\lambda, 1]$ such that

$$\begin{aligned} \forall x \in X \quad \exists y \in Tx \quad \exists v \in \mathcal{S}(y, Ty) \quad \forall u \in \mathcal{S}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \lambda d(x, y)], \end{aligned} \quad (1.14)$$

then $\text{Fix}(T) \neq \emptyset$.

(b) If there exist $\lambda \in [0, 1)$, $b \in (\lambda, 1]$ such that

$$\begin{aligned} \forall x \in X \quad \forall y \in Tx \quad \exists v \in \mathcal{S}(y, Ty) \quad \forall u \in \mathcal{S}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \lambda d(x, y)], \end{aligned} \quad (1.15)$$

then $\text{Fix}(T) = \text{End}(T) \neq \emptyset$.

In 1997, Zabrejko [16] introduced the K -metric and K -normed linear spaces and showed the existence and uniqueness of fixed-points for operators which act in K -metric or K -normed linear spaces. Huang and Zhang [17] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space, and they showed some fixed-point theorems of contractive type mappings on cone metric spaces. The category of cone metric spaces is larger than metric spaces. Subsequently, many authors like Abbas and Jungck [18] have generalized the results of Huang and Zhang [17] and studied the existence of common fixed-points of a pair of self-mappings satisfying a contractive type condition in the framework of normal cone metric spaces. However, authors like Rezapour and Hamlbarani [19] studied the existence of common fixed-points of a pair of self- and non-self-mappings satisfying a contractive type condition in the situation in which the cone does not need to be normal. Many authors studied this subject and many results on fixed-point theory are proved (see, e.g., [12, 19–39]).

Definition 1.12 (see [17]). Let E be a real Banach space and P a nonempty subset of E . $P \neq \{\theta\}$, where θ denotes the zero element of E , is called a cone if and only if:

- (i) P is closed;
- (ii) $a, b \in \mathbb{R}^+$, $a, b > 0$, $x, y \in P \Rightarrow ax + by \in P$;
- (iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

For given a cone $P \subset E$, we define a partial ordering with respect to P by $x \preceq y$ or $x \succcurlyeq y$ if and only if $y - x \in P$ for all $x, y \in E$. The real Banach space E equipped with the partial order induced by P is denoted by $(E, \preceq)$. We shall write $x < y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ or $y \gg x$ will stand for $y - x \in \text{int}(P)$, where $\text{int}(P)$ denotes the interior of P .

Proposition 1.13 (see [40]). Suppose P is a cone in a real Banach space E . Then one has the following.

- (i) If $e \preceq f$ and $f \ll g$, then $e \ll g$.
- (ii) If $e \ll f$ and $f \preceq g$, then $e \ll g$.
- (iii) If $e \ll f$ and $f \ll g$, then $e \ll g$.
- (iv) If $a \in P$ and $a \ll e$ for each $e \in \text{int}(P)$, then $a = \theta$.

Proposition 1.14 (see [41]). Suppose $e \in \text{int}(P)$, $\theta \preceq a_n$ and $a_n \rightarrow \theta$. Then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll e$ for all $n \geq n_0$.

The cone P is called normal if there exists a real number $K > 0$ such that, for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|. \quad (1.16)$$

The least positive number K , satisfying the above is called the normal constant of P .

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y, \quad (1.17)$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 1.15 (see [17]). Let X be a nonempty set, E a real Banach space, and P a cone in E . Suppose that the mapping $d : X \times X \rightarrow (E, \preceq)$ satisfies

- (i) $\theta \preceq d(x, y)$, for all $x, y \in X$;
- (ii) $d(x, y) = \theta$ if and only if $x = y$;
- (iii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (iv) $d(x, y) + d(y, z) \succeq d(x, z)$, for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space.

Definition 1.16 (see [17]). Let (X, d) be a cone metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x) \ll c, \quad \forall n > n_0, \quad (1.18)$$

then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x .

Definition 1.17 (see [17]). Let (X, d) be a cone metric space, and let $\{x_n\}$ be a sequence in X . We say that $\{x_n\}$ is a Cauchy sequence if, for any $c \in E$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \ll c, \quad \forall n, m > n_0, \quad (1.19)$$

Definition 1.18 (see [17]). Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X , then X is called a complete cone metric space.

Remark 1.19 (see [17]). If P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \rightarrow \theta$ as $n \rightarrow \infty$. Further, in the case $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow \theta$ as $m, n \rightarrow \infty$.

