Research Article

σ -Approximately Contractible Banach Algebras

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We investigate σ -approximate contractibility and σ -approximate amenability of Banach algebras, which are extensions of usual notions of contractibility and amenability, respectively, where σ is a dense range or an idempotent bounded endomorphism of the corresponding Banach algebra.

1. Introduction

For a Banach algebra $\mathcal{A}$, an $\mathcal{A}$ -bimodule will always refer to a Banach $\mathcal{A}$ -bimodule $\mathcal{X}$, that is, a Banach space which is algebraically an $\mathcal{A}$ -bimodule, and for which there is a constant $c \geq 0$ such that for $a \in \mathcal{A}, x \in \mathcal{X}$, we have

$$\|a \cdot x\| \leq c\|a\|\|x\|, \quad \|x \cdot a\| \leq c\|a\|\|x\|. \quad (1.1)$$

A derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ is a linear map, always taken to be continuous, satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}). \quad (1.2)$$

A Banach algebra $\mathcal{A}$ is amenable if for any $\mathcal{A}$ -bimodule $\mathcal{X}$, any derivation $D : \mathcal{A} \rightarrow \mathcal{X}^*$ is inner, that is, there exists $x^* \in \mathcal{X}^*$, with

$$D(a) = a \cdot x^* - x^* \cdot a = \delta_{x^*}(a) \quad (a \in \mathcal{A}). \quad (1.3)$$

Let $\mathcal{A}$ be a Banach algebra and σ a bounded endomorphism of $\mathcal{A}$, that is, a bounded Banach algebra homomorphism from $\mathcal{A}$ into $\mathcal{A}$. A σ -derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$ -bimodule $\mathcal{X}$ is a bounded linear map $D : \mathcal{A} \rightarrow \mathcal{X}$ satisfying

$$D(ab) = \sigma(a) \cdot D(b) + D(a) \cdot \sigma(b) \quad (a, b \in \mathcal{A}). \quad (1.4)$$

For each $x \in \mathcal{X}$, the mapping

$$\delta_x^\sigma : \mathcal{A} \longrightarrow \mathcal{X} \quad (1.5)$$

defined by $\delta_x^\sigma(a) = \sigma(a) \cdot x - x \cdot \sigma(a)$, for all $a \in \mathcal{A}$, is a σ -derivation called an inner σ -derivation.

Remark 1.1. Throughout this paper, we will assume that $\mathcal{A}$ is a Banach algebra, and σ is a bounded endomorphism of $\mathcal{A}$ unless otherwise specified. Also, we write $(\sigma\text{-a.i})$ for σ -approximately inner, $(\sigma\text{-a.a})$ for σ -approximately amenable, and $(\sigma\text{-a.c})$ for σ -approximately contractible.

The basic definition for the present paper is as follows.

Definition 1.2. A σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ is $\sigma\text{-a.i}$, if there exists a net $(x_\alpha) \subseteq \mathcal{X}$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a)$, the limit being in norm and we write $D = \lim \delta_{x_\alpha}^\sigma$. Note that we do not suppose (x_α) to be bounded.

Definition 1.3. A Banach algebra $\mathcal{A}$ is called $\sigma\text{-a.c}$ if for any $\mathcal{A}$ -bimodule $\mathcal{X}$, every σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ is $\sigma\text{-a.i}$.

Definition 1.4. A Banach algebra $\mathcal{A}$ is called $\sigma\text{-a.a}$ if for any $\mathcal{A}$ -bimodule $\mathcal{X}$, every σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}^*$ is $\sigma\text{-a.i}$.

Definition 1.5. Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{X}$ and $\mathcal{Y}$ be Banach $\mathcal{A}$ -bimodules. The linear map $T : \mathcal{X} \rightarrow \mathcal{Y}$ is called a $\sigma\text{-}\mathcal{A}$ -bimodule homomorphism if

$$T(a \cdot x) = \sigma(a) \cdot T(x), \quad T(x \cdot a) = T(x) \cdot \sigma(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \quad (1.6)$$

2. Basic Properties

Proposition 2.1. *Let $\mathcal{A}$ be a $\sigma\text{-a.c}$ Banach algebra. Then $\sigma(\mathcal{A})$ has a left and right approximate identity.*

Proof. Consider $\mathcal{X} = \mathcal{A}$ as a Banach $\mathcal{A}$ -bimodule with the trivial right action, that is,

$$a \cdot x = ax, \quad x \cdot a = 0 \quad (a \in \mathcal{A}, x \in \mathcal{X}). \quad (2.1)$$

Then $D : \mathcal{A} \rightarrow \mathcal{X}$ defined by $D(a) = \sigma(a)$ is a σ -derivation, and so there is a net $\{u_\alpha\} \subseteq \mathcal{X}(=\mathcal{A})$ such that $D = \lim_\alpha \delta_{u_\alpha}^\sigma$. Hence for each $a \in \mathcal{A}$,

$$\sigma(a) = D(a) = \lim_\alpha \delta_{u_\alpha}^\sigma(a) = \lim_\alpha \sigma(a) \cdot u_\alpha - u_\alpha \cdot \sigma(a) = \lim_\alpha \sigma(a)u_\alpha, \quad (2.2)$$

which shows that $\{u_\alpha\}$ is a right approximate identity for $\sigma(\mathcal{A})$. Similarly, one can find a left approximate identity for $\sigma(\mathcal{A})$. $\square$

Corollary 2.2. *Let $\mathcal{A}$ be a σ -a.c Banach algebra and σ a continuous epimorphism of $\mathcal{A}$. Then $\mathcal{A}$ has a left and right approximate identity.*

Proposition 2.3. *Let φ be a bounded endomorphism of Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is σ -a.c, then $\mathcal{A}$ is $(\varphi\sigma)$ -a.c.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule and let $D : \mathcal{A} \rightarrow \mathcal{X}$ be a $(\varphi\sigma)$ -derivation. Then $(\mathcal{X}, *)$ is an $\mathcal{A}$ -bimodule with the following module actions:

$$a * x = \varphi(a) \cdot x, \quad x * a = x \cdot \varphi(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \quad (2.3)$$

For each $a, b \in \mathcal{A}$, we have

$$D(ab) = (\varphi\sigma(a)) \cdot D(b) + D(a) \cdot (\varphi\sigma(b)) = \sigma(a) * D(b) + D(a) * \sigma(b). \quad (2.4)$$

Thus $D : \mathcal{A} \rightarrow (\mathcal{X}, *)$ is a continuous σ -derivation. Since $\mathcal{A}$ is σ -a.c, there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that $D = \lim_\alpha \delta_{x_\alpha}^\sigma$. In fact,

$$\begin{aligned} D(a) &= \lim_\alpha (\sigma(a) * x_\alpha - x_\alpha * \sigma(a)) \\ &= \lim_\alpha (\varphi\sigma(a) \cdot x_\alpha - x_\alpha \cdot \varphi\sigma(a)) \\ &= \lim_\alpha \delta_{x_\alpha}^{\varphi\sigma}(a) \quad (a \in \mathcal{A}). \end{aligned} \quad (2.5)$$

Therefore, D is a $(\varphi\sigma)$ -a.i and so $\mathcal{A}$ is $(\varphi\sigma)$ -a.c. $\square$

Corollary 2.4. *Let $\mathcal{A}$ be an a.c Banach algebra. Then $\mathcal{A}$ is σ -a.c for each bounded endomorphism σ of $\mathcal{A}$.*

Proposition 2.5. *Let $\mathcal{A}$ be a σ -a.c Banach algebra, where σ is a bounded epimorphism of $\mathcal{A}$. Then $\mathcal{A}$ is a.c.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule and let $d : \mathcal{A} \rightarrow \mathcal{X}$ be a continuous derivation. It is easy to see that $d\sigma$ is a σ -derivation. Since $\mathcal{A}$ is σ -a.c, there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that

$d\sigma(a) = \lim_{\alpha} \sigma(a)x_{\alpha} - x_{\alpha}\sigma(a)$. Now for $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ such that $b = \sigma(a)$, and, therefore,

$$\begin{aligned} d(b) &= d(\sigma(a)) = \lim_{\alpha} x_{\alpha}\sigma(a) - \sigma(a)x_{\alpha} \\ &= \lim_{\alpha} x_{\alpha}b - bx_{\alpha}, \end{aligned} \tag{2.6}$$

which shows that d is approximately inner and so $\mathcal{A}$ is a.c. $\square$

Corollary 2.6. *Let φ be a bounded endomorphism of Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is σ -a.a then it is $(\varphi\sigma)$ -a.a too.*

Corollary 2.7. *Let $\mathcal{A}$ be an a.a Banach algebra. For each bounded endomorphism σ , $\mathcal{A}$ is σ -a.a.*

Corollary 2.8. *Let $\mathcal{A}$ be a σ -a.a Banach algebra, where σ is a bounded epimorphism of $\mathcal{A}$. Then $\mathcal{A}$ is a.a.*

Proposition 2.9. *Suppose that $\mathcal{B}$ is a Banach algebra and $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a continuous epimorphism. If $\mathcal{A}$ is a.c, then $\mathcal{B}$ is σ -a.c for each bounded endomorphism σ of $\mathcal{B}$.*

Proof. Let $\sigma : \mathcal{B} \rightarrow \mathcal{B}$ be a bounded endomorphism of $\mathcal{B}$ and $\mathcal{X}$ a Banach $\mathcal{B}$ -bimodule, then $(\mathcal{X}, *)$ is an $\mathcal{A}$ -bimodule with the following module actions:

$$a * x = \sigma(\varphi(a)) \cdot x, \quad x * a = x \cdot \sigma(\varphi(a)) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{2.7}$$

Now let $D : \mathcal{B} \rightarrow \mathcal{X}$ be a continuous σ -derivation. It is easy to check that $D\varphi : \mathcal{A} \rightarrow (\mathcal{X}, *)$ is a derivation. Since $\mathcal{A}$ is approximately contractible, there exists a net $\{x_{\alpha}\} \subseteq \mathcal{X}$ such that $D\varphi(a) = \lim_{\alpha} \delta_{x_{\alpha}}(a)$. We have

$$\begin{aligned} D(\varphi(a)) &= D\varphi(a) = \lim_{\alpha} \delta_{x_{\alpha}}(a) = \lim_{\alpha} (a * x_{\alpha} - x_{\alpha} * a) \\ &= \lim_{\alpha} \sigma(\varphi(a))x_{\alpha} - x_{\alpha}\sigma(\varphi(a)) \quad (a \in \mathcal{A}). \end{aligned} \tag{2.8}$$

