Research Article

Differential Subordination Results for Certain Integrodifferential Operator and Its Applications

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Received 8 October 2012; Accepted 27 November 2012

Academic Editor: Josip E. Pecaric

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We introduce an integrodifferential operator $J_{s,b}(f)$ which plays an important role in the Geometric Function Theory. Some theorems in differential subordination for $J_{s,b}(f)$ are used. Applications in Analytic Number Theory are also obtained which give new results for Hurwitz-Lerch Zeta function and Polylogarithmic function.

1. Introduction

Let $A$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

Also, let $\mu$ denote the class of analytic functions in the form

$$r(z) = 1 + \sum_{k=1}^{\infty} a_k z^k.$$
We begin by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ defined by (cf., e.g., [1, P. 121 et seq.])

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s}, \quad (1.3)$$

($b \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, \ldots\}, s \in \mathbb{C}$ when $z \in U$, Re$(s) > 1$ when $|z| = 1$)

which contains important functions of Analytic Number Theory, as the Polylogarithmic function:

$$L_i(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z\Phi(z, s, 1), \quad (1.4)$$

($s \in \mathbb{C}$ when $z \in U$, Re$(s) > 1$ when $|z| = 1$).

Several properties of $\Phi(z, s, b)$ can be found in the recent papers, for example Choi et al. [2], Ferreira and López [3], Gupta et al. [4], and Luo and Srivastava [5]. See, also [6–16].

Recently, Srivastava and Attiya [8] introduced the operator $J_{s,b}(f)$ which makes a connection between Geometric Function Theory and Analytic Number Theory, defined by

$$J_{s,b}(f)(z) = G_{s,b}(z) \ast f(z), \quad (1.5)$$

($z \in U; f \in A; b \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$),

where

$$G_{s,b}(z) = (1 + b)^s [\Phi(z, s, b) - b^{-s}] \quad (1.6)$$

and $\ast$ denotes the Hadamard product (or convolution).

Furthermore, Srivastava and Attiya [8] showed that

$$J_{s,b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + b}{k + b}\right)^s a_k z^k. \quad (1.7)$$
As special cases of $J_{s,b}(f)$, Srivastava and Attiya [8] introduced the following identities:

\[ J_{0,b}(f)(z) = f(z), \]
\[ J_{1,0}(f)(z) = \int_0^z \frac{f(t)}{t} \, dt = A(f)(z), \]
\[ J_{1,1}(f)(z) = \frac{2}{z} \int_0^z f(t) \, dt = \mathcal{L}(f)(z), \]
\[ J_{1,\gamma}(f)(z) = \frac{1 + \gamma}{z^\gamma} \int_0^z f(t) t^{1-\gamma} \, dt = \mathcal{L}_\gamma(f)(z) \quad (\gamma \text{ real}; \gamma > -1), \]
\[ J_{0,1}(f)(z) = \frac{2\sigma}{z \Gamma(\sigma)} \int_0^z \left( \log \left( \frac{z}{t} \right) \right)^{\sigma-1} f(t) \, dt = I^\sigma(f)(z) \quad (\sigma \text{ real}; \sigma > 0), \]

where, the operators $A(f)$ and $\mathcal{L}(f)$ are the integral operators introduced earlier by Alexander [17] and Libera [18], respectively, $\mathcal{L}_\gamma(f)$ is the generalized Bernardi operator, $\mathcal{L}_\gamma(f)(\gamma \in \mathbb{N} = \{1, 2, \ldots\})$ introduced by Bernardi [19], and $I^\sigma(f)$ is the Jung-Kim-Srivastava integral operator introduced by Jung et al. [20].

Moreover, in [8], Srivastava and Attiya defined the operator $J_{s,b}(f)$ for $b \in \mathbb{C} \setminus \mathbb{Z}^-$, by using the following relationship:

\[ J_{s,0}(f)(z) = \lim_{b \to 0} J_{s,b}(f)(z). \]  

Some applications of the operator $J_{s,b}(f)$ to certain classes in Geometric Function Theory can be found in [21, 22].

In our investigations we need the following definitions and lemma.