The purpose of this paper is to present the fixed-points and endpoints of set-valued contractions concerning with the stronger Meir-Keeler cone-type mappings in cone metric spaces. Our results generalize the recent results of Kadelburg and Radenović [42–44] and Wardowski [15].

2. Main Results

In this section, we first introduce the following notion of stronger Meir-Keeler cone-type mapping.

Definition 2.1. Let (X, d) be a cone metric space with cone P , and let

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1]. \quad (2.1)$$

Then the function ψ is called a stronger Meir-Keeler cone-type mapping, if, for each $\eta \in \text{int}(P)$, there exists $\delta \gg \theta$ such that, for $x, y \in X$ with $\eta \preccurlyeq d(x, y) \ll \delta + \eta$, there exists $\gamma_\eta \in [0, b)$ such that $\psi(d(x, y)) < \gamma_\eta$.

We now are in a position to present the following fixed-point theorem.

Theorem 2.2. Let (X, d) be a complete cone metric space, and let P be a normal cone in E and $T : X \rightarrow C(X)$. Assume that a function $I : X \rightarrow \mathbb{R}$ defined by $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$ is lower semicontinuous. The following holds.

(*) There exists a stronger Meir-Keeler cone-type mapping

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1] \quad (2.2)$$

such that

$$\begin{aligned} \forall x \in X \quad \exists y \in Tx \quad \exists v \in \mathfrak{D}(y, Ty) \quad \forall u \in \mathfrak{D}(x, Tx), \\ [bd(x, y) \preccurlyeq u] \wedge [v \preccurlyeq \psi(d(x, y))d(x, y)]. \end{aligned} \quad (2.3)$$

Then $\text{Fix}(T) \neq \emptyset$.

Proof. Given $x_1 \in X$, we define the sequence $\{x_n\}$ recursively as follows. Take any $u_1 \in \mathfrak{D}(x_1, Tx_1)$. By (*), there exist $x_2 \in Tx_1$ and $u_2 \in \mathfrak{D}(x_2, Tx_2)$ such that

$$\begin{aligned} bd(x_1, x_2) \preccurlyeq u_1, \\ u_2 \preccurlyeq \psi(d(x_1, x_2))d(x_1, x_2), \quad \psi(d(x_1, x_2)) < b. \end{aligned} \quad (2.4)$$

Thus,

$$u_1 - u_2 \succcurlyeq [b - \psi(d(x_1, x_2))]d(x_1, x_2), \quad u_2 \preccurlyeq \frac{\psi(d(x_1, x_2))}{b}u_1. \quad (2.5)$$

If $d(x_1, x_2) = \theta$, then $x_1 = x_2 \in Tx_1$, and we are done. Assume that $d(x_1, x_2) \gg \theta$. Put $d(x_1, x_2) = \eta_1$. By the definition of the stronger Meir-Keeler cone-type mapping ψ , corresponding to η_1 use, there exists $\delta_1 \gg \theta$ and $\gamma_{\eta_1} \in [0, b)$ with $\eta_1 \preccurlyeq d(x_1, x_2) \ll \delta_1 + \eta_1$ such that $\psi(d(x_1, x_2)) < \gamma_{\eta_1}$. Taking into account (2.5), we have that

$$u_1 - u_2 \succcurlyeq [b - \gamma_{\eta_1}]d(x_1, x_2), \quad u_2 \preccurlyeq \frac{\gamma_{\eta_1}}{b}u_1. \quad (2.6)$$

By the same process, for $x_2 \in X$, there exist $x_3 \in Tx_2$ and $u_3 \in \mathfrak{D}(x_3, Tx_3)$ such that

$$\begin{aligned} bd(x_2, x_3) &\preceq u_2, \\ u_3 &\preceq \psi(d(x_2, x_3))d(x_2, x_3), \quad \psi(d(x_2, x_3)) < b. \end{aligned} \quad (2.7)$$

Thus,

$$u_2 - u_3 \succ [b - \psi(d(x_2, x_3))]d(x_2, x_3), \quad (2.8)$$

$$u_3 \preceq \frac{\psi(d(x_2, x_3))}{b}u_2 \preceq \frac{\psi(d(x_1, x_2)) \cdot \psi(d(x_2, x_3))}{b^2}u_1. \quad (2.9)$$