Since φ is an epimorphism, so for each $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that $b = \varphi(a)$, and we have

$$D(b) = \lim_{\alpha} \sigma(b)x_{\alpha} - x_{\alpha}\sigma(b), \tag{2.9}$$

which shows that D is σ -a.i and so $\mathcal{B}$ is σ -a.c. $\square$

Proposition 2.10. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, and let σ and τ be bounded endomorphism of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a bounded epimorphism such that $\varphi\sigma = \tau\varphi$. If $\mathcal{A}$ is σ -a.c, then $\mathcal{B}$ is τ -a.c.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{B}$ -bimodule and $D : \mathcal{B} \rightarrow \mathcal{X}$ a continuous τ -derivation. Then $(\mathcal{X}, *)$ is an $\mathcal{A}$ -bimodule with the following actions:

$$a * x = \varphi(a) \cdot x, \quad x * a = x \cdot \varphi(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \quad (2.10)$$

It is easy to check that $D\varphi : \mathcal{A} \rightarrow (\mathcal{X}, *)$ is a σ -derivation. Since $\mathcal{A}$ is σ -a.c, there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that $D\varphi(a) = \lim_\alpha \delta_{x_\alpha}^\sigma(a)$, so we have

$$\begin{aligned} D(\varphi(a)) &= \lim_\alpha \sigma(a) * x_\alpha - x_\alpha * \sigma(a) \\ &= \lim_\alpha \varphi(\sigma(a)) \cdot x_\alpha - x_\alpha \cdot \varphi(\sigma(a)) \\ &= \lim_\alpha \tau(\varphi(a)) \cdot x_\alpha - x_\alpha \cdot \tau(\varphi(a)) \quad (a \in \mathcal{A}). \end{aligned} \quad (2.11)$$

Since φ is epimorphism, so $D(b) = \lim_\alpha \tau(b)x_\alpha - x_\alpha \tau(b)$ for all $b \in \mathcal{B}$, and hence $\mathcal{B}$ is τ -a.c. $\square$

3. σ -Approximate Contractibility for Unital Banach Algebras

In this section we state some properties of σ -approximate contractibility when $\mathcal{A}$ has an identity. First we express the following proposition that one can see its proof in [1, Proposition 3.3], and bring some corollaries when $\sigma(\mathcal{A})$ is dense in $\mathcal{A}$.

Proposition 3.1. *Let $\mathcal{A}$ be a unital Banach algebra with unit e , $\sigma(\mathcal{A})$ dense in $\mathcal{A}$, $\mathcal{X}$ a Banach $\mathcal{A}$ -bimodule, and $D : \mathcal{A} \rightarrow \mathcal{X}$ a σ -derivation. Then, there is a σ -derivation $D_1 : \mathcal{A} \rightarrow e \cdot \mathcal{X} \cdot e$ and $\eta \in \mathcal{X}$, such that $D = D_1 + \delta_\eta$.*

The following definition extends the definition of the unital Banach $\mathcal{A}$ -module in the classical sense.

Definition 3.2. Let $\mathcal{A}$ be a unital Banach algebra with identity e . Banach $\mathcal{A}$ -bimodule $\mathcal{X}$ is called σ -unital if $\mathcal{X} = \sigma(e) \cdot \mathcal{X} \cdot \sigma(e)$.

Corollary 3.3. *Let $\mathcal{A}$ be a unital Banach algebra and $\sigma(\mathcal{A})$ dense in $\mathcal{A}$. Then, $\mathcal{A}$ is σ -a.c (resp., σ -a.a) if and only if for all σ -unital Banach $\mathcal{A}$ -bimodule $\mathcal{X}$, every σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ (resp., $D : \mathcal{A} \rightarrow \mathcal{X}^*$) is σ -a.i.*

Proof. Since $\sigma(e)$ is a unit for $\sigma(\mathcal{A})$, and $\sigma(\mathcal{A})$ is dense in $\mathcal{A}$, we see that $\sigma(e) = e$, so that $e \cdot \mathcal{X} \cdot e$ is a σ -unital Banach $\mathcal{A}$ -bimodule. Now by Proposition 3.1, the proof is complete. $\square$

Corollary 3.4. *Suppose that $\mathcal{A}$ is a unital Banach algebra and $\sigma(\mathcal{A})$ is dense in $\mathcal{A}$. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule and $D : \mathcal{A} \rightarrow \mathcal{X}^*$ a σ -derivation. If $\mathcal{A}$ is σ -a.a, then there exists a net $(\eta_\alpha) \subseteq e \cdot \mathcal{X}^* \cdot e$, and $\eta \in \mathcal{X}^*$, such that $D = \lim_\alpha \delta_{\eta_\alpha}^\sigma + \delta_\eta$.*

Proof. By Proposition 3.1, $D = D_1 + \delta_\eta$ such that $\eta \in \mathcal{X}^*$ and $D_1 : \mathcal{A} \rightarrow e \cdot \mathcal{X}^* \cdot e$ is a σ -derivation. Since $e \cdot \mathcal{X}^* \cdot e \cong (e \cdot \mathcal{X} \cdot e)^*$ and $\mathcal{A}$ is σ -a.a, $D_1 : \mathcal{A} \rightarrow (e \cdot \mathcal{X} \cdot e)^*$ is σ -a.i. Hence $D_1 = \lim_\alpha \delta_{\eta_\alpha}^\sigma$ for some net $(\eta_\alpha) \subseteq e \cdot \mathcal{X}^* \cdot e$. $\square$

In the following proposition we consider σ -approximate contractibility when σ is an idempotent endomorphism of $\mathcal{A}$. We can see the proof of the following proposition in [1, Proposition 4.1].

Proposition 3.5. *Assume that $\mathcal{A}$ has an element e which is a unit for $\sigma(\mathcal{A})$ and $\mathcal{X}$ is a Banach $\mathcal{A}$ -bimodule. If σ is a bounded idempotent endomorphism of $\mathcal{A}$, then for each σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ there exists a σ -derivation $D_1 : \mathcal{A} \rightarrow \sigma(e) \cdot \mathcal{X} \cdot \sigma(e)$ and $\eta \in \mathcal{X}$, such that $D = D_1 + \delta_\eta$.*

Corollary 3.6. *Assume that $\mathcal{A}$ has an element e which is a unit for $\sigma(\mathcal{A})$ and σ is a bounded idempotent endomorphism of $\mathcal{A}$, then $\mathcal{A}$ is σ -a.c (resp., σ -a.a) if and only if for all σ -unital Banach $\mathcal{A}$ -bimodule, $\mathcal{X}$, every σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ (resp., $D : \mathcal{A} \rightarrow \mathcal{X}^*$) is σ -a.i.*

Lemma 3.7. *Assume that $\mathcal{A}$ is a unital Banach algebra with the identity e , and $(\mathcal{X}, *)$ is a σ -unital Banach $\mathcal{A}$ -bimodule with the following module actions:*

$$a * x = \sigma(a)x, \quad x * a = x\sigma(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}). \quad (3.1)$$

If $D : \mathcal{A} \rightarrow \mathcal{X}^*$ is a σ -derivation, then $D(e) = 0$.

Proof. We have $D(e) = D(ee) = \sigma(e)D(e) + D(e)\sigma(e)$ and

$$\begin{aligned} \langle e * x, D(e)\sigma(e) \rangle &= \langle x, D(e)\sigma(e) * e \rangle = \langle x, D(e)\sigma(e)\sigma(e) \rangle \\ &= \langle x, D(e)\sigma(e) \rangle = \langle e * x, D(e) \rangle \quad (x \in \mathcal{X}). \end{aligned} \quad (3.2)$$

Hence $D(e)\sigma(e) = D(e)$ and so $\sigma(e)D(e) = 0$. Hence $D(e) = 0$. $\square$

Proposition 3.8. *Let σ be a bounded idempotent endomorphism of Banach algebra $\mathcal{A}$. If $\mathcal{A}$ is σ -a.a, then $\mathcal{A}^\#$ is $\widehat{\sigma}$ -a.a, where $\widehat{\sigma}$ is the endomorphism of $\mathcal{A}^\#$ induced by σ , that is, $\widehat{\sigma}(a + \alpha) = \sigma(a) + \alpha$.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}^\#$ -bimodule and $D : \mathcal{A}^\# \rightarrow \mathcal{X}^*$ a continuous $\widehat{\sigma}$ -derivation. By Proposition 3.5, there exists $\eta \in \mathcal{X}^*$ and $D_1 : \mathcal{A}^\# \rightarrow \widehat{\sigma}(e) \cdot \mathcal{X}^* \cdot \widehat{\sigma}(e)$ such that $D = D_1 + \delta_\eta$. Set $d : D_1|_{\mathcal{A}} : \mathcal{A} \rightarrow \widehat{\sigma}(e) \cdot \mathcal{X}^* \cdot \widehat{\sigma}(e)$. It is easy to check that d is a σ -derivation. Since $\mathcal{A}$ is σ -a.a, there exists a net $(x_\gamma^*) \subseteq \mathcal{X}^*$ such that $d = \lim_\gamma \delta_{x_\gamma^*}^\sigma$. Hence $D_1(a) = \lim_\gamma \sigma(a)x_\gamma^* - x_\gamma^*\sigma(a)$, ($a \in \mathcal{A}$). Since $\widehat{\sigma}(e) \cdot \mathcal{X}^* \cdot \widehat{\sigma}(e)$ is $\widehat{\sigma}$ -unital, by Lemma 3.7, $D_1(e) = 0$ and for each $a + \alpha \in \mathcal{A}^\#$ we have

$$\begin{aligned} D_1(a + \alpha) &= D_1(a) + \alpha D_1(e) = D_1(a) = \lim_\gamma \sigma(a)x_\gamma^* - x_\gamma^*\sigma(a) \\ &= \lim_\gamma (\widehat{\sigma}(a + \alpha) - \alpha)x_\gamma^* - x_\gamma^*(\widehat{\sigma}(a + \alpha) - \alpha) \\ &= \lim_\gamma \varphi(a + \alpha)x_\gamma^* - x_\gamma^*\varphi(a + \alpha). \end{aligned} \quad (3.3)$$

This shows that D_1 is $\widehat{\sigma}$ -a.i, and so $\mathcal{A}^\#$ is $\widehat{\sigma}$ -a.a. $\square$

