Research Article

$N_\theta$-Ward Continuity

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A function $f$ is continuous if and only if $f$ preserves convergent sequences; that is, $(f(\alpha_n))$ is a convergent sequence whenever $(\alpha_n)$ is convergent. The concept of $N_\theta$-ward continuity is defined in the sense that a function $f$ is $N_\theta$-ward continuous if it preserves $N_\theta$-quasi-Cauchy sequences; that is, $(f(\alpha_n))$ is an $N_\theta$-quasi-Cauchy sequence whenever $(\alpha_n)$ is $N_\theta$-quasi-Cauchy. A sequence $(\alpha_k)$ of points in $\mathbb{R}$, the set of real numbers, is $N_\theta$-quasi-Cauchy if $\lim_{r \to \infty} \sum_{k \in I_r} |\Delta \alpha_k| = 0$, where $\Delta \alpha_k = \alpha_{k+1} - \alpha_k$, $I_r = (k_r, \infty)$, and $\theta = (k_r)$ is a lacunary sequence, that is, an increasing sequence of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \to \infty$. A new type compactness, namely, $N_\theta$-ward compactness, is also, defined and some new results related to this kind of compactness are obtained.

1. Introduction

It is well known that a real function $f$ is continuous if and only if, for each point $\alpha_0$ in the domain, $\lim_{n \to \infty} f(\alpha_n) = f(\alpha_0)$ whenever $\lim_{n \to \infty} \alpha_n = \alpha_0$. This is equivalent to the statement that $(f(\alpha_n))$ is a convergent sequence whenever $(\alpha_n)$ is. This is also equivalent to the statement that $(f(\alpha_n))$ is a Cauchy sequence whenever $(\alpha_n)$ is Cauchy provided that the domain of the function is either whole $\mathbb{R}$ or a bounded and closed subset of $\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. These well known results for continuity for real functions in terms of sequences suggested to introduce and study new types of continuities such as slowly oscillating continuity [1], quasi-slowly oscillating continuity [2], $\delta$-quasi-slowly oscillating continuity [3], forward continuity [4], statistical ward continuity [5] which enabled some authors to obtain some characterizations of uniform continuity in terms of sequences in the sense that a function preserves either quasi-Cauchy sequences or slowly oscillating sequences (see [6–8]).

The purpose of this paper is to introduce a new kind of continuity and a new type of compactness, namely, $N_\theta$-ward continuity and $N_\theta$-ward compactness, respectively, in the senses that a function $f$ is $N_\theta$-ward continuous if $f$ preserves $N_\theta$-quasi-Cauchy sequences,
and a subset $A$ of $\mathbb{R}$ is $N_{\theta}$-ward compact if any sequence of points in $A$ has an $N_{\theta}$-quasi-Cauchy subsequence and to investigate relations among this kind of continuity, compactness, and some other types of continuities.

2. Preliminaries

We will use boldface letters $\mathbf{a}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for sequences $\mathbf{a} = (a_k)$, $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, and $\mathbf{z} = (z_n)$ of points in $\mathbb{R}$ for the sake of abbreviation. $s$ and $c$ will denote the set of all sequences and the set of convergent sequences of points in $\mathbb{R}$.

A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. A subset $A$ of $\mathbb{R}$ is bounded if $|a| \leq M$ for all $a \in A$ where $M$ is a positive real constant number. This is equivalent to the statement that any sequence of points in $A$ has a Cauchy subsequence. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence $\{a_n\}$ of points in $\mathbb{R}$ is quasi-Cauchy if $\Delta a_n = a_{n+1} - a_n$ is a null sequence where $\Delta a_n = a_{n+1} - a_n$. These sequences were named as quasi-Cauchy by Burton and Coleman [8, page 328], while they were called as forward convergent to 0 sequences in [4, page 226].

It is known that a sequence $\{a_n\}$ of points in $\mathbb{R}$ is slowly oscillating if

$$\lim_{\lambda \to 1^+} \lim_{n \to \infty} \max_{n+1 \leq k \leq \lfloor \lambda n \rfloor} |a_k - a_n| = 0,$$

where $\lfloor \lambda n \rfloor$ denotes the integer part of $\lambda n$ (see [9, Definition 2 page 947]). Any Cauchy sequence is slowly oscillating, and any slowly oscillating sequence is quasi-Cauchy. There are quasi-Cauchy sequences which are not Cauchy. For example, the sequence $\{(\sqrt{n})\}$ is quasi-Cauchy, but not Cauchy. Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences, and fails for slowly oscillating sequences as well. A counter example for the case, quasi-Cauchy, is again the sequence $\{a_n\} = (\sqrt{n})$ with the subsequence $\{a_{\omega n}\} = (\omega n)$ for $\omega > 1$. A counter example for the case slowly oscillating is the sequence $(\log_{10} n)$ with the subsequence $(n)$. Furthermore we give more examples without neglecting: the sequences $(\sum_{k=1}^{n} 1/n, (\ln n), (\ln(\ln n)), (\ln(\ln(\ln n))), \ldots, (\ln(\ln(\ln(\ln(\ln n)))))$ and combinations like that are all slowly oscillating, but not Cauchy. The bounded sequence $(\cos(6 \log(n + 1)))$ is slowly oscillating, but not Cauchy. The sequences $(\cos(\pi \sqrt{n}))$ and $(\sum_{k=1}^{n} (1/k)(\sum_{j=1}^{k} (1/j)))$ are quasi-Cauchy, but not slowly oscillating.

