Research Article

Univalence Criteria for Two Integral Operators

Laura Filofteia Stanciu¹ and Daniel Breaz²

¹ Department of Mathematics, University of Pitești, Târgul din Vale Street No. 1,
 110040 Pitești, Argeș, Romania

² Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, Street N. Iorga,
 No. 11-13, 510000 Alba Iulia, Romania

Correspondence should be addressed to Daniel Breaz, breazdaniel@yahoo.com

Received 6 October 2011; Accepted 17 December 2011

Academic Editor: Ibrahim Sadek

Copyright © 2012 L. F. Stanciu and D. Breaz. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study the univalence conditions for two integral operators to be univalent in the open unit disk. Many known univalence conditions are written to prove our main results.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}, \quad (1.2)$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0. \quad (1.3)$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$.

In [1], for $0 < b \leq 1$, Silverman considered the class:

$$G_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf'(z)}{f(z)} \right|, \quad z \in \mathbb{U} \right\}. \quad (1.4)$$

Here, in our present investigation, we consider two general families of integral operators:

$$I(f_1, \dots, f_n; g_1, \dots, g_n)(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{(\alpha_i-1)/M_i} (g'_i(t))^{\gamma_i} dt \right)^{1/\beta}, \quad (1.5)$$

$\alpha_i, \gamma_i \in \mathbb{C}; \beta \in \mathbb{C} \setminus \{0\}; f_i, g_i \in \mathcal{A}, M_i \geq 1$ for all $i \in \{1, 2, \dots, n\}$;

$$J(f_1, \dots, f_n; g_1, \dots, g_n)(z) = \left(\left(1 + \sum_{i=1}^n \alpha_i \right) \int_0^z \prod_{i=1}^n (f_i(t))^{\alpha_i} (g'_i(t))^{\gamma_i} dt \right)^{1/(1+\sum_{i=1}^n \alpha_i)}, \quad (1.6)$$

$\alpha_i, \gamma_i \in \mathbb{C}; f_i, g_i \in \mathcal{A}$ for all $i \in \{1, 2, \dots, n\}$.

Many authors have studied the problem of integral operators which preserve the class $\mathcal{S}$ (see, e.g., [2–5]).

In the present paper, we study the univalence conditions involving the general families of integral operators defined by (1.5) and (1.6).

In the proof of our main results (Theorem 2.1 and Theorem 3.1), we need the following univalence criterion. The univalence criterion, asserted by Theorem 1.1, is a generalization of Ahlfors' and Becker's univalence criterion; it was proven by Pescar [6].

Theorem 1.1 (see Pescar [6]). *Let $\beta \in \mathbb{C}$ with $\operatorname{Re} \beta > 0$, $c \in \mathbb{C}$ with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies*

$$\left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zf''(z)}{\beta f'(z)} \right| \leq 1, \quad (1.7)$$

for all $z \in \mathbb{U}$, then the integral operator,

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t) dt \right)^{1/\beta}, \quad (1.8)$$

is in the class $\mathcal{S}$.

Finally, in the present investigation, one also needs the familiar Schwarz Lemma (see, for details, [7]).

Lemma 1.2 ((General Schwarz Lemma) (see [7])). *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M . If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R). \quad (1.9)$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m, \quad (1.10)$$

where θ is constant.

2. Univalence Conditions for $I(f_1, \dots, f_n; g_1, \dots, g_n)(z)$

Theorem 2.1. Let $M_i \geq 1$ ($i \in \{1, 2, \dots, n\}$) and $\beta, \alpha_i, \gamma_i$ be complex numbers with $\operatorname{Re} \beta \geq 0$ and

$$\operatorname{Re} \beta \geq \sum_{i=1}^n [3|\alpha_i - 1| + |\gamma_i|(2b_i + 1)], \quad (2.1)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [3|\alpha_i - 1| + |\gamma_i|(2b_i + 1)]. \quad (2.2)$$

If for all $i \in \{1, 2, \dots, n\}$, $f_i \in \mathcal{A}$ satisfy the conditions:

$$|f_i(z)| \leq M_i \quad (z \in \mathbb{U}), \quad \left| \frac{z^2 f'_i(z)}{f_i^2(z)} - 1 \right| \leq \frac{2M_i - 1}{M_i} \quad (z \in \mathbb{U}), \quad (2.3)$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ with

$$\left| \frac{z g'_i(z)}{g_i(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (2.4)$$

then the integral operator $I(f_1, \dots, f_n; g_1, \dots, g_n)(z)$ defined by (1.5) is in the class $\mathcal{S}$.