Proposition 3.9. *Let σ be a bounded endomorphism of Banach algebra $\mathcal{A}$. If $\mathcal{A}^\#$ is $\widehat{\sigma}$ -a.a, then $\mathcal{A}$ is σ -a.a.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule and $D : \mathcal{A} \rightarrow \mathcal{X}^*$ a continuous σ -derivation. $\mathcal{X}$ is a Banach $\mathcal{A}^\#$ -bimodule with the following module actions:

$$(a + \alpha) \cdot x = a \cdot x + \alpha x, \quad x \cdot (a + \alpha) = x \cdot a + \alpha x, \quad (3.4)$$

for all $a \in \mathcal{A}, x \in \mathcal{X}, \alpha \in \mathbb{C}$. Define $D^\# : \mathcal{A}^\# \rightarrow \mathcal{X}^*$ with $D^\#(a + \alpha) = D(a)$. Clearly, $D^\#$ is a continuous $\hat{\sigma}$ -derivation. Hence, there is a net $(x_\gamma^*) \subseteq \mathcal{X}^*$ such that $D^\# = \lim_\gamma \hat{\sigma} x_\gamma^*$. Hence, for each $a \in \mathcal{A}$ we have

$$D(a) = D^\#(a + \alpha) = \lim_\gamma \hat{\sigma}(a + \alpha)x_\gamma^* - x_\gamma^* \hat{\sigma}(a + \alpha) = \lim_\gamma \sigma(a)x_\gamma^* - x_\gamma^* \sigma(a) \quad (3.5)$$

which shows that D is σ -a.i and so $\mathcal{A}$ is σ -a.a. $\square$

4. σ -Approximate Amenability When $\mathcal{A}$ Has a Bounded Approximate Identity

Lemma 4.1. *Let $\mathcal{A}$ be a Banach algebra with bounded approximate identity and $\mathcal{X}$ a Banach $\mathcal{A}$ -bimodule with trivial left or right action, then every σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}^*$ is σ -inner.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule with trivial left action. Hence, $\mathcal{X}^*$ is a Banach $\mathcal{A}$ -bimodule with trivial right action, that is,

$$x^* \cdot a = 0, \quad a \cdot x^* = ax^* \quad (x^* \in \mathcal{X}^*, a \in \mathcal{A}). \quad (4.1)$$

Let $D : \mathcal{A} \rightarrow \mathcal{X}^*$ be a continuous σ -derivation and (e_α) a bounded approximate identity of $\mathcal{A}$. By Banach Alaoglu's Theorem, $(D(e_\alpha))$ has a subnet $(D(e_\beta))$ such that $D(e_\beta) \xrightarrow{w^*} x_0^*$, for some $x_0^* \in \mathcal{X}^*$. Since $a \cdot e_\beta \xrightarrow{\|\cdot\|} a$ and D is continuous, $D(a \cdot e_\beta) \xrightarrow{\|\cdot\|} D(a)$. Hence, $D(a \cdot e_\beta) \xrightarrow{w^*} D(a)$.

On the other hand, $D(a \cdot e_\beta) = \sigma(a)D(e_\beta) \xrightarrow{w^*} \sigma(a)x_0^*$ and so $D(a) = \sigma(a)x_0^*$. Hence, $D(a) = \sigma(a)x_0^* - x_0^*\sigma(a)$ and D is σ -inner. $\square$

The following definitions extends the definition of the neo-unital and essential Banach $\mathcal{A}$ -bimodule in the classical sense.

Definition 4.2. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule. Then $\mathcal{X}$ is called σ -neo-unital (σ -pseudo-unital), if $\mathcal{X} = \sigma(\mathcal{A}) \cdot \mathcal{X} \cdot \sigma(\mathcal{A})$. Similarly, one defines σ -neo-unital left and right Banach modules.

Definition 4.3. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule. Then $\mathcal{X}$ is called σ -essential if $\mathcal{X} = \sigma(\mathcal{A})\mathcal{X}\sigma(\mathcal{A}) = \overline{\text{span}} \sigma(\mathcal{A}) \cdot \mathcal{X} \cdot \sigma(\mathcal{A})$. Similarly, one defines σ -essential left and right Banach modules.

We recall that a bounded approximate identity in Banach algebra $\mathcal{A}$ for Banach $\mathcal{A}$ -bimodule $\mathcal{X}$ is a bounded net (e_α) in $\mathcal{A}$ such that for each $x \in \mathcal{X}$, $e_\alpha x \rightarrow x$ and $xe_\alpha \rightarrow x$.

Proposition 4.4. *Assume that $\mathcal{A}$ has a left bounded approximate identity, σ is a bounded idempotent endomorphism of $\mathcal{A}$, and $\mathcal{X}$ is a left Banach $\mathcal{A}$ -module. Then $\mathcal{X}$ is σ -neo-unital if and only if $\mathcal{X}$ is σ -essential.*

Proof. Let $\mathcal{X}$ be a σ -essential Banach $\mathcal{A}$ -bimodule. Since σ is idempotent, $\sigma(\mathcal{A})$ is Banach subalgebra of $\mathcal{A}$. Let $(e_\alpha) \subseteq \mathcal{A}$ be left approximate identity with bound m . First suppose that $z \in \text{span } \sigma(\mathcal{A}) \cdot \mathcal{X}$, so there exist $a_1, \dots, a_n \in \mathcal{A}$, $x_1, \dots, x_n \in \mathcal{X}$ such that $z = \sum_{i=1}^n \sigma(a_i)x_i$. For $1 \leq i \leq n$, $e_\alpha a_i \rightarrow a_i$ and, therefore, $\sigma(e_\alpha)z \rightarrow z$.

Now suppose that $z \in \sigma(\mathcal{A})\mathcal{X}$. There exists $\{z_n\} \subseteq \text{span } \sigma(\mathcal{A}) \cdot \mathcal{X}$ such that $z_n \rightarrow z$. Thus,

$$\exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \left(n \geq n_0; \|z_n - z\| < \frac{\varepsilon}{2(\|\sigma\|m+1)} \right) \quad (4.2)$$

On the other hand, for each $n \in \mathbb{N}$ we have $\sigma(e_\alpha)z_n \xrightarrow{\alpha} z_n$ and so $\sigma(e_\alpha)z_{n_0} \xrightarrow{\alpha} z_{n_0}$. Therefore,

$$\exists \alpha_0; \quad \forall \alpha \left(\alpha \geq \alpha_0; \|\sigma(e_\alpha)z_{n_0} - z_{n_0}\| < \frac{\varepsilon}{2} \right). \quad (4.3)$$

Now we have

$$\begin{aligned} \|\sigma(e_\alpha)z - z\| &\leq \|\sigma(e_\alpha)z - \sigma(e_\alpha)z_{n_0} + \sigma(e_\alpha)z_{n_0} - z_{n_0} + z_{n_0} - z\| \\ &\leq \|\sigma\| \|e_\alpha\| \|z - z_{n_0}\| + \|\sigma(e_\alpha)z_{n_0} - z_{n_0}\| + \|z_{n_0} - z\| \\ &< (\|\sigma\| \|e_\alpha\| + 1) \|z - z_{n_0}\| + \frac{\varepsilon}{2} \\ &< (\|\sigma\|m + 1) \frac{\varepsilon}{(\|\sigma\|m + 1)2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned} \quad (4.4)$$

which shows that $(\sigma(e_\alpha)) \subseteq \sigma(\mathcal{A})$ is a left bounded approximate identity for $\mathcal{X}$. Now by Cohen factorization Theorem, $\mathcal{X} = \sigma(\mathcal{A}) \cdot \mathcal{X}$. So $\mathcal{X}$ is σ -neo-unital. The other side is trivial. $\square$

Corollary 4.5. *Every σ -neo-unital left Banach $\mathcal{A}$ -module is essential.*

Proof. Let $\mathcal{X}$ be a σ -neo-unital left Banach $\mathcal{A}$ -module. We have $\mathcal{X} = \sigma(\mathcal{A}) \cdot \mathcal{X} \subseteq \mathcal{A} \cdot \mathcal{X} \subseteq \mathcal{A}\mathcal{X} \subseteq \mathcal{X}$ so $\mathcal{X} = \mathcal{A}\mathcal{X}$. $\square$

Proposition 4.6. *Let $\mathcal{A}$ be a Banach algebra with a left bounded approximate identity, σ be a bounded idempotent endomorphism of $\mathcal{A}$, and $\mathcal{X}$ a left Banach $\mathcal{A}$ -module. Then $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is closed weakly complemented submodule of $\mathcal{X}$.*

Proof. Set $\mathcal{Y} = \sigma(\mathcal{A})\mathcal{X}$, since $\mathcal{A}$ has a left bounded approximate identity, by Cohen factorization Theorem $\mathcal{A}^2 = \mathcal{A}$, and we have $\sigma(\mathcal{A})\mathcal{Y} = \sigma(\mathcal{A})\sigma(\mathcal{A})\mathcal{X} = \sigma(\mathcal{A}^2)\mathcal{X} = \sigma(\mathcal{A})\mathcal{X} = \mathcal{Y}$, which shows that $\mathcal{Y}$ is σ -essential by Proposition 4.4, $\mathcal{Y}$ is σ -neo unital that is, $\mathcal{Y} = \sigma(\mathcal{A}) \cdot \mathcal{Y}$. Hence, $\sigma(\mathcal{A})\mathcal{X} = \mathcal{Y} = \sigma(\mathcal{A}) \cdot \mathcal{Y} \subseteq \sigma(\mathcal{A}) \cdot \mathcal{X}$ and so $\sigma(\mathcal{A})\mathcal{X} = \sigma(\mathcal{A}) \cdot \mathcal{X}$. Thus $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is closed submodule of $\mathcal{X}$.