**Definition 1.1.** Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be subordinate to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $U$, with $w(0) = 0$ and $|w(z)| \leq 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

**Definition 1.2.** Let $\Psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be analytic in domain $D$, and let $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ with $(p(z), z p'(z)) \in D$ when $z \in U$, then we say that $p(z)$ satisfies a first order differential subordination if

\[ \Psi(p(z), z p'(z); z) \prec h(z) \quad (z \in U). \]  

The univalent function $q(z)$ is called dominant of the differential subordination (1.10), if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.10), if $\tilde{q}(z) \prec q(z)$ for all dominant of (1.10), then we say that $\tilde{q}(z)$ is the best dominant of (1.10).

**Lemma 1.3** (see [8]). If $z \in U$, $f \in A$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$ and $s \in \mathbb{C}$, then

\[ zJ_{s+1,b}(f)(z) = (1 + b)J_{s,b}(f)(z) - bJ_{s+1,b}(f)(z). \]
The purpose of the present paper is to extend the use of \( J_{s,b}(f) \) as integrodifferential operator, and some theorems in differential subordination for \( J_{s,b}(f) \) are used. Applications in Analytic Number Theory are also obtained which give new results for Hurwitz-Lerch Zeta function and Polylogarithmic function.

2. Making Use of \( J_{s,b}(f) \) as a Differential Operator

From the definition of \( J_{s,b}(f) \) in (1.5) and using (1.7), we obtain the following identities.

For \( z \in U, f \in A, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( b \in \mathbb{C} \setminus \mathbb{Z}^- \), we have

\[
J_{-1,0}(f)(z) = zf'(z), \\
J_{-1,1}(f)(z) = \frac{1}{2} \{f(z) + zf'(z)\}, \\
J_{-1,1/(1-\lambda)}(f)(z) = \lambda f(z) + (1-\lambda)zf'(z) \quad (\lambda \neq 1), \\
J_{-n,0}(f)(z) = D^n(f)(z), \\
J_{-n/(1-\lambda)-1}(f)(z) = D^n_1(f)(z) \quad (\lambda \neq 0), \\
J_{-n,\lambda}(f)(z) = I^n_{\lambda}(f)(z) \quad (\lambda > -1), \\
J_{-n,1}(f)(z) = I_n(f)(z),
\]

(2.1)

where \( D^n(f) \) is the Salagean differential operator which introduced by Sälagean [23], \( D^n_1(f) \) is the generalized of operator, \( D^n_1(f) (\lambda > 0; \text{real}) \) introduced by Al-Oboudi [24], \( I^n_{\lambda}(f) \) was studied by Cho and Srivastava [25] and by Cho and Kim [26], and the operator \( I_n(f) \) was studied by Uralegaddi and Somanatha [27].

Also, we note that

\[
J_{-n,0}(f)(z) = Li_n(z) \ast f(z) \quad (n \in \mathbb{N}_0; f \in A), \\
J_{-n,1}(f)(z) = \frac{Li_n(z)}{z} \ast f(z) \quad (n \in \mathbb{N}_0; f \in A),
\]

(2.2)

where \( Li_n(z) \) is the Polylogarithmic function defined by (1.4).

Now, we prove the following lemma.

Lemma 2.1. If \( z \in U, f \in A, n \in \mathbb{N}_0 \) and \( b \in \mathbb{C} \setminus \mathbb{Z}^- \), then

\[
J_{-n,b}(f)(z) = \frac{1}{(1+b)^n} (zD + b)^n f(z) \left( D := \frac{d}{dz} \right),
\]

(2.3)
where \((zD + b)^n = (zD + b) \circ (zD + b) \circ \cdots \circ (zD + b)\) to \(n\)-times, and \(\circ\) denotes the composition \((I \circ f)(z) = I(f(z))\).