Now, put $d(x_2, x_3) = \eta_2$. By the definition of the stronger Meir-Keeler cone-type mapping ψ , corresponding to η_2 use, there exists $\delta_2 \gg \theta$ and $\gamma_{\eta_2} \in [0, b)$ with $\eta_2 \preceq d(x_2, x_3) \ll \delta_2 + \eta_2$ such that $\psi(d(x_2, x_3)) < \gamma_{\eta_2}$. And we put $\alpha_1 = \gamma_{\eta_1}$ and $\alpha_2 = \max\{\gamma_{\eta_1}, \gamma_{\eta_2}\}$. Taking into account (2.9), we have that

$$u_2 - u_3 \succ [b - \gamma_{\eta_2}]d(x_2, x_3) \succ [b - \alpha_2]d(x_2, x_3), \quad (2.10)$$

$$u_3 \preceq \frac{\gamma_{\eta_2}}{b}u_2 \preceq \frac{\gamma_{\eta_1} \cdot \gamma_{\eta_2}}{b^2}u_1 \preceq \frac{\alpha_2^2}{b^2}u_1. \quad (2.11)$$

We continue in this manner. Inductively, for $x_n \in X$, there exist $x_{n+1} \in Tx_n$ and $u_{n+1} \in \mathfrak{D}(x_{n+1}, Tx_{n+1})$ such that

$$\begin{aligned} bd(x_n, x_{n+1}) &\preceq u_n, \\ u_{n+1} &\preceq \psi(d(x_n, x_{n+1}))d(x_n, x_{n+1}), \quad \psi(d(x_n, x_{n+1})) < b. \end{aligned} \quad (2.12)$$

Thus,

$$u_n - u_{n+1} \succ [b - \psi(d(x_n, x_{n+1}))]d(x_n, x_{n+1}), \quad (2.13)$$

$$u_{n+1} \preceq \frac{\psi(d(x_n, x_{n+1}))}{b}u_n \preceq \dots \preceq \frac{\psi(d(x_1, x_2)) \cdot \psi(d(x_2, x_3)) \cdots \psi(d(x_n, x_{n+1}))}{b^n}u_1. \quad (2.14)$$

Put $d(x_n, x_{n+1}) = \eta_n$. By the definition of the stronger Meir-Keeler cone-type mapping ψ , corresponding to η_n use, there exist $\delta_n \gg \theta$ and $\gamma_{\eta_n} \in [0, b)$ with $\eta_n \preceq d(x_n, x_{n+1}) \ll \delta_n + \eta_n$ such that $\psi(d(x_n, x_{n+1})) < \gamma_{\eta_n}$. And we put $\alpha_n = \gamma_{\eta_n}$ and $\alpha_n = \max\{\gamma_{\eta_1}, \gamma_{\eta_2}, \dots, \gamma_{\eta_n}\}$. Taking into account (2.14), we have that

$$u_n - u_{n+1} \succ [b - \alpha_n]d(x_n, x_{n+1}), \quad (2.15)$$

$$u_{n+1} \preceq \frac{\alpha_n^n}{b^n}u_1. \quad (2.16)$$

Let $m, n \in \mathbb{N}$ such that $m > n$. Taking into account (2.15) and (2.16), we obtain that

$$\begin{aligned}
 d(x_n, x_m) &\preccurlyeq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\
 &\preccurlyeq \sum_{i=n}^{m-1} \frac{1}{[b - \alpha_i]} (u_i - u_{i+1}) \\
 &\preccurlyeq \sum_{i=n}^{m-1} \frac{1}{[b - \alpha_m]} (u_i - u_{i+1}) \\
 &\preccurlyeq \frac{1}{[b - \alpha_m]} (u_n - u_m) \\
 &\preccurlyeq \frac{1}{[b - \alpha_m]} \frac{\alpha_n^n}{b^n} u_1.
 \end{aligned} \tag{2.17}$$

Let $e \gg \theta$ be given. Since $\alpha_n < b$ and $\alpha_m < b$, we get

$$\lim_{n \rightarrow \infty} \frac{1}{[b - \alpha_m]} \frac{\alpha_n^n}{b^n} u_1 = \theta. \tag{2.18}$$

Thus, there exists $n_0 \in \mathbb{N}$ such that, for all $m, n \geq n_0$,

$$\frac{1}{[b - \alpha_m]} \frac{\alpha_n^n}{b^n} u_1 \ll e, \tag{2.19}$$

and we also conclude that $d(x_n, x_m) \ll e$ for all $m, n \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $v \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = v. \tag{2.20}$$