Now we prove that $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is weakly complemented in $\mathcal{X}$. Let (e_α) be a left approximate identity in $\mathcal{A}$ with bound m , and define a net (T_α) in $\mathcal{B}(\mathcal{X}^*)$ by setting $T_\alpha(x^*) = x^* \cdot \sigma(e_\alpha)$ ($x^* \in \mathcal{X}^*$). We have $\|T_\alpha\| \leq \|\sigma\|m$. Thus (T_α) is a bounded net in $\mathcal{B}(\mathcal{X}^*)$ since $\mathcal{B}(\mathcal{X}^*) = (\mathcal{X}^* \otimes \mathcal{X})^*$ and ball $\mathcal{B}(\mathcal{X}^*)$ is w^* -compact, so there exists $T \in \mathcal{B}(\mathcal{X}^*)$ such that we

may suppose that $w^* - \lim_{\alpha} T_{\alpha} = T$ and $\|T\| \leq \|\sigma\|m$. For each $a \in \mathcal{A}$, $x \in \mathcal{X}$, and $x^* \in \mathcal{X}^*$, we have

$$\begin{aligned} \langle \sigma(a) \cdot x, T(x^*) \rangle &= \lim_{\alpha} \langle \sigma(a) \cdot x, x^* \cdot \sigma(e_{\alpha}) \rangle \\ &= \lim_{\alpha} \langle \sigma(e_{\alpha}) \sigma(a) \cdot x, x^* \rangle \\ &= \langle \sigma(a) \cdot x, x^* \rangle, \end{aligned} \quad (4.5)$$

and so $x^* - Tx^* \in (\sigma(\mathcal{A}) \cdot \mathcal{X})^{\perp}$. On other hand, for each $x^* \in \mathcal{X}^*$,

$$T^2 x^* = T(Tx^*) = \lim_{\alpha} T(x^*) \sigma(e_{\alpha}) = \lim_{\alpha} x^* \sigma(e_{\alpha}) = T(x^*). \quad (4.6)$$

Thus T is projection, and $I_{\mathcal{X}^*} - T : \mathcal{X}^* \rightarrow (\sigma(\mathcal{A}) \cdot \mathcal{X})^{\perp}$ is projection. So $\sigma(\mathcal{A}) \cdot \mathcal{X}$ is weakly complemented in $\mathcal{X}$ and, we have $\mathcal{X}^* = (\sigma(\mathcal{A}) \cdot \mathcal{X})^{\perp} \oplus (\sigma(\mathcal{A}) \cdot \mathcal{X})^*$. $\square$

Corollary 4.7. *Let $\mathcal{A}$ have a bounded approximate identity, and let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule and σ a bounded idempotent endomorphism of $\mathcal{A}$. Then*

- (i) $\sigma(\mathcal{A}) \cdot \mathcal{X} \cdot \sigma(\mathcal{A})$ is a closed weakly complemented submodule of $\mathcal{X}$,
- (ii) $\mathcal{A}$ is σ -a.a if and only if for every σ -neo-unital Banach $\mathcal{A}$ -bimodule $\mathcal{X}$, every σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}^*$ is σ -approximately inner.

Proof. Set $\mathcal{Y} = \sigma(\mathcal{A}) \cdot \mathcal{X}$. By Proposition 4.6, $\mathcal{Y}$ is a closed and weakly complemented submodule of $\mathcal{X}$, and $T : \mathcal{X}^* \rightarrow \mathcal{Y}^*$ and $I - T : \mathcal{X}^* \rightarrow \mathcal{Y}^{\perp}$ are projection maps. Let $D : \mathcal{A} \rightarrow \mathcal{X}^*$ be a σ -derivation, so ToD and $(I - T)oD$ are σ -derivations and $D = (ToD) + (I - T)oD$. Since $A \cdot (X/Y) = \{0\}$ by Lemma 4.1, $(I - T)oD$ is σ -inner. So there exists $J_0 \in \mathcal{Y}^{\perp}$ such that $(I - T)oD = \delta_{J_0}^{\sigma}$. Thus $D = ToD + \delta_{J_0}^{\sigma}$ and so D is σ -a.i if and only if $ToD : \mathcal{A} \rightarrow \mathcal{Y}^*$ is σ -a.i.

Now let $\mathcal{Z} = \mathcal{Y} \cdot \sigma(\mathcal{A})$. By Proposition 4.6, $\mathcal{Z}$ is a closed weakly complemented in $\mathcal{Y}$, and $T' : \mathcal{Y}^* \rightarrow \mathcal{Z}^*$ and $I - T' : \mathcal{Y}^* \rightarrow \mathcal{Z}^{\perp}$ are projection maps. Assume that $D_1 : \mathcal{A} \rightarrow \mathcal{Y}^*$ is a σ -derivation, thus $T'oD$ and $(I - T')oD$ are σ -derivations, and we have $D_1 = T'oD_1 + (I - T') \cdot D_1$. Since $(\mathcal{Y}/\mathcal{Z}) \cdot \mathcal{A} = \{0\}$, by Lemma 4.1, $(I - T') \cdot D_1$ is σ -inner and so there exists $z_0 \in \mathcal{Z}^{\perp}$ such that $(I - T')oD_1 = \delta_{z_0}^{\sigma}$. Therefore, $D_1 = T'oD_1 + \delta_{z_0}^{\sigma}$. Thus, D_1 is σ -a.i if and only if $T'oD_1$ is σ -a.i. Set $DoT = D_1$. Thus, $D = T'oD_1 + \delta_{z_0}^{\sigma} + \delta_{J_0}^{\sigma}$. Therefore, D is σ -a.i, if and only if $T'oD_1 : \mathcal{A} \rightarrow \mathcal{Z}^* = (\sigma(\mathcal{A}) \cdot \mathcal{X} \cdot \sigma(\mathcal{A}))^*$ is σ -a.i. Recall that $\mathcal{Z}$ is σ -neo-unital. Thus, $\mathcal{A}$ is σ -a.a if and only if for every σ -neo-unital Banach $\mathcal{A}$ -bimodul, $\mathcal{X}$, every σ -derivation $D : A \rightarrow X^*$ is σ -a.i. $\square$

Corollary 4.8. *Let $\mathcal{A}$ have a bounded approximate identity, and let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule and σ a bounded idempotent endomorphism of $\mathcal{A}$. Then $\mathcal{A}$ is σ -a.a if and only if for every σ -essential Banach $\mathcal{A}$ -bimodule $\mathcal{X}$, every σ -derivation $D : \mathcal{A} \rightarrow \mathcal{X}^*$ is σ -approximately inner.*

Proposition 4.9. *Suppose that σ is a bounded idempotent endomorphism of $\mathcal{A}$ and define $\hat{\sigma} : \mathcal{A}^{\#} \rightarrow \mathcal{A}^{\#}$ with $\hat{\sigma}(a + \alpha) = \sigma(a) + \alpha$. The following statements are equivalent.*

- (1) $\mathcal{A}$ is σ -a.a.
- (2) There is a net $(\mu_{\alpha}) \subseteq (A^{\#} \hat{\otimes} \mathcal{A}^{\#})^{**}$ such that for each $a \in \mathcal{A}^{\#}$, $\hat{\sigma}(a) \cdot \mu_{\alpha} - \mu_{\alpha} \cdot \hat{\sigma}(a) \rightarrow 0$ and $\pi^{**}(\mu_{\alpha}) \rightarrow \hat{e}$.

(3) There is a net $(\mu'_\alpha) \subseteq (A^\# \hat{\otimes} \mathcal{A}^\#)^{**}$ such that for each $a \in \mathcal{A}^\#$, $\hat{\sigma}(a) \cdot \mu_\alpha - \mu_\alpha \cdot \hat{\sigma}(a) \rightarrow 0$ and for every α , $\pi^{**}(\mu'_\alpha) = \hat{e}$.

Proof. (1 $\Rightarrow$ 3) Suppose that $\mathcal{A}$ is σ -a.a, by Proposition 3.8, $\mathcal{A}^\#$ is $\hat{\sigma}$ -a.a. Let $u = e \otimes e \in \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$. $\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ is a Banach $\mathcal{A}^\#$ -bimodule with the following module actions:

$$a \cdot (b \otimes c) = \hat{\sigma}(a)(b \otimes c), \quad (b \otimes c) \cdot a = (b \otimes c)\hat{\sigma}(a) \quad (a, b, c \in \mathcal{A}^\#). \quad (4.7)$$

Set $\delta_{\hat{u}} : \mathcal{A}^\# \rightarrow \ker \pi^{**}$ with definition $\delta_{\hat{u}}(a) = \hat{\sigma}(a) \cdot \hat{u} - \hat{u} \cdot \hat{\sigma}(a)$ ($a \in \mathcal{A}^\#$). $\delta_{\hat{u}}$ is $\hat{\sigma}$ -derivation. Recall that $\ker \pi^{**} = (\ker \pi)^{**}$. Since $\mathcal{A}^\#$ is $\hat{\sigma}$ -a.a, thus there exists $(e_\alpha) \subseteq \ker \pi^{**}$ such that

$$\delta_{\hat{u}}(a) = \lim_{\alpha} \hat{\sigma}(a)e_\alpha - e_\alpha \hat{\sigma}(a) \quad (a \in \mathcal{A}^\#). \quad (4.8)$$

Set $\mu'_\alpha = \hat{u} - e_\alpha \in (\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)^{**}$. We have

$$\hat{\sigma}(a)\mu'_\alpha - \mu'_\alpha \hat{\sigma}(a) = \hat{\sigma}(a)\hat{u} - \hat{u}\hat{\sigma}(a) - (\hat{\sigma}(a)e_\alpha - e_\alpha \hat{\sigma}(a)) \longrightarrow 0, \quad (4.9)$$

and for each α ,

$$\pi^{**}(\mu'_\alpha) = \pi^{**}(\hat{u} - e_\alpha) = \pi^{**}(\hat{u}) - \pi^{**}(e_\alpha) = \pi(u) = e. \quad (4.10)$$

(3 $\Rightarrow$ 2) is clear.

(2 $\Rightarrow$ 1) By Proposition 3.9, it is sufficient to show that $A^\#$ is $\hat{\sigma}$ -a.a.