Proof. Putting \(s = -n (n \in \mathbb{N}_0)\) in (1.11), we have

\[
(1 + b)(J_{-n,b})(f)(z) = \left[\frac{d}{dz}J_{-n+1,b}(f)(z) + bJ_{-n+1,b}(f)(z)\right]
\]

\[
= (zD + b)J_{-n+1,b}(f)(z) \quad \left(D := \frac{d}{dz}\right),
\]

therefore,

\[
J_{-n,b}(f)(z) = \frac{1}{1+b}(zD + b)J_{-n+1,b}(f)(z).
\]

Noting that the relation (2.5) is a recurrence relation, by using mathematical induction, we get (2.3), which completes the proof of the lemma.

Putting \(f(z) = f_0(z) = z/(1-z)\) in Lemma 2.1, we obtain the following properties for both Hurwitz-Lerch Zeta function \(\Phi(z,s,b)\) and Polylogarithmic function \(Li_s(z)\).

**Corollary 2.2.** Let \(\Phi(z,s,b)\) and \(Li_s(z)\) be the Hurwitz-Lerch Zeta function and Polylogarithmic function defined by (1.3) and (1.4), respectively, then we have

\[
\Phi(z,-n,b) = b^n + \left(z \frac{d}{dz} + b\right)^n \left(\frac{z}{1-z}\right) \quad (|z| < 1),
\]

\[
Li_{-n}(z) = z \left\{1 + \left(z \frac{d}{dz} + 1\right)^n \left(\frac{z}{1-z}\right)\right\} \quad (|z| < 1),
\]

where \(b \in \mathbb{C} \setminus \mathbb{Z}_0^-\) and \(n \in \mathbb{N}_0\).

**Example 2.3.** Using Corollary 2.2, we have the following well known results for \(z(z \in \mathbb{C}; |z| < 1)\).

(i) \(\Phi(z,0,b) = 1/(1-z)\).

(ii) \(\Phi(z,-1,b) = b + ((1+b)z - bz^2)/(1-z)^2\).

(iii) \(\Phi(z,-2,b) = b^2 + ((1+b)^2z + (1-2b-2b^2)z^2 + b^2z^3)/(1-z)^3\).

(iv) \(Li_0(z) = z/(1-z)\).

(v) \(Li_{-1}(z) = z/(1-z)^2\).

(vi) \(Li_{-2}(z) = z(1+z)/(1-z)^3\).

**3. Applications of Differential Subordination for** \(J_{s,b}(f)\)

To prove our results, we need the following lemmas due to Hallenbeck and Ruscheweyh [28] and Miller and Mocanu [29], respectively, see also Miller and Mocanu [30].
Lemma 3.1. Let $h(z)$ be convex univalent in $U$, with $h(0) = 1$, $\gamma \neq 0$ and $\Re(\gamma) \geq 0$. If $q(z) \in \mu$ and
\[
q(z) + \frac{zq'(z)}{\gamma} \prec h(z),
\] (3.1)
then
\[
q(z) \prec S(z) \prec h(z),
\] (3.2)
where
\[
S(z) = \frac{\gamma}{2\pi} \int_0^z h(t)t^{\gamma-1}dt.
\] (3.3)

The function $S(z)$ is convex univalent and is the best dominant.

Lemma 3.2. Let $\lambda > 0$, and let $\beta_0 = \beta_0(\lambda)$ be the root of the equation as follows:
\[
\beta \pi = \frac{3\pi}{2} - \tan^{-1}(\lambda \beta).
\] (3.4)

In addition, let $\alpha = \alpha(\beta, \lambda) = \beta + (2/\pi)\tan^{-1}(\lambda \pi)$, for $0 < \beta \leq \beta_0$.

If $p(z) \in \mu$ and
\[
p(z) + \lambda z p'(z) \prec \left[\frac{1+z}{1-z}\right]^\alpha
\] (3.5)
then
\[
p(z) \prec \left[\frac{1+z}{1-z}\right]^{\beta}.
\] (3.6)

Now, we define the function $L(f)(z) := L_{s,b,\lambda}(f)(z)$ as the following:
\[
L(f)(z) = (1 - \lambda - \lambda b) J_{s,b}(f)(z) + \lambda (1 + b) J_{s-1,b}(f)(z) \quad (z \in U),
\] (3.7)
\[
(z \in U; f \in A; b \in \mathbb{C} \setminus \mathbb{Z}; \{s, \lambda \in \mathbb{C}; \lambda \neq 0; \Re \lambda \geq 0\}).
\]