By the definition of $\mathfrak{D}(x, Tx)$, since $u_n \in \mathfrak{D}(x_n, Tx_n)$, there exists a sequence $\{z_n\}$ such that $z_n \in Tx_n$, $u_n = d(x_n, z_n)$ for all $n \in \mathbb{N}$. From the convergence of the sequence $\{u_n\}$ and from the lower semicontinuity of the function I , we obtain that

$$\inf_{y \in Tv} \|d(v, y)\| \leq \liminf_{n \rightarrow \infty} \|d(x_n, y)\| \leq \liminf_{n \rightarrow \infty} \|d(x_n, z_n)\| = 0. \tag{2.21}$$

Thus,

$$\inf_{y \in Tv} \|d(v, y)\| = 0. \tag{2.22}$$

We claim that $v \in Tv$. To prove this, on the contrary, assume that $v \notin Tv$. Then by (2.22), there exists a sequence $\{y_n\} \subset Tv$ such that $\lim_{n \rightarrow \infty} \|d(v, y_n)\| = 0$, and hence $\lim_{n \rightarrow \infty} d(y_n, v) = \theta$.

Thus, for any $m, n \geq 0$,

$$d(y_m, y_n) \preceq d(y_m, v) + d(v, y_n). \quad (2.23)$$

Let $\tau \gg \theta$ be given. Then there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $d(y_m, p) \ll 1/2\tau$ and $d(y_n, p) \ll 1/2\tau$, and so

$$d(y_m, y_n) \preceq d(y_m, v) + d(v, y_n) \ll \frac{1}{2}\tau + \frac{1}{2}\tau = \tau. \quad (2.24)$$

This implies that $\{y_n\}$ is a Cauchy sequence in X . Since X is complete, there exists $\mu \in X$ such that

$$\lim_{n \rightarrow \infty} y_n = \mu. \quad (2.25)$$

By the closedness of the Tv , we deduce that $\mu \in Tv$. Then, for any $n \in \mathbb{N}$,

$$d(\mu, v) \preceq d(v, y_n) + d(y_n, \mu). \quad (2.26)$$

Let $\omega \gg \theta$ be given. Since $\lim_{n \rightarrow \infty} d(y_n, v) = \theta$ and $\lim_{n \rightarrow \infty} y_n = \mu$, we can deduce that there exists $M \in \mathbb{N}$ such that, for all $n > M$, $d(y_n, v) \ll 1/2\omega$ and $d(y_n, \mu) \ll 1/2\omega$, and hence

$$d(\mu, v) \ll \frac{1}{2}\omega + \frac{1}{2}\omega = \omega. \quad (2.27)$$

By Proposition 1.13, we obtain that $d(\mu, v) = \theta$ that is, $\mu = v$, which is a contradiction. Therefore, $v \in Tv$. $\square$

Applying Remark 1.7, it is easy to establish the following corollary.

Corollary 2.3. *Let (X, d) be a complete cone metric space, P a normal cone in E , and $T : X \rightarrow C(X)$. Assume that a function $I : X \rightarrow \mathbb{R}$ defined by: $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$ is lower semicontinuous. The following holds.*

(**) *There exists a mapping*

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1] \quad (2.28)$$

such that

$$\limsup_{s \rightarrow t} \psi(s) < b, \quad \forall t \in \text{int}(P) \cup \{\theta\}, \quad (2.29)$$

$$\begin{aligned} \forall x \in X \quad \exists y \in Tx \quad \exists v \in \mathfrak{D}(y, Ty) \quad \forall u \in \mathfrak{D}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \psi(d(x, y))d(x, y)]. \end{aligned} \quad (2.30)$$

Then $\text{Fix}(T) \neq \emptyset$.

Corollary 2.4. *Let (X, d) be a complete cone metric space, P a normal cone in E , and $T : X \rightarrow X$. Assume that a function $I : X \rightarrow \mathbb{R}$ defined by $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$ is lower semicontinuous. The following holds.*

(***) *There exists a stronger Meir-Keeler cone-type mapping*

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1], \quad (2.31)$$

such that

$$d(Tx, T^2x) \preceq \psi(d(x, Tx))d(x, Tx), \quad \forall x \in X. \quad (2.32)$$

Then $\text{Fix}(T) \neq \emptyset$.

Example 2.5. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E, x, y \geq 0\}$, $X = \{a, b, c\} \subset E$ where $a = (0, 0)$, $b = (0, 1)$, $c = (1, 0)$, $d : X \times X \rightarrow E$ defined by $d(a, b) = (1, 2/3)$, $d(c, a) = (4/3, 1)$, $d(b, c) = (7/3, 5/3)$, with $d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$. Let $T : X \rightarrow X$ be defined by $Ta = b$, $Tb = b$, and $Tc = a$. Then $T^2a = b$, $T^2b = b$, $T^2c = b$. Take $\psi(t) = 5/6$ for all $t \in \text{int}(P) \cup \{\theta\}$. Calculation shows that, for $x, y \in X$,

$$d(Tx, T^2x) \preceq \frac{5}{6}d(x, Tx) = \psi(d(x, Tx))d(x, Tx) \quad (2.33)$$

that is, according to Corollary 2.4 we have that $\text{Fix}(T) \neq \emptyset$.