Let $D : A^\# \rightarrow \mathcal{X}^*$ be a derivation. By Corollary 4.7, we may take $\mathcal{X}$ to be σ -neo-unital. We run the standard argument, so for each $\alpha \in I$, set $f_\alpha(x) = \mu_\alpha(\psi_x)$, where for $a, b \in \mathcal{A}^\#$, $x \in \mathcal{X}$, we have $\psi_x(a \otimes b) = \langle x, \hat{\sigma}(a)D(b) \rangle$. Then, $(m_\alpha^\gamma) \subset A^\# \hat{\otimes} \mathcal{A}^\#$ converging ω^* to μ_α ($\alpha \in I$) and noting that for $m \in \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$, $a \in \mathcal{A}^\#$, $x \in \mathcal{X}$, then

$$\psi_{\hat{\sigma}(a)x - x\hat{\sigma}(a)}(m) = (\hat{\sigma}(a)\psi_x - \psi_x\hat{\sigma}(a))(m) - \langle x, \hat{\sigma}(\pi(m))D(a) \rangle. \quad (4.11)$$

Since $\mathcal{X}$ is $\hat{\sigma}$ -neo-unital, so $\mathcal{X} = \mathcal{X}\hat{\sigma}(\mathcal{A}^\#)$. So for each $a \in A$ and $x \in \mathcal{X}$, we have

$$\begin{aligned} \langle \hat{\sigma}(a)x - x\hat{\sigma}(a), f_\alpha \rangle &= \langle \psi_{\hat{\sigma}(a)x - x\hat{\sigma}(a)}, \mu_\alpha \rangle \\ &= \lim_{\gamma} \langle m_\alpha^\gamma, \psi_{\hat{\sigma}(a)x - x\hat{\sigma}(a)} \rangle \\ &= \langle \hat{\sigma}(a)\psi_x - \psi_x\hat{\sigma}(a), \mu_\alpha \rangle - \lim_{\gamma} \langle x, \hat{\sigma}(\pi(m_\alpha^\gamma))D(a) \rangle \\ &= \langle \psi_x, \mu_\alpha \hat{\sigma}(a) - \hat{\sigma}(a)\mu_\alpha \rangle - \langle x, \pi^{**}(\mu_\alpha)D(a) \rangle. \end{aligned} \quad (4.12)$$

Thus,

$$\begin{aligned}
& \|\langle x, \widehat{\sigma}(a)f_\alpha - f_\alpha \widehat{\sigma}(a) \rangle - \langle x, D(a) \rangle\| \\
& \leq \|\langle \psi_x, \widehat{\sigma}(a)\mu_\alpha - \mu_\alpha \widehat{\sigma}(a) \rangle\| + \|x\| \|\pi^{**}(\mu_\alpha) - \widehat{e}\| \|D(a)\| \\
& \leq \|D\| \cdot \|x\| \|\widehat{\sigma}(a)\mu_\alpha - \mu_\alpha \widehat{\sigma}(a)\| + \|x\| \|\pi^*(\mu_\alpha) - \widehat{e}\| \|D(a)\|,
\end{aligned} \tag{4.13}$$

and, therefore, $D = \lim_\alpha \delta_{f_\alpha}^{\widehat{\sigma}}$. It follows that $\mathcal{A}^\#$ is $\widehat{\sigma}$ -a.a and so $\mathcal{A}$ is σ -a.a. $\square$

Proposition 4.10. *Suppose that $\mathcal{A}$ is σ -a.a, and let*

$$\Sigma : 0 \longrightarrow \mathcal{X}^* \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \longrightarrow 0, \tag{4.14}$$

be an admissible short exact sequence of left $\mathcal{A}$ -module and left σ - $\mathcal{A}$ -module homomorphism. Then Σ , σ -approximately split, that is, there is a net $G_\alpha : \mathcal{Z} \rightarrow \mathcal{Y}$ of right inverse maps to g such that $\lim_\alpha (\sigma(a)G_\alpha - G_\alpha \sigma(a)) = 0$ for $a \in \mathcal{A}$, and a net $F_\alpha : \mathcal{Y} \rightarrow \mathcal{X}^*$ of left inverse maps to f such that $\lim_\alpha (\sigma(a)f_\alpha - f_\alpha \sigma(a)) = 0$ for $a \in \mathcal{A}$.

Proof. Following the proof of [2, Theorem 2.3], for a right inverse G for g , σ -approximate amenability gives a net $(\varphi_\alpha) \subseteq \mathcal{B}(\mathcal{Z}, \mathcal{X}^*)$ such that

$$\sigma(a) \cdot G - G \cdot \sigma(a) = \lim_\alpha (\sigma(a) \cdot f G_\alpha - f G_\alpha \cdot \sigma(a)) \quad (a \in \mathcal{A}). \tag{4.15}$$

Setting $G_\alpha = G - f\varphi_\alpha$ gives the required net. Applying the same argument as [2, Proposition 1.1] provides (F_α) . $\square$

We recall that if $\mathcal{A}$ is a Banach algebra with a weak left (right) approximate identity, then $\mathcal{A}$ has a left (right) approximate identity [1, Lemma 2.2].

Corollary 4.11. *Suppose that Banach algebra $\mathcal{A}$ is σ -a.a, then $\sigma(\mathcal{A})$ has left and right approximate identities.*

Corollary 4.12. *Suppose that Banach algebra $\mathcal{A}$ is σ -a.a and σ is a bounded epimorphism of $\mathcal{A}$, then $\mathcal{A}$ has left and right approximate identities.*

Lemma 4.13. *Let σ be a bounded idempotent endomorphism of Banach algebra $\mathcal{A}$ and $\mathcal{X}$ a σ -neounital Banach $\mathcal{A}$ -module. If $(e_\alpha)_\alpha$ is a bounded approximate identity in $\mathcal{A}$, then $(\sigma(e_\alpha))_\alpha$ is a bounded approximate identity for $\mathcal{X}$.*

Proof. For every $a \in \mathcal{A}$ we have $e_\alpha \sigma(a) \rightarrow \sigma(a)$. Since σ is idempotent, $\sigma(e_\alpha) \sigma(a) \rightarrow \sigma(a)$. For each $x \in \mathcal{X}$, there exists $a \in \mathcal{A}$ and $y \in \mathcal{X}$ such that $x = \sigma(a) \cdot y$. Therefore,

$$\sigma(e_\alpha) \cdot x = \sigma(e_\alpha) \sigma(a) \cdot y \rightarrow \sigma(a) \cdot y = x, \tag{4.16}$$

which shows that $(\sigma(e_\alpha))$ is a bounded approximate identity for $\mathcal{X}$. $\square$

It is often convenient to extend a derivation to a large algebra. If a Banach algebra I is contained as a closed ideal in another Banach algebra $\mathcal{A}$, then the strict topology on $\mathcal{A}$ with respect to I is defined through the family of seminorms $(P_i)_{i \in I}$, where

$$P_i(a) := \|ai\| + \|ia\| \quad (a \in \mathcal{A}). \quad (4.17)$$

Note that the strict topology is Hausdorff only if $\{a \in \mathcal{A} : a \cdot I = I \cdot a = \{0\}\} = \{0\}$ [3].

Proposition 4.14. *Let $\mathcal{A}$ be a Banach algebra and I a closed ideal in $\mathcal{A}$. let σ be a bounded idempotent endomorphism of $\mathcal{A}$ and I has a bounded approximate identity. Let $\mathcal{X}$ be a σ -neo-unital Banach I -module and $D : I \rightarrow \mathcal{X}^*$ a σ -derivation. Then, $\mathcal{X}$ is a Banach $\mathcal{A}$ -bimodule in a canonical fashion, and there is a unique σ -derivation $\tilde{D} : \mathcal{A} \rightarrow \mathcal{X}^*$ such that*

$$(i) \quad \tilde{D}|_I = D,$$

$$(ii) \quad \tilde{D} \text{ is continuous with respect to the strict topology on } \mathcal{A} \text{ and the } \omega^* \text{-topology on } \mathcal{X}^*.$$

Proof. Since $\mathcal{X}$ is a σ -neo-unital Banach I -module, so for each $x \in \mathcal{X}$, there exists $i \in I$ and $y \in \mathcal{X}$ such that $x = \sigma(i) \cdot y$. Define $a \cdot x = \sigma(ai) \cdot y$ ($a \in \mathcal{A}$).

We claim that $a \cdot x$ is well defined, that is, independent of the choices of i and y . Let $i' \in I$ and $y' \in \mathcal{X}$ be such that $x = \sigma(i') \cdot y'$, and let $(e_\alpha)_\alpha$ be a bounded approximate identity for I . For each $a \in \mathcal{A}$ and $x \in \mathcal{X}$ we have

$$\begin{aligned} a \cdot x &= \sigma(ai) \cdot y = \lim_{\alpha} \sigma(ae_\alpha i) \cdot y \\ &= \lim_{\alpha} \sigma(ae_\alpha) \sigma(i) \cdot y = \lim_{\alpha} \sigma(ae_\alpha) x \\ &= \lim_{\alpha} \sigma(ae_\alpha) \sigma(i') \cdot y' = \lim_{\alpha} \sigma(ae_\alpha i') \cdot y' \\ &= \sigma(ai') \cdot y'. \end{aligned} \quad (4.18)$$

It is obvious that this operation of $\mathcal{A}$ on $\mathcal{X}$ turns $\mathcal{X}$ into a left Banach $\mathcal{A}$ -module. Similarly, one defines a right Banach $\mathcal{A}$ -module structure on $\mathcal{X}$. So that, eventually, $\mathcal{X}$ becomes a Banach $\mathcal{A}$ -bimodule. To extend D , let

$$\tilde{D} : \mathcal{A} \longrightarrow \mathcal{X}^*, \quad a \longrightarrow \omega^* - \lim_{\alpha} (D(ae_\alpha) - \sigma(a) \cdot D(e_\alpha)). \quad (4.19)$$

We claim that $\tilde{D}$ is well-defined, that is, the limit in (4.19) does exist. Let $x \in \mathcal{X}$, and let $i \in I$ and $y \in \mathcal{X}$ such that $x = y \cdot \sigma(i)$. By Lemma 4.13, $\sigma(e_\alpha)$ is bounded approximate identity for $\mathcal{X}$, and we have

$$\begin{aligned} \langle x, D(ae_\alpha) - \sigma(a) \cdot D(e_\alpha) \rangle &= \langle y \cdot \sigma(i), D(ae_\alpha) - \sigma(a) \cdot D(e_\alpha) \rangle \\ &= \langle y, \sigma(i) D(ae_\alpha) - \sigma(ia) \cdot D(e_\alpha) \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle y, D(iae_\alpha) - D(i)\sigma(ae_\alpha) - D(iae_\alpha) + D(ia)\sigma(e_\alpha) \rangle \\
&= \langle \sigma(e_\alpha) \cdot y, D(ia) \rangle - \langle \sigma(ae_\alpha) \cdot y, D(i) \rangle \\
&\xrightarrow{\alpha} \langle y, D(ia) \rangle - \langle \sigma(a) \cdot y, D(i) \rangle \quad (a \in \mathcal{A}).
\end{aligned} \tag{4.20}$$

So the limit in (4.19) exists. Furthermore, for $i \in I$,

$$\begin{aligned}
\tilde{D}(i) &= \omega^* - \lim_{\alpha} (D(ie_\alpha) - \sigma(i) \cdot D(e_\alpha)) \\
&= \omega^* - \lim_{\alpha} D(ie_\alpha) - D(ie_\alpha) + D(i)\sigma(e_\alpha) = D(i),
\end{aligned} \tag{4.21}$$

so $\tilde{D}$ is an extension of D . Also for $a \in \mathcal{A}$ and $i \in I$ we have

$$\begin{aligned}
(\tilde{D}a) \cdot \sigma(i) &= \omega^* - \lim_{\alpha} (D(ae_\alpha) \cdot \sigma(i) - \sigma(a) \cdot D(e_\alpha) \cdot \sigma(i)) \\
&= \omega^* - \lim_{\alpha} (D(ae_\alpha i) - \sigma(ae_\alpha) \cdot D(i) - \sigma(a) \cdot D(e_\alpha i) + \sigma(a)\sigma(e_\alpha) \cdot D(i)) \\
&= \omega^* - \lim_{\alpha} (D(ae_\alpha i) - \sigma(a) \cdot D(e_\alpha i)) = D(ai) - \sigma(a) \cdot D(i).
\end{aligned} \tag{4.22}$$

We claim that $\tilde{D}$ is continuous with respect to the strict topology on $\mathcal{A}$ and the ω^* -topology on $\mathcal{K}^*$.