Theorem 3.3. Let the function $L(f)(z)$ defined by (3.7) and for some $\alpha (0 \leq \alpha < 1)$. If
\[
\Re\left\{\frac{L(f)(z)}{z}\right\} > \alpha,
\] (3.8)
then

$$\text{Re}\left\{ \frac{J_{s,b}(f)(z)}{z} \right\} > (2\alpha - 1) + 2(1 - \alpha) \frac{1}{z} F_1 \left( 1; \frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1 \right).$$

(3.9)

The constant \((2\alpha - 1) + 2(1 - \alpha) \frac{1}{z} F_1 (1, 1/\lambda; (1/\lambda) + 1, -1)\) is the best estimate.

**Proof.** Defining the function \(q(z) = J_{s,b}(f)(z)/z\), then we have \(q(z) \in \mu\).

If we take \(\gamma = 1/\lambda\), and the convex univalent function \(h(z)\) defined by

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad 0 \leq \alpha < 1,$$

(3.10)

then, we have

$$q(z) + \frac{zq'(z)}{\gamma} = (1 - \lambda) \frac{J_{s,b}(f)(z)}{z} + \lambda J_{s,b}'(f)(z).$$

(3.11)

Using Lemma 1.3 and (3.7), therefore (3.11) can be written as

$$q(z) + \frac{zq'(z)}{\gamma} = \frac{L(f)(z)}{z},$$

(3.12)

then,

$$q(z) + \frac{zq'(z)}{\gamma} < h(z),$$

(3.13)

where \(h(z)\) is defined by (3.10) satisfying \(h(0) = 1\).

Applying Lemma 3.1, we obtain that \(J_{s,b}(f)(z)/z < S(z)\), where the convex univalent function \(S(z)\) defined by

$$S(z) = \frac{1}{\lambda z^{1/\lambda}} \int_0^z \frac{1 + (2\alpha - 1)\frac{t}{1 + t}}{1 + t} t^{(1/\lambda) - 1} dt.$$

(3.14)

Since \(\text{Re}\{h(z)\} > 0\) and \(S(z) < h(z)\), we have \(\text{Re}\{S(z)\} > 0\).

This implies that

$$\inf_{z \in U} \text{Re}\{S(z)\} = S(1) = (2\alpha - 1) + \frac{2}{\lambda}(1 - \alpha) \int_0^1 \frac{u^{(1/\lambda) - 1}}{1 + u} du$$

$$= (2\alpha - 1) + 2(1 - \alpha) \int_0^1 \frac{dt}{1 + t^\lambda}$$

$$= (2\alpha - 1) + 2(1 - \alpha) \frac{1}{\lambda} F_1 \left( 1; \frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1 \right).$$

(3.15)
Hence, the constant \(2\alpha - 1 + 2(1 - \alpha) \tfrac{1}{\lambda_1} F_1(1, 1/\lambda; 1/\lambda + 1, -1)\) cannot be replaced by any larger one.

This completes the proof of Theorem 3.3. \(\square\)

**Theorem 3.4.** Let the function \(L(f)(z)\) with \(\lambda > 0\) real, defined by (3.7), and let \(\beta_0\) satisfy the following equation:

\[
\beta_0 \pi + \tan^{-1} \left( \frac{\beta_0}{2} \right) = \frac{3\pi}{2}.
\] (3.16)

If

\[
\frac{L(f)(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\beta + (2/\pi)\tan^{-1}(\lambda\beta)},
\] (3.17)

then

\[
\frac{J_{s,b}(f)(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\beta} (0 < \beta \leq \beta_0).
\] (3.18)

**Proof.** Defining the function \(p(z) = J_{s,b}(f)(z)/z \in \mu\), then we have

\[
p(z) + \lambda z p'(z) = (1 - \lambda) \frac{J_{s,b}(f)(z)}{z} + \lambda J_{s,b}'(f)(z).
\] (3.19)

Using Lemma 1.3 and (3.7), therefore (3.11) can be written as

\[
p(z) + \lambda z p'(z) = \frac{L(f)(z)}{z}.
\] (3.20)

This completes the proof of Theorem 3.4 after applying Lemma 3.2 \(\square\)

**4. Applications in Analytic Number Theory**

Putting \(f(z) = f_0(z) = z/(1 - z)\) in Theorem 3.3, then we have the following property of Hurwitz-Lerch Zeta function.