Theorem 2.6. *Let (X, d) be a complete cone metric space, P a normal cone in E , and $T : X \rightarrow K(X)$. Assume that a function $I : X \rightarrow \mathbb{R}$ defined by $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$, is lower semicontinuous. The following hold:*

(A) *There exists a stronger Meir-Keeler cone-type mapping*

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1] \quad (2.34)$$

such that

$$\begin{aligned} \forall x \in X \quad \exists y \in Tx \quad \exists v \in \mathcal{S}(y, Ty) \quad \forall u \in \mathcal{S}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \psi(d(x, y))d(x, y)]. \end{aligned} \quad (2.35)$$

Then $\text{Fix}(T) \neq \emptyset$.

(B) *There exists a stronger Meir-Keeler cone-type mapping*

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1] \quad (2.36)$$

such that

$$\begin{aligned} \forall x \in X \quad \forall y \in Tx \quad \exists v \in \mathcal{S}(y, Ty) \quad \forall u \in \mathcal{S}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \psi(d(x, y))d(x, y)]. \end{aligned} \quad (2.37)$$

Then $\text{Fix}(T) = \text{End}(T) \neq \emptyset$.

Proof. (A) By the same proof of the part (i) of Theorem 3.2 in [15], we have that $\mathcal{S}(x, Tx) \neq \emptyset$ for all $x \in X$. The remainder proof is similar to Theorem 2.2, we can deduce that $\text{Fix}(T) \neq \emptyset$.

(B) By (A), we obtain that the existence of $v \in X$ such that, $v \in Tv$. Taking any $y \in Tv$, we have that, for all $u \in \mathcal{S}(v, Tv)$, $bd(v, y) \preceq u$. Since $v \in Tv$, we get $\theta \in \mathcal{S}(v, Tv)$, and hence $bd(v, y) \preceq \theta$, which gives $v = y$. Thus $Tv = \{v\}$. $\square$

Example 2.7. Let $X = [0, 1]$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E, x, y \geq 0\}$. Let $d(x, y) = (|x - y|, 0)$ for all $x, y \in X$, and let $T : X \rightarrow K(X)$, $Tx = \{(1/2)x\}$. Then $\mathcal{S}(x, Tx) = \{((1/2)x, 0) : x \in X\}$, $I(x) = (1/2)x$. Let $\psi(t) = 2/3$ for all $t \in \text{int}(P) \cup \{\theta\}$. It is easy to check that all conditions of Theorem 2.6 are satisfied and that $\text{Fix}(T) = \text{End}(T) = \{0\} \neq \emptyset$.

Corollary 2.8. Let (X, d) be a complete cone metric space, P a normal cone in E , and $T : X \rightarrow K(X)$. Assume that a function $I : X \rightarrow \mathbb{R}$ defined by $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$ is lower semicontinuous. The following hold:

(C) There exists a mapping

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1], \quad (2.38)$$

such that

$$\begin{aligned} \limsup_{s \rightarrow t} \psi(s) < b, \quad \forall t \in \text{int}(P) \cup \{\theta\}, \\ \forall x \in X \quad \exists y \in Tx \quad \exists v \in \mathcal{S}(y, Ty) \quad \forall u \in \mathcal{S}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \psi(d(x, y))d(x, y)]. \end{aligned} \quad (2.39)$$

Then $\text{Fix}(T) \neq \emptyset$.

(D) There exists a mapping

$$\psi : \text{int}(P) \cup \{\theta\} \longrightarrow [0, b), \quad \text{where } b \in (0, 1], \quad (2.40)$$

such that

$$\begin{aligned} \limsup_{s \rightarrow t} \psi(s) < b, \quad \forall t \in \text{int}(P) \cup \{\theta\}, \\ \forall x \in X \quad \forall y \in Tx \quad \exists v \in \mathcal{S}(y, Ty) \quad \forall u \in \mathcal{S}(x, Tx), \\ [bd(x, y) \preceq u] \wedge [v \preceq \psi(d(x, y))d(x, y)]. \end{aligned} \quad (2.41)$$

Then $\text{Fix}(T) = \text{End}(T) \neq \emptyset$.

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