Let $a_n \xrightarrow{\text{strict}} a$ in $\mathcal{A}$.

$$\forall i \in I, \quad \|a_n i\| + \|ia_n\| \longrightarrow \|ai\| + \|ia\|. \tag{4.23}$$

For each $x \in \mathcal{K}$,

$$\begin{aligned}
&\left| \langle x, \tilde{D}(a_n) \rangle - \langle x, \tilde{D}(a) \rangle \right| \\
&= \lim_{\alpha} |\langle x, D(a_n e_\alpha) - \sigma(a_n) \cdot D(e_\alpha) \rangle - \langle x, D(a e_\alpha) - \sigma(a) \cdot D(e_\alpha) \rangle| \\
&= \lim_{\alpha} |\langle x, D(a_n e_\alpha) - D(a e_\alpha) - \sigma(a_n) D(e_\alpha) + \sigma(a) D(e_\alpha) \rangle| \\
&\leq \lim_{\alpha} \|x\| \| (D(a_n e_\alpha) - D(a e_\alpha)) - (\sigma(a_n) \cdot D(e_\alpha) - \sigma(a) \cdot D(e_\alpha)) \| \\
&\leq \lim_{\alpha} \|x\| (\|D\| \|a_n e_\alpha - a e_\alpha\| + \|\sigma(a_n) - \sigma(a)\| \|D(e_\alpha)\|) \\
&\leq \lim_{\alpha} \|x\| (\|D\| \|a_n - a\| \|e_\alpha\| + \|\sigma\| \|a_n - a\| \|D(e_\alpha)\|) \longrightarrow 0,
\end{aligned} \tag{4.24}$$

so $\tilde{D}$ is continuous.

It remains to show that $\tilde{D}$ is a σ -derivation. From the definition of the strict topology, we have $ae_\alpha \rightarrow a$ in the strict topology for all $a \in \mathcal{A}$ because $\|ae_\alpha i\| + \|iae_\alpha\| \xrightarrow{\alpha} \|ai\| + \|ia\|$ ($i \in I$) and so $\tilde{D}(ae_\alpha) \xrightarrow{w^*} \tilde{D}(a)$. Therefore,

$$\begin{aligned}
 \tilde{D}(ab) &= \omega^* - \lim_{\alpha} \lim_{\beta} \tilde{D}((ae_\alpha)(be_\beta)) \\
 &= \omega^* - \lim_{\alpha} \lim_{\beta} D((ae_\alpha)(be_\beta)) \\
 &= \omega^* - \lim_{\alpha} \lim_{\beta} (\sigma(ae_\alpha)D(be_\beta) + D(ae_\alpha) \cdot \sigma(be_\beta)) \\
 &= \omega^* - \lim_{\alpha} \lim_{\beta} (\sigma(ae_\alpha)\tilde{D}(be_\beta) + \tilde{D}(ae_\alpha) \cdot \sigma(be_\beta)) \\
 &= \sigma(a)\tilde{D}(b) + \tilde{D}(a)\sigma(b),
 \end{aligned} \tag{4.25}$$

that is, $\tilde{D}$ is σ -derivation. $\square$

Corollary 4.15. *Suppose that $\mathcal{A}$ is σ -a.a, where σ is bounded idempotent endomorphism of $\mathcal{A}$, I is a closed ideal in $\mathcal{A}$. If I has a bounded approximate identity, then I is σ -a.a.*

Proof. Suppose that I has a bounded approximate identity, $\mathcal{K}$ is a σ -neo-unital Banach I -bimodule, and $D : I \rightarrow \mathcal{K}^*$ is a σ -derivation. By Proposition 4.14, $\mathcal{K}$ becomes to a Banach $\mathcal{A}$ -bimodule and D has a unique extension $\tilde{D} : \mathcal{A} \rightarrow \mathcal{K}^*$ which is a σ -derivation. Since $\mathcal{A}$ is σ -a.a,

$$\exists \{x_\alpha^*\} \subseteq \mathcal{K}^* \text{ s.t. } \tilde{D}(a) = \lim_{\alpha} \sigma(a) \cdot x_\alpha^* - x_\alpha^* \cdot \sigma(a) \quad (a \in \mathcal{A}). \tag{4.26}$$

So we have $D(i) = \tilde{D}(i) = \lim_{\alpha} \sigma(i) \cdot x_\alpha^* - x_\alpha^* \cdot \sigma(i)$, which shows that $D = \lim_{\alpha} \delta_{x_\alpha^*}^\sigma$ is σ -a.i, and I is σ -a.a. $\square$

Corollary 4.16. *Let $\mathcal{A}$ be an a.a Banach algebra and I a closed ideal of $\mathcal{A}$. Then $\mathcal{A}/I$ is σ -a.a for each bounded endomorphism σ of $\mathcal{A}/I$.*

Proposition 4.17. *Let I be a closed ideal of $\mathcal{A}$ such that $\sigma(I) \subseteq I$. If $\mathcal{A}$ is σ -a.a, then $\mathcal{A}/I$ is $\hat{\sigma}$ -a.c, where $\hat{\sigma}$ is an endomorphism of $\mathcal{A}/I$ induced by σ (i.e., $\hat{\sigma}(a + I) = \sigma(a) + I$ for $a \in \mathcal{A}$).*

Proof. Let $\mathcal{K}$ be a Banach $\mathcal{A}/I$ -bimodule and $D : \mathcal{A}/I \rightarrow \mathcal{K}$ a $\hat{\sigma}$ -derivation. Then $\mathcal{K}$ becomes an $\mathcal{A}$ -bimodule with the following module actions:

$$a \cdot x = \pi(a) \cdot x, \quad x \cdot a = x \cdot \pi(a) \quad (a \in \mathcal{A}, x \in \mathcal{K}), \tag{4.27}$$

where π is the canonical homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{A}/I$. It is easy to see that $D \circ \pi : \mathcal{A} \rightarrow \mathcal{X}$ becomes a σ -derivation. Since $\mathcal{A}$ is σ -a.c., there exists a net $\{x_\alpha\} \subseteq \mathcal{X}$ such that $D \circ \pi(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a)$ ($a \in \mathcal{A}$). Therefore, for each ($a \in A$),

$$\begin{aligned} D(a + I) &= D \circ \pi(a) = \lim_\alpha \sigma(a) \cdot x_\alpha - x_\alpha \cdot \sigma(a) \\ &= \lim_\alpha \pi(\sigma(a)) \cdot x_\alpha - x_\alpha \cdot \pi(\sigma(a)) \\ &= \lim_\alpha (\sigma(a) + I) \cdot x_\alpha - x_\alpha \cdot (\sigma(a) + I) \\ &= \lim_\alpha \widehat{\sigma}(a + I)x_\alpha - x_\alpha \widehat{\sigma}(a + I). \end{aligned} \tag{4.28}$$

Thus, $\mathcal{A}/I$ is $\widehat{\sigma}$ -a.c. □

Proposition 4.18. *Suppose that I is a closed ideal in $\mathcal{A}$. If I is σ -amenable and $\mathcal{A}/I$ is a.a., then $\mathcal{A}$ is σ -a.a.*

Proof. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule and $D : \mathcal{A} \rightarrow \mathcal{X}^*$ a σ -derivation. $\mathcal{X}$ is a Banach I -bimodule too.

Clearly, $d = D|_I : I \rightarrow \mathcal{X}^*$ is a σ -derivation, and by σ -amenability of I there exists $x_0^* \in \mathcal{X}^*$ such that $D = \delta_{x_0^*}^\sigma$, and, therefore, for each $i \in I$ we have $d(i) = \sigma(i) \cdot x_0^* - x_0^* \cdot \sigma(i)$. Set $D_1 = D - \delta_{x_0^*}^\sigma$. Clearly, D_1 is σ -derivation and $D_1|_I = 0$. Now let $\mathcal{X}_0 = \overline{\text{span}}(\mathcal{X} \cdot \sigma(I) \cup \sigma(I) \cdot \mathcal{X})$. $(\mathcal{X}/\mathcal{X}_0)$ is a Banach $\mathcal{A}/I$ -bimodule via the following module actions:

$$(a + I)(x + \mathcal{X}_0) = \sigma(a)x + \mathcal{X}_0, \quad (x + \mathcal{X}_0)(a + I) = x\sigma(a) + \mathcal{X}_0 \quad (x \in \mathcal{X}, a \in \mathcal{A}). \tag{4.29}$$

Now we define

$$\tilde{D} : \frac{\mathcal{A}}{I} \longrightarrow \left(\frac{\mathcal{X}}{\mathcal{X}_0} \right)^*; \quad \langle x + \mathcal{X}_0, \tilde{D}(a + I) \rangle = \langle x, D_1(a) \rangle \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{4.30}$$

Let $a + I = a' + I$ and $x + \mathcal{X}_0 = x' + \mathcal{X}_0$ for some $a, a' \in \mathcal{A}$ and $x, x' \in \mathcal{X}$. So $a - a' \in I$, and we have $D_1(a - a') = 0$. Thus, $D_1(a) = D_1(a')$. Now we have

$$\langle x + \mathcal{X}_0, \tilde{D}(a + I) \rangle = \langle x' + \mathcal{X}_0, \tilde{D}(a' + I) \rangle. \tag{4.31}$$