Proof. We begin by setting

$$h(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{(\alpha_i-1)/M_i} (g'_i(t))^{\gamma_i} dt, \quad (2.5)$$

and then we calculate for $h(z)$ the derivatives of the first and second orders.

From (2.5), we obtain

$$\begin{aligned} h'(z) &= \prod_{i=1}^n \left(\frac{f_i(z)}{z} \right)^{(\alpha_i-1)/M_i} (g'_i(z))^{\gamma_i}, \\ \frac{z h''(z)}{h'(z)} &= \sum_{i=1}^n \left[\frac{\alpha_i - 1}{M_i} \left(\frac{z f'_i(z)}{f_i(z)} - 1 \right) + \gamma_i \frac{z g''_i(z)}{g'_i(z)} \right] \\ &= \sum_{i=1}^n \left[\frac{\alpha_i - 1}{M_i} \left(\frac{z f'_i(z)}{f_i(z)} - 1 \right) \right. \\ &\quad \left. + \gamma_i \left(\frac{z g''_i(z)}{g'_i(z)} - \frac{z g'_i(z)}{g_i(z)} + 1 \right) + \gamma_i \left(\frac{z g'_i(z)}{g_i(z)} - 1 \right) \right]. \end{aligned} \quad (2.6)$$

Thus, we have

$$\begin{aligned}
\left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{i=1}^n \left[\frac{|\alpha_i - 1|}{M_i} \left(\left| \frac{zf'_i(z)}{f_i(z)} \right| + 1 \right) \right. \\
&\quad \left. + |\gamma_i| \left| \frac{zg''_i(z)}{g'_i(z)} - \frac{zg'_i(z)}{g_i(z)} + 1 \right| + |\gamma_i| \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| \right] \\
&\leq \sum_{i=1}^n \left[\frac{|\alpha_i - 1|}{M_i} \left(\left(\left| \frac{z^2 f'_i(z)}{f_i^2(z)} - 1 \right| + 1 \right) \left| \frac{f_i(z)}{z} \right| + 1 \right) \right. \\
&\quad \left. + |\gamma_i| \left| \frac{zg''_i(z)}{g'_i(z)} - \frac{zg'_i(z)}{g_i(z)} + 1 \right| + |\gamma_i| \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| \right].
\end{aligned} \tag{2.7}$$

From the hypothesis (2.3) of Theorem 2.1, we have

$$\begin{aligned}
|f_i(z)| &\leq M_i \quad (z \in \mathbb{U}; M_i \geq 1), \\
\left| \frac{z^2 f'_i(z)}{f_i^2(z)} - 1 \right| &\leq \frac{2M_i - 1}{M_i} \quad (z \in \mathbb{U}; M_i \geq 1)
\end{aligned} \tag{2.8}$$

for all $i \in \{1, 2, \dots, n\}$.

By applying the General Schwarz Lemma, we thus obtain

$$|f_i(z)| \leq M_i |z| \quad (z \in \mathbb{U}; i \in \{1, 2, \dots, n\}). \tag{2.9}$$

Since $g_i \in G_{b_i}$, $0 < b_i \leq 1$ for all $i \in \{1, 2, \dots, n\}$, from (1.4), (2.4), we obtain

$$\begin{aligned}
\left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{i=1}^n \left[3|\alpha_i - 1| + |\gamma_i| b_i \left| \frac{zg'_i(z)}{g_i(z)} \right| + |\gamma_i| \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| \right] \\
&\leq \sum_{i=1}^n \left[3|\alpha_i - 1| + |\gamma_i| b_i \left(\left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| + 1 \right) + |\gamma_i| \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| \right] \\
&\leq \sum_{i=1}^n \left[3|\alpha_i - 1| + |\gamma_i| b_i \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| + |\gamma_i| b_i + |\gamma_i| \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| \right] \\
&\leq \sum_{i=1}^n \left[3|\alpha_i - 1| + (|\gamma_i| b_i + |\gamma_i|) \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| + |\gamma_i| b_i \right] \\
&\leq \sum_{i=1}^n [3|\alpha_i - 1| + |\gamma_i| (2b_i + 1)],
\end{aligned} \tag{2.10}$$

which readily shows that

$$\begin{aligned} \left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| &\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^n [3|\alpha_i - 1| + |\gamma_i|(2b_i + 1)] \\ &\leq |c| + \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [3|\alpha_i - 1| + |\gamma_i|(2b_i + 1)] \\ &\leq 1, \end{aligned} \quad (2.11)$$

where we have also used the hypothesis (2.2) of Theorem 2.1.