**Corollary 4.1.** Let the function \(G_{s,b}(z)\) defined by (1.6). If

\[
\Re \left\{ \frac{(1 - \lambda - \lambda b)G_{s,b}(z) + \lambda (1 + b)G_{s-1,b}(z)}{z} \right\} > \alpha,
\] (4.1)

then

\[
\Re \left\{ \frac{G_{s,b}(z)}{z} \right\} > (2\alpha - 1) + 2(1 - \alpha) \tfrac{1}{\lambda} F_1 \left( 1, 1; \frac{1}{\lambda}; 1 + 1, -1 \right),
\] (4.2)

where \(z \in \mathbb{U}, 0 \leq \alpha < 1, b \in \mathbb{C} \setminus \mathbb{Z}^{-}\) and \(\{s, \lambda \in \mathbb{C}; \lambda \neq 0, \Re \lambda \geq 0\}\).
The constant \((2\alpha - 1) + 2(1 - \alpha) \binom{2}{1} F_1\left(\frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1\right)\) is the best estimate.

Putting \(f(z) = f_0(z) = z/(1 - z)\) in Theorem 3.4, then we have another property of Hurwitz-Lerch Zeta function.

**Corollary 4.2.** Let the function \(G_{s,b}(z)\) defined by (1.6), and let \(\beta_0\) satisfy the following equation:

\[
\beta_0 \pi + \tan^{-1}(\lambda \beta_0) = \frac{3\pi}{2}. \tag{4.3}
\]

If

\[
\left(1 - \lambda - \lambda b\right) G_{s,b}(z) + \lambda (1 + b) G_{s-1,b}(z) \prec \left[\frac{1 + z}{1 - z}\right]^{\beta + (2/\pi) \tan^{-1}(\lambda \beta)}, \tag{4.4}
\]

then

\[
\frac{G_{s,b}(z)}{z} \prec \left[\frac{1 + z}{1 - z}\right]^\beta \quad (0 < \beta \leq \beta_0), \tag{4.5}
\]

where \(z \in U\), \(b \in \mathbb{C} \setminus \mathbb{Z}^-, s \in \mathbb{C}\) and \(\lambda > 0; \text{ real}\).

Putting \(f(z) = f_0(z) = z/(1 - z)\) and \(b = 1\) in Theorem 3.3, then we have the following property of Polylogarithmic function.

**Corollary 4.3.** Let the function \(H_s(z)\) defined by

\[
H_s(z) = 2^s \left[\frac{L_h(z)}{z} - 1\right]. \tag{4.6}
\]

If

\[
\text{Re}\left\{\frac{(1 - 2\lambda) H_s(z) + 2\lambda H_{s-1}(z)}{z}\right\} > \alpha, \tag{4.7}
\]

then

\[
\text{Re}\left\{\frac{H_s(z)}{z}\right\} > (2\alpha - 1) + 2(1 - \alpha) \binom{2}{1} F_1\left(\frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1\right), \tag{4.8}
\]

where \(z \in U\), \(0 \leq \alpha < 1\) and \(\{s, \lambda \in \mathbb{C}; \lambda \neq 0; \text{ Re } \lambda \geq 0\}\).

The constant \((2\alpha - 1) + 2(1 - \alpha) \binom{2}{1} F_1\left(\frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1\right)\) is the best estimate.