Thus, $\langle x, D_1(a) \rangle = \langle x', D_1(a') \rangle = \langle x', D_1(a) \rangle$, and, therefore,

$$\langle x - x', D_1(a) \rangle = 0. \tag{4.32}$$

It is enough to show that $D_1(a)$ is zero on $\mathcal{X}_0$. Suppose that $\sigma(i)x \in \mathcal{X}_0$, we have

$$\begin{aligned} \langle \sigma(i)x, D_1(a) \rangle &= \langle x, D_1(a)\sigma(i) \rangle = \langle x, D_1(ai) - \sigma(a)D_1(i) \rangle = 0, \\ \langle x\sigma(i), D_1(a) \rangle &= \langle x, \sigma(i)D_1(a) \rangle = \langle x, D_1(ia) - D_1(i)\sigma(a) \rangle = 0. \end{aligned} \tag{4.33}$$

So for all $a \in \mathcal{A}$, $D_1(a) = 0$ on $\sigma(I) \cdot \mathcal{X} \cup \mathcal{X} \cdot \sigma(I)$ and so for all $a \in \mathcal{A}$, $D_1(a) = 0$ on $\mathcal{X}_0$. Since $x - x' \in \mathcal{X}_0$, therefore $\langle x - x', D_1(a) \rangle = 0$ which shows that D_1 is well defined. We claim that $\tilde{D}$ is a derivation;

$$\begin{aligned}
\langle x + \mathcal{X}_0, \tilde{D}((a + I)(b + I)) \rangle &= \langle x, D_1(ab) \rangle \\
&= \langle x, \sigma(a)D_1(b) + D_1(a)\sigma(b) \rangle \\
&= \langle x\sigma(a), D_1(b) \rangle + \langle \sigma(b)x, D_1(a) \rangle \\
&= \langle x\sigma(a) + \mathcal{X}_0, \tilde{D}(b + I) \rangle \\
&\quad + \langle \sigma(b)x + \mathcal{X}_0, \tilde{D}(a + I) \rangle \\
&= \langle (x + \mathcal{X}_0)(a + I), \tilde{D}(b + I) \rangle \\
&\quad + \langle (b + I)(x + \mathcal{X}_0), \tilde{D}(a + I) \rangle.
\end{aligned} \tag{4.34}$$

So there exists a net $(\varphi_\alpha) \subseteq (\mathcal{X}/\mathcal{X}_0)^*$ such that $\tilde{D} = \lim_\alpha \delta_{\varphi_\alpha}$. Let $q : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{X}_0$ be the quotient map. For every α , $(\varphi_\alpha \circ q) \in \mathcal{X}^*$. Set $(x_\alpha^*) = (\varphi_\alpha \circ q) \subseteq \mathcal{X}^*$. We have

$$\begin{aligned}
\langle x, D_1(a) \rangle &= \langle x + \mathcal{X}_0, \tilde{D}(a + I) \rangle \\
&= \left\langle x + \mathcal{X}_0, \lim_\alpha (a + I)\varphi_\alpha - \varphi_\alpha(a + I) \right\rangle \\
&= \lim_\alpha \langle x\sigma(a) + \mathcal{X}_0, \varphi_\alpha \rangle - \langle \sigma(a)x + \mathcal{X}_0, \varphi_\alpha \rangle \\
&= \lim_\alpha \langle q(x\sigma(a)), \varphi_\alpha \rangle - \langle q(\sigma(a)x), \varphi_\alpha \rangle \\
&= \lim_\alpha \varphi_\alpha \circ q(x\sigma(a) - \sigma(a)x) = \langle x\sigma(a) - \sigma(a)x, x_\alpha^* \rangle \\
&= \lim_\alpha \langle x, \sigma(a)x_\alpha^* - x_\alpha^*\sigma(a) \rangle \\
&= \left\langle x, \lim_\alpha \delta_{x_\alpha^*}^\sigma(a) \right\rangle.
\end{aligned} \tag{4.35}$$

So $D_1 = D - \delta_{x^*}^\sigma = \lim_\alpha \delta_{x_\alpha^*}^\sigma$, and, therefore, $D = \lim_\alpha \delta_{(x_\alpha^* - x_0^*)}^\sigma$. Which shows that D is σ -a.i and so $\mathcal{A}$ is σ -a.a. $\square$

Example 4.19. Let $\mathcal{A}$ be a Banach algebra and let $0 \neq \varphi \in \text{Ball}(\mathcal{A}^*)$. Then $\mathcal{A}$ with the product $a \cdot a' = \varphi(a)a'$ becomes a Banach algebra. We denote this algebra with $\mathcal{A}_\varphi$. It is easy to see that $\mathcal{A}_\varphi$ has a left identity e , while it has not right approximate identity, so $\mathcal{A}_\varphi$ is not contractible and is not approximately contractible. Also $\mathcal{A}_\varphi$ is biprojective. Now suppose that $\sigma : \mathcal{A}_\varphi \rightarrow \mathcal{A}_\varphi$ be defined by $\sigma(a) = \varphi(a)e$. We have

$$\sigma^2(a) = \sigma(\varphi(a)e) = \varphi(a)\sigma(e) = \varphi(a)\varphi(e)e = \varphi(a)e = \sigma(a). \tag{4.36}$$

Thus σ is idempotent. It is easy to see that e is identity for $\sigma(\mathcal{A}_\varphi)$, and since $\mathcal{A}$ is biprojective by [1, Corollary 5.3], $\mathcal{A}_\varphi$ is σ -biprojective. Thus by [1, Theorem 4.3], $\mathcal{A}_\varphi$ is σ -contractible and so $\mathcal{A}_\varphi$ is σ -a.c.

It is easy to see that $\ker \varphi$ and all subspaces of $\ker \varphi$ are all ideals of $\mathcal{A}_\varphi$ and $\sigma(\ker \varphi) \subseteq \ker \varphi$ so $\sigma(I) \subseteq I$ for each ideal of $\mathcal{A}$. Therefore, by Proposition 4.17, $\mathcal{A}_\varphi/I$ is $\widehat{\sigma}$ -a.c for each ideal I of $\mathcal{A}_\varphi$, where $\widehat{\sigma}(a + I) = \sigma(a) + I = \varphi(a)e + I$.

Corollary 4.20. *Suppose that σ is a bounded idempotent endomorphism of Banach algebra $\mathcal{A}$. Then $\mathcal{A}$ is σ -a.a if and only if there are nets (μ''_α) in $(\mathcal{A} \widehat{\otimes} \mathcal{A})^{**}$ and $(F_\alpha), (G_\alpha) \subseteq \mathcal{A}^{**}$, such that for each $a \in \mathcal{A}$,*

$$(1) \sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \rightarrow 0,$$

$$(2) \sigma(a) \cdot F_\alpha \rightarrow \sigma(a), G_\alpha \cdot \sigma(a) \rightarrow \sigma(a),$$

$$(3) \pi^{**}(\mu''_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) \rightarrow 0.$$

Proof. Suppose that $\mathcal{A}$ is σ -a.a, take the net (μ_α) given in Proposition 4.9 and write

$$\mu_\alpha = \mu''_\alpha - F_\alpha \otimes \widehat{e} - \widehat{e} \otimes G_\alpha + c_\alpha \widehat{e} \otimes \widehat{e}, \quad (4.37)$$

where $(\mu''_\alpha) \subseteq (A \widehat{\otimes} A)^{**}$, $(F_\alpha), (G_\alpha) \subseteq A^{**}$, and $(c_\alpha) \subseteq \mathbb{C}$. Applying π^{**} , $\pi^{**}(\mu''_\alpha) - F_\alpha - G_\alpha + c_\alpha \widehat{e} \rightarrow \widehat{e}$, hence $c_\alpha \rightarrow 1$, then

$$\pi^{**}(\mu''_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) + \widehat{e} \cdot \sigma(a) \longrightarrow \widehat{e} \cdot \sigma(a) \quad (a \in \mathcal{A}). \quad (4.38)$$

So we have (iii) further, by Proposition 4.9, for $a \in \mathcal{A}^\#$,

$$\begin{aligned} & \widehat{\sigma}(a) \cdot \mu''_\alpha - \widehat{\sigma}(a) \cdot F_\alpha \otimes \widehat{e} - \widehat{\sigma}(a) \otimes G_\alpha + \widehat{\sigma}(a) \otimes \widehat{e} \\ & + \mu''_\alpha \cdot \widehat{\sigma}(a) + F_\alpha \otimes \widehat{\sigma}(a) + \widehat{e} \otimes G_\alpha \cdot \widehat{\sigma}(a) - \widehat{e} \otimes \widehat{\sigma}(a) \longrightarrow 0. \end{aligned} \quad (4.39)$$

Thus $\widehat{\sigma}(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \widehat{\sigma}(a) + F_\alpha \otimes \widehat{\sigma}(a) - \widehat{\sigma}(a) \otimes G_\alpha \rightarrow 0$, and $\widehat{\sigma}(a) \cdot F_\alpha \rightarrow \widehat{\sigma}(a)$, $G_\alpha \cdot \widehat{\sigma}(a) \rightarrow \widehat{\sigma}(a)$. So for $a \in \mathcal{A}$,

$$\begin{aligned} & \sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \longrightarrow 0, \\ & \sigma(a) \cdot F_\alpha \longrightarrow \sigma(a), \quad G_\alpha \cdot \sigma(a) \longrightarrow \sigma(a). \end{aligned} \quad (4.40)$$

Conversely, set $c_\alpha = 1$ and $\mu_\alpha = \mu''_\alpha - F_\alpha \otimes \hat{e} - \hat{e} \otimes G_\alpha + \hat{e} \otimes \hat{e}$. We have

$$\begin{aligned}
 \hat{\sigma}(a + \alpha) \cdot \mu_\alpha - \mu_\alpha \cdot \hat{\sigma}(a + \alpha) &= (\sigma(a) + \alpha) \cdot \mu_\alpha - \mu_\alpha \cdot (\sigma(a) + \alpha) \\
 &= \sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) + a\mu_\alpha - \alpha\mu_\alpha \\
 &= \sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) \\
 &= \sigma(a) \cdot \mu''_\alpha - \sigma(a)F_\alpha \otimes e - \sigma(a) \otimes G_\alpha \\
 &\quad + \sigma(a) \otimes e(-\mu''_\alpha \cdot \sigma(a)) \\
 &\quad + F_\alpha \otimes \sigma(a) + e \otimes G_\alpha \sigma(a) - e \otimes \sigma(a)) \\
 &= \sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) \\
 &\quad + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \rightarrow 0 \quad (a \in \mathcal{A}).
 \end{aligned} \tag{4.41}$$

So $\hat{\sigma}(a) \cdot \mu_\alpha - \mu_\alpha \cdot \hat{\sigma}(a) \rightarrow 0$ ($a \in \mathcal{A}^\#$). Also

$$\begin{aligned}
 \pi^{**}(\mu_\alpha) \cdot \sigma(a) &= \pi^{**}(\mu''_\alpha - F_\alpha \otimes \hat{e} - \hat{e} \otimes G_\alpha + \hat{e} \otimes \hat{e})\sigma(a) \\
 &= \pi^{**}(\mu''_\alpha)\sigma(a) - F_\alpha \cdot \sigma(a) \\
 &\quad - G_\alpha \cdot \sigma(a) + \sigma(a) \rightarrow \sigma(a) \quad (a \in \mathcal{A}),
 \end{aligned} \tag{4.42}$$

and so $\pi^{**}(\mu_\alpha) \rightarrow \hat{e}$. Now, by Proposition 4.9, $\mathcal{A}$ is σ -a.a. □

For σ -approximate contractibility we have the following parallel result.