Finally, by applying Theorem 1.1, we conclude that the integral operator $I(f_1, \dots, f_n; g_1, \dots, g_n)(z)$ defined by (1.5) is in the class $\mathcal{S}$. This evidently completes the proof of Theorem 2.1. $\square$

Setting $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let $M_i \geq 1$ ($i \in \{1, 2, \dots, n\}$) and β, γ_i be complex numbers with $\operatorname{Re} \beta \geq 0$ and*

$$\operatorname{Re} \beta \geq \sum_{i=1}^n [|\gamma_i|(2b_i + 1)] \quad (2.12)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [|\gamma_i|(2b_i + 1)]. \quad (2.13)$$

If for all $i \in \{1, 2, \dots, n\}$, $g_i \in G_{b_i}$, $0 < b_i \leq 1$ with

$$\left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (2.14)$$

then the integral operator,

$$I(f_1, \dots, f_n; g_1, \dots, g_n)(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^n (g'_i(t))^{\gamma_i} dt \right)^{1/\beta}, \quad (2.15)$$

is in the class $\mathcal{S}$.

Setting $\gamma_i = 1$ for all $i \in \{1, 2, \dots, n\}$ in Theorem 2.1, we have the following result.

Corollary 2.3. *Let $M_i \geq 1$ ($i \in \{1, 2, \dots, n\}$) and β, α_i be complex numbers with $\operatorname{Re} \beta \geq 0$ and*

$$\operatorname{Re} \beta \geq \sum_{i=1}^n [3|\alpha_i - 1| + (2b_i + 1)], \quad (2.16)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [3|\alpha_i - 1| + (2b_i + 1)]. \quad (2.17)$$

If for all $i \in \{1, 2, \dots, n\}$, $f_i \in \mathcal{A}$ satisfy the conditions:

$$|f_i(z)| \leq M_i \quad (z \in \mathbb{U}), \quad \left| \frac{z^2 f'_i(z)}{f_i^2(z)} - 1 \right| \leq \frac{2M_i - 1}{M_i} \quad (z \in \mathbb{U}) \quad (2.18)$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ with

$$\left| \frac{z g'_i(z)}{g_i(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (2.19)$$

then the integral operator

$$I(f_1, \dots, f_n; g_1, \dots, g_n)(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{(\alpha_i-1)/M_i} (g'_i(t)) dt \right)^{1/\beta} \quad (2.20)$$

is in the class $\mathcal{S}$.

Setting $n = 1$ in Theorem 2.1, we have the following result.

Corollary 2.4. Let $M \geq 1$ and β, α, γ be complex numbers with $\operatorname{Re} \beta \geq 0$ and

$$\operatorname{Re} \beta \geq [3|\alpha - 1| + |\gamma|(2b + 1)], \quad (2.21)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} [3|\alpha - 1| + |\gamma|(2b + 1)]. \quad (2.22)$$

If the function $f \in \mathcal{A}$ satisfies the conditions:

$$|f(z)| \leq M \quad (z \in \mathbb{U}), \quad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq \frac{2M - 1}{M} \quad (z \in \mathbb{U}), \quad (2.23)$$

and $g \in G_b$, $0 < b \leq 1$ with

$$\left| \frac{z g'(z)}{g(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (2.24)$$

then the integral operator,

$$I(f; g)(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} \right)^{(\alpha-1)/M} (g'(t))^Y dt \right)^{1/\beta}, \quad (2.25)$$

is in the class $\mathcal{S}$.

3. Univalence Conditions for $J(f_1, \dots, f_n; g_1, \dots, g_n)(z)$

Theorem 3.1. Let $M_i \geq 1$ ($i \in \{1, 2, \dots, n\}$) and $\beta, \alpha_i, \gamma_i$ be complex numbers, $\beta = (1 + \sum_{i=1}^n \alpha_i)$, $\operatorname{Re} \beta \geq 0$ and

$$\operatorname{Re} \beta \geq \sum_{i=1}^n [|\alpha_i| + |\gamma_i|(b_i + 1)(2M_i + 1) + |\gamma_i|b_i], \quad (3.1)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [|\alpha_i| + |\gamma_i|(b_i + 1)(2M_i + 1) + |\gamma_i|b_i]. \quad (3.2)$$

If for all $i \in \{1, 2, \dots, n\}$, $f_i \in \mathcal{A}$ satisfy the condition:

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (3.3)$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ with

$$\left| \frac{z^2 g'_i(z)}{g_i^2(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (3.4)$$

$$|g_i(z)| \leq M_i \quad (z \in \mathbb{U}; i \in \{1, 2, \dots, n\}), \quad (3.5)$$

then the integral operator $J(f_1, \dots, f_n; g_1, \dots, g_n)(z)$ defined by (1.6) is in the class $\mathcal{S}$.