Putting \(f(z) = f_0(z) = z/(1 - z)\) and \(b = 1\) in Theorem 3.4, then we have the following property of Polylogarithmic function.
Corollary 4.4. Let the functions \( G_{s,b}(z) \) and \( H_{s}(z) \) defined by (1.6) and (4.6), respectively, and let \( \beta_0 \) satisfy the following:

\[
\beta_0 \pi + \tan^{-1}(\lambda \beta_0) = \frac{3\pi}{2}.
\]

If

\[
\frac{(1 - 2\lambda)H_{s}(z) + 2\lambda H_{s-1}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\beta + (2/\pi)\tan^{-1}(\lambda \beta)},
\]

then

\[
\frac{G_{s,b}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\beta}, \quad (0 < \beta \leq \beta_0),
\]

where \( z \in \mathbb{U}, \ s \in \mathbb{C} \) and \( \lambda > 0; \ \text{real} \).

Setting \( f(z) = f_0(z) = z/(1 - z), \ b = 1 \) and \( \lambda = 1/2 \) in Theorem 3.3, then we have the following property of Polylogarithmic function.

Corollary 4.5. Let the function \( H_{s}(z) \) defined by (4.6).

If

\[
\text{Re}\left\{ \frac{H_{s-1}(z)}{z} \right\} > \alpha,
\]

then

\[
\text{Re}\left\{ \frac{H_{s}(z)}{z} \right\} > 2(2\ln 2 - 1)\alpha + (3 - 4\ln 2),
\]

where \( z \in \mathbb{U}, \ 0 \leq \alpha < 1 \) and \( s \in \mathbb{C} \).

The constant \( 2(2\ln 2 - 1)\alpha + (3 - 4\ln 2) \) is the best estimate.

Taking \( f(z) = f_0(z) = z/(1 - z), \ b = 1 \) and \( \lambda = 1/2 \) in Theorem 3.4, then we have the following property of polylogarithmic function.

Corollary 4.6. Let the function \( H_{s}(z) \) defined by (4.6).

If

\[
\frac{H_{s-1}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\beta + (2/\pi)\tan^{-1}(\beta)},
\]

then

\[
\frac{H_{s}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\beta}, \quad (0 < \beta \leq 1.3148754023 \ldots),
\]

where \( z \in \mathbb{U} \) and \( s \in \mathbb{C} \).
Corollary 4.7. Let the function $H_s(z)$ defined by (4.6) as follows:

If

$$\frac{H_{s-1}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{3/2},$$

then

$$\text{Re}\left\{ \frac{H_{s+n}(z)}{z} \right\} > 1 - (4 \ln 2 - 2)^n \quad (n \in \mathbb{N}_0),$$

where $z \in \mathbb{U}$ and $s \in \mathbb{C}$.

Proof. Let $H_{s-1}(z)$ satisfy the condition (4.16). Also, putting $f(z) = f_0(z) = z/(1 - z)$, $b = 1$, $\lambda = 1/2$ and $\beta = 1$ in Theorem 3.4.

Using (4.16), then we have

$$\frac{H_s(z)}{z} < \left[ \frac{1 + z}{1 - z} \right],$$

therefore

$$\text{Re}\left\{ \frac{H_s(z)}{z} \right\} > 0.$$  \hspace{1cm} (4.19)

Corollary 4.5, gives

$$\text{Re}\left\{ \frac{H_{s+1}(z)}{z} \right\} > 3 - 4 \ln 2.$$  \hspace{1cm} (4.20)

Applied (4.11) again and to $n$-times, which gives (4.17). This completes the proof of Corollary 4.7. \hfill \Box

Finally, we can put Corollary 4.7 in the following form.

Corollary 4.8. Let the function $H_s(z)$ defined by (4.6).

If

$$\left| \text{Arg} \left( \frac{H_{s-1}(z)}{z} \right) \right| < \frac{3\pi}{4},$$

then

$$\text{Re}\left\{ \frac{H_{s+n}(z)}{z} \right\} > 1 - (4 \ln 2 - 2)^n \quad (n \in \mathbb{N}_0),$$

where $z \in \mathbb{U}$ and $s \in \mathbb{C}$. 
Acknowledgments

This research was funded by the Deanship of Scientific Research (DSR), King Abdul-Aziz University, Jeddah, Saudi Arabia, under Grant no. 103-130-D1432. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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