Proposition 4.21. $\mathcal{A}$ is σ -a.c if and only if any of the following equivalent conditions hold:

- (1) there is a net $(\mu_\alpha) \subset \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ such that for each $a \in \mathcal{A}^\#$, $\sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) \rightarrow 0$ and $\pi(\mu_\alpha) \rightarrow e$;
- (2) there is a net $(\mu'_\alpha) \subset \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ such that for each $a \in \mathcal{A}^\#$, $\sigma(a) \cdot \mu'_\alpha - \mu'_\alpha \cdot \sigma(a) \rightarrow 0$ and $\pi(\mu'_\alpha) = e$;
- (3) there are nets $(\mu''_\alpha) \subset \mathcal{A} \hat{\otimes} \mathcal{A}$, $(F_\alpha), (G_\alpha) \subset \mathcal{A}$, such that for each $a \in \mathcal{A}$,
 - (i) $\sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) + F_\alpha \otimes \sigma(a) - \sigma(a) \otimes G_\alpha \rightarrow 0$;
 - (ii) $\sigma(i) \cdot F_\alpha \rightarrow \sigma(a), G_\alpha \cdot \sigma(a) \rightarrow \sigma(a)$;
 - (iii) $\pi(\mu''_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) \rightarrow 0$.

We know Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ has bounded approximate diagonal [3].

Proposition 4.22. Banach algebra $\mathcal{A}$ is σ -amenable if and only if $\mathcal{A}$ has bounded approximate σ -diagonal, that is, there is a bounded net $(\mu_\alpha) \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$ such that for each $a \in \mathcal{A}$, $\sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) \rightarrow 0$ and $\pi(\mu_\alpha) \cdot \sigma(a) \rightarrow \sigma(a)$.

Proposition 4.23. If Banach algebra $\mathcal{A}$ is σ -amenable, then $\mathcal{A}$ is σ -a.c.

Proof. Suppose that $\mathcal{A}$ is σ -amenable. Then there exists a bounded net (μ_α) in $\mathcal{A} \otimes \mathcal{A}$ such that for each $a \in \mathcal{A}$,

$$\sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) \longrightarrow 0, \quad \pi(\mu_\alpha) \cdot \sigma(a) \longrightarrow \sigma(a). \quad (4.43)$$

Set $f_\alpha = \pi(\mu_\alpha)$. It is easy to see that (f_α) is a bounded approximate identity. Then $\mu''_\alpha = \mu_\alpha + f_\alpha \otimes f_\alpha$ and $F_\alpha = G_\alpha = f_\alpha$ satisfy (i)–(iii) of Proposition 4.21, because

- (i) $\sigma(a) \cdot \mu''_\alpha - \mu''_\alpha \cdot \sigma(a) + f_\alpha \otimes \sigma(a) - \sigma(a) \otimes f_\alpha = \sigma(a) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(a) + \sigma(a) f_\alpha \otimes f_\alpha - f_\alpha \otimes \sigma(a) + f_\alpha \otimes \sigma(a) - \sigma(a) \otimes f_\alpha \rightarrow 0 \ (a \in \mathcal{A}),$
- (ii) $\sigma(a) \cdot f_\alpha = \sigma(a) \cdot \pi(\mu_\alpha) \rightarrow \sigma(a), \ f_\alpha \cdot \sigma(a) = \pi(\mu_\alpha) \cdot \sigma(a) \rightarrow \sigma(a),$
- (iii) $\pi(\mu''_\alpha) \cdot \sigma(a) = \pi(\mu_\alpha + f_\alpha \otimes f_\alpha) \cdot \sigma(a) = f_\alpha \cdot \sigma(a) + f_\alpha^2 \cdot \sigma(a).$

So

$$\pi(\mu''_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) = f_\alpha \cdot \sigma(a) + f_\alpha^2 \cdot \sigma(a) - f_\alpha \cdot \sigma(a) - f_\alpha \cdot \sigma(a) \longrightarrow 0. \quad (4.44)$$

Note that f_α^2 is a bounded approximate identity too, thus, by Proposition 4.21, $\mathcal{A}$ is σ -a.c. $\square$

Corollary 4.24. *Suppose that $\mathcal{A}$ is a σ -a.a Banach algebra where σ is an idempotent endomorphism of $\mathcal{A}$ and I is a closed two-sided ideal of $\mathcal{A}$ which $\sigma(I)$ has a bounded approximate identity and $\sigma(I) \subseteq I$. Then, I is σ -a.a.*

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity in $\sigma(I)$, so $\{\hat{e}_\alpha\}$ is bounded net in $\sigma(I)^{**}$, and so by Banach-Alaoglu theorem there exists a subnet $\{\hat{e}_\beta\} \subseteq \{\hat{e}_\alpha\}$ and $E \in \sigma(I)^{**}$ such that $\hat{e}_\beta \xrightarrow{w^*} E$. E is a right identity in $\sigma(I)^{**}$ because for each $F \in \sigma(I)^{**}$ and $f \in \sigma(I)^*$,

$$\langle f, F \square E \rangle = \langle f \cdot F, E \rangle = \lim_\beta \langle e_\beta, fF \rangle = \lim_\beta \langle e_\beta f, F \rangle = \langle f, F \rangle. \quad (4.45)$$

Also E acts as an identity on $\sigma(I)$ itself. Let $(\mu_\alpha), (F_\alpha), (G_\alpha)$ be the nets given by Corollary 4.20 for $\mathcal{A}$. Define $\mu'_\alpha = E \cdot \mu_\alpha \cdot E \in (I \hat{\otimes} I)^{**}$, $F'_\alpha = E \cdot F_\alpha \in I^{**}$, and $G'_\alpha = G_\alpha \cdot E \in I^{**}$. Then, for $i \in I$,

(i) we consider

$$\begin{aligned} & \sigma(i) \cdot \mu'_\alpha - \mu'_\alpha \cdot \sigma(i) + F'_\alpha \otimes \sigma(i) - \sigma(i) \otimes G'_\alpha \\ &= \sigma(i) \cdot E \cdot \mu_\alpha \cdot E - E \cdot \mu_\alpha \cdot E \cdot \sigma(i) + E \cdot F_\alpha \otimes \sigma(i) - \sigma(i) \otimes G_\alpha \cdot E \\ &= \sigma(i) \cdot \mu_\alpha \cdot E - E \cdot \mu_\alpha \sigma(i) + E \cdot F_\alpha \otimes \sigma(i) - \sigma(i) \otimes G_\alpha \cdot E \\ &= E \cdot \sigma(i) \cdot \mu_\alpha \cdot E - E \cdot \mu_\alpha \cdot \sigma(i) \cdot E \\ &\quad + E \cdot F_\alpha \otimes \sigma(i) \cdot E - E \cdot \sigma(i) \otimes G_\alpha \cdot E \\ &= E(\sigma(i) \cdot \mu_\alpha - \mu_\alpha \cdot \sigma(i) + F_\alpha \otimes \sigma(i) - \sigma(i) \otimes G_\alpha) \cdot E \longrightarrow 0, \end{aligned} \quad (4.46)$$

(ii) we consider

$$\begin{aligned}\sigma(i) \cdot F'_\alpha &= \sigma(i) \cdot E \cdot F_\alpha = \sigma(i) \cdot F_\alpha \longrightarrow \sigma(i), \\ G'_\alpha \cdot \sigma(i) &= G_\alpha \cdot E \cdot \sigma(i) = G_\alpha \cdot \sigma(i) \longrightarrow \sigma(i)\end{aligned}\tag{4.47}$$

(iii) we consider

$$\begin{aligned}\pi^{**}(\mu'_\alpha) \cdot \widehat{\sigma(a)} - F'_\alpha \cdot \widehat{\sigma(a)} - G'_\alpha \cdot \widehat{\sigma(a)} \\ &= \pi^{**}(E \cdot \mu_\alpha \cdot E) \cdot \sigma(a) - E \cdot F_\alpha \cdot \sigma(a) - G_\alpha \cdot E \cdot \sigma(a) \\ &= E \cdot \pi^{**}(\mu_\alpha) \cdot E \cdot \sigma(a) - E \cdot F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) \\ &= E \cdot \pi^{**}(\mu_\alpha) \cdot \sigma(a) - E \cdot F_\alpha \cdot \sigma(a) - G_\alpha \cdot \sigma(a) - E \cdot G_\alpha \sigma(a) + E \cdot G_\alpha \sigma(a) \\ &= E \cdot (\pi^{**}(\mu_\alpha) \cdot \sigma(a) - F_\alpha \cdot \sigma(a) - G_\alpha \sigma(a)) + (E - \hat{e})G_\alpha \sigma(a) \longrightarrow 0.\end{aligned}\tag{4.48}$$

An alternative proof would be to follow the standard argument stated in Corollary 4.15. $\square$

References

- [1] P. C. Curtis Jr. and R. J. Loy, "The structure of amenable Banach algebras," *Journal of the London Mathematical Society*, vol. 40, no. 1, pp. 89–104, 1989.
- [2] M. Eshaghi Gordji, "Point derivations on second duals and unitization of Banach algebras," *Nonlinear Functional Analysis and Applications*, vol. 13, no. 2, pp. 271–275, 2008.
- [3] M. Eshaghi Gordji, "Homomorphisms, amenability and weak amenability of Banach algebras," *Vietnam Journal of Mathematics*, vol. 36, no. 3, pp. 253–260, 2008.