Proof. We begin by observing that the integral operator $J(f_1, \dots, f_n; g_1, \dots, g_n)(z)$ defined by (1.6) can be rewritten as follows:

$$J(f_1, \dots, f_n; g_1, \dots, g_n)(z) = \left(\left(1 + \sum_{i=1}^n \alpha_i \right) \int_0^z t^{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\alpha_i} (g'_i(t))^{\gamma_i} dt \right)^{1/(1+\sum_{i=1}^n \alpha_i)}, \quad (3.6)$$

where $f_i \in \mathcal{A}$ for all $i \in \{1, 2, \dots, n\}$.

Defining the function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\alpha_i} (g'_i(t))^{\gamma_i} dt, \quad (3.7)$$

we take the same steps as in the proof of Theorem 2.1, and we have

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^n \left[|\alpha_i| \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| + (|\gamma_i|b_i + |\gamma_i|) \left| \frac{zg'_i(z)}{g_i(z)} - 1 \right| + |\gamma_i|b_i \right]. \quad (3.8)$$

Thus, we have

$$\begin{aligned} & \left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \\ & \leq |c| + \frac{1}{|\beta|} \sum_{i=1}^n \left[|\alpha_i| + (|\gamma_i|b_i + |\gamma_i|) \left(\left| \frac{z^2 g'_i(z)}{g_i^2(z)} \right| \left| \frac{g_i(z)}{z} \right| + 1 \right) + |\gamma_i|b_i \right]. \end{aligned} \quad (3.9)$$

Furthermore, from the hypothesis (3.4) of Theorem 3.1, we have

$$\begin{aligned} & \left| \frac{z^2 g'_i(z)}{g_i^2(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \\ & |g_i(z)| \leq M_i \quad (z \in \mathbb{U}; i \in \{1, 2, \dots, n\}). \end{aligned} \quad (3.10)$$

By applying the General Schwarz Lemma, we obtain

$$|g_i(z)| \leq M_i |z| \quad (z \in \mathbb{U}; i \in \{1, 2, \dots, n\}). \quad (3.11)$$

So, we obtain

$$\begin{aligned} & \left| c|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| \\ & \leq |c| + \frac{1}{|\beta|} \sum_{i=1}^n \left[|\alpha_i| + (|\gamma_i|b_i + |\gamma_i|) \left(\left(\left| \frac{z^2 g'_i(z)}{g_i^2(z)} - 1 \right| + 1 \right) M_i + 1 \right) + |\gamma_i|b_i \right] \\ & \leq |c| + \frac{1}{|\beta|} \sum_{i=1}^n [|\alpha_i| + (|\gamma_i|b_i + |\gamma_i|)(2M_i + 1) + |\gamma_i|b_i] \\ & \leq |c| + \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [|\alpha_i| + (|\gamma_i|b_i + |\gamma_i|)(2M_i + 1) + |\gamma_i|b_i] \\ & \leq 1. \end{aligned} \quad (3.12)$$

Finally, by applying Theorem 1.1, we conclude that the integral operator $J(f_1, \dots, f_n; g_1, \dots, g_n)(z)$ defined by (1.6) is in the class $\mathcal{S}$. This evidently completes the proof of Theorem 3.1. $\square$

Setting $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ in Theorem 3.1, we have

Corollary 3.2. *Let $M_i \geq 1$ ($i \in \{1, 2, \dots, n\}$) and β, γ_i be complex numbers, $\beta = (1 + n)$, $\operatorname{Re} \beta \geq 0$ and*

$$\operatorname{Re} \beta \geq \sum_{i=1}^n [1 + |\gamma_i|(b_i + 1)(2M_i + 1) + |\gamma_i|b_i], \quad (3.13)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [1 + |\gamma_i|(b_i + 1)(2M_i + 1) + |\gamma_i|b_i]. \quad (3.14)$$

If for all $i \in \{1, 2, \dots, n\}$, $f_i \in \mathcal{A}$ satisfy the condition:

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (3.15)$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ with

$$\left| \frac{z^2 g'_i(z)}{g_i^2(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad |g_i(z)| \leq M_i \quad (z \in \mathbb{U}), \quad (3.16)$$

then the integral operator,

$$J(f_1, \dots, f_n; g_1, \dots, g_n)(z) = \left((1 + n) \int_0^z \prod_{i=1}^n (f_i(t)) (g'_i(t))^{\gamma_i} dt \right)^{1/(1+n)}, \quad (3.17)$$

is in the class $\mathcal{S}$.

Setting $\gamma_i = 1$ for all $i \in \{1, 2, \dots, n\}$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Let $M_i \geq 1$ ($i \in \{1, 2, \dots, n\}$) and β, α_i be complex numbers, $\beta = (1 + \sum_{i=1}^n \alpha_i)$, $\operatorname{Re} \beta \geq 0$ and*

$$\operatorname{Re} \beta \geq \sum_{i=1}^n [|\alpha_i| + (b_i + 1)(2M_i + 1) + b_i], \quad (3.18)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} \sum_{i=1}^n [|\alpha_i| + (b_i + 1)(2M_i + 1) + b_i]. \quad (3.19)$$

If for all $i \in \{1, 2, \dots, n\}$, $f_i \in \mathcal{A}$ satisfy the condition:

$$\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad (3.20)$$

and $g_i \in G_{b_i}$, $0 < b_i \leq 1$ with

$$\left| \frac{z^2 g'_i(z)}{g_i^2(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad |g_i(z)| \leq M_i \quad (z \in \mathbb{U}), \quad (3.21)$$

then the integral operator,

$$J(f_1, \dots, f_n; g_1, \dots, g_n)(z) = \left(\left(1 + \sum_{i=1}^n \alpha_i \right) \int_0^z \prod_{i=1}^n (f_i(t))^{\alpha_i} (g'_i(t)) dt \right)^{1/(1+\sum_{i=1}^n \alpha_i)}, \quad (3.22)$$

is in the class $\mathcal{S}$.

Setting $n = 1$ in Theorem 3.1, we have the following result.

Corollary 3.4. Let $M \geq 1$ and β, α, γ be complex numbers, $\beta = (1 + \alpha)$, $\operatorname{Re} \beta \geq 0$ and

$$\operatorname{Re} \beta \geq [|\alpha| + |\gamma|(b + 1)(2M + 1) + |\gamma|b], \quad (3.23)$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re} \beta} [|\alpha| + |\gamma|(b + 1)(2M + 1) + |\gamma|b]. \quad (3.24)$$

If one has that the function $f \in \mathcal{A}$ satisfies the condition:

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 \quad (z \in \mathbb{U}), \quad (3.25)$$

and $g \in G_b$, $0 < b \leq 1$ with

$$\left| \frac{z^2 g'(z)}{g^2(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq M \quad (z \in \mathbb{U}), \quad (3.26)$$

then the integral operator,

$$J(f; g)(z) = \left((1 + \alpha) \int_0^z (f(t))^\alpha (g'(t))^\gamma dt \right)^{1/(1+\alpha)}, \quad (3.27)$$

is in the class $\mathcal{S}$.

Acknowledgment

This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007–2013 co-financed by the European Social Fund-Investing in People.

References

- [1] H. Silverman, "Convex and starlike criteria," *International Journal of Mathematics and Mathematical Sciences*, vol. 22, no. 1, pp. 75–79, 1999.
- [2] A. Baricz and B. A. Frasin, "Univalence of integral operators involving Bessel functions," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 371–376, 2010.
- [3] V. Pescar, "On the univalence of some integral operators," *General Mathematics*, vol. 14, no. 2, pp. 77–84, 2006.
- [4] D. Blezu and R. N. Pascu, "Univalence criteria for integral operators," *Glasnik Matematički. Serija III*, vol. 36, no. 2, pp. 241–245, 2001.
- [5] D. Breaz and N. Breaz, "Univalence conditions for certain integral operators," *Universitatis Babeş-Bolyai, Studia, Mathematica*, vol. 47, no. 2, pp. 9–15, 2002.
- [6] V. Pescar, "A new generalization of Ahlfors's and Becker's criterion of univalence," *Malaysian Mathematical Society, Bulletin (Second Series)*, vol. 19, no. 2, pp. 53–54, 1996.
- [7] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, NY, USA, 1952.