By a method of sequential convergence, or briefly a method, we mean a linear function $G$ defined on a subspace of $s$, denoted by $cG$, into $\mathbb{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be $G$-convergent to $\ell$ if $a \in cG$ and $G(a) = \ell$ [10]. In particular, $\lim$ denotes the limit function $\lim a = \lim_n a_n$ on the space $c$ of convergent sequences of points in $\mathbb{R}$. A method $G$ is called regular if $c \subset cG$; that is, every convergent sequence $\mathbf{a} = (a_n)$ is $G$-convergent with $G(\mathbf{a}) = \lim a$. A point $\ell$ in $\mathbb{R}$ is in the $G$-sequential closure of a subset $A$ of $\mathbb{R}$ if there is a sequence $\mathbf{x} = (x_n)$ of points in $A$ such that $G(\mathbf{x}) = \ell$. A subset $A$ is called $G$-sequentially closed if it contains all of the points in its $G$-sequential closure.

Consider an infinite matrix $\mathbf{A} = (a_{nk})_{n,k=1}^{\infty}$ of real numbers. Then, for any sequence $\mathbf{x} = (x_n)$ the sequence $\mathbf{Ax}$ is defined as

$$\mathbf{Ax} = \left( \sum_{k=1}^{\infty} a_{nk}x_k \right)_n$$

(2.2)
provided that each of the series converges. A sequence \( x \) is called \( \lambda \)-convergent (or \( \lambda \)-summable) to \( \ell \) if \( \lambda x \) exists and is convergent with

\[
\lim \lambda x = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} x_k = \ell. \tag{2.3}
\]

Then \( \ell \) is called the \( \lambda \)-limit of \( x \). We have thus defined a method of sequential convergence, that is, \( G(x) = \lim \lambda x \), called a matrix method or a summability matrix.

The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence. A sequence \( (\alpha_k) \) of points in \( \mathbb{R} \) is called statistically convergent to an element \( \ell \) of \( \mathbb{R} \) if for each \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \frac{1}{n} | \{ k \leq n : |\alpha_k - \ell| \geq \varepsilon \} | = 0, \tag{2.4}
\]

and this is denoted by \( st-lim_{k \to \infty} \alpha_k = \ell \) (see [11–15]). This defines a method of sequential convergence, that is, \( G(\alpha) := st-lim_{k \to \infty} \alpha_k \).

Now we recall the concepts of ward compactness, and slowly oscillating compactness: a subset \( A \) of \( \mathbb{R} \) is called ward compact if whenever \( (\alpha_n) \) is a sequence of points in \( A \), there is a quasi-Cauchy subsequence \( z = (z_k) = (\alpha_{n_k}) \) of \( (\alpha_n) \) [4]. A subset \( A \) of \( \mathbb{R} \) is called slowly oscillating compact if whenever \( (\alpha_n) \) is a sequence of points in \( A \), there is a slowly oscillating subsequence \( z = (z_k) = (\alpha_{n_k}) \) of \( (\alpha_n) \) [1].

A function \( f \) is called \( G \)-sequentially continuous at \( u \in \mathbb{R} \) if, given a sequence \( \alpha = (\alpha_n) \) of points in \( \mathbb{R} \), \( G(\alpha) = u \) implies that \( G(f(\alpha)) = f(u) \).

Recently, Cakalli (see [16, page 594], [17]) gave a sequential definition of compactness, which is a generalization of ordinary sequential compactness, as in the following: a subset \( A \) of \( \mathbb{R} \) is \( G \)-sequentially compact if for any sequence \( (\alpha_k) \) of points in \( A \) there exists a subsequence \( z \) of the sequence such that \( G(z) \in A \). His idea enables us obtaining new kinds of compactness via most of the nonmatrix sequential convergence methods as well as all matrix sequential convergence methods.

3. \( N_\theta \)-Quasi-Cauchy Sequences

A lacunary sequence \( \theta = (k_r) \) is an increasing sequence \( \theta = (k_r) \) of positive integers such that \( k_0 = 0 \) and \( h_r : k_r - k_{r-1} \to \infty \). The intervals determined by \( \theta \) will be denoted by \( I_r = (k_{r-1}, k_r] \), and the ratio \( k_r / k_{r-1} \) will be abbreviated by \( q_r \). Sums of the form \( \sum_{k_{r-1}+1}^{k_r} |\alpha_k| \) frequently occur, and will often be written for convenience as \( \sum_{k \in I_r} |\alpha_k| \). Throughout this paper, we will assume that \( \inf q_r, q_r > 1 \).

The notion of \( N_\theta \) convergence was introduced and studied by Freedman et al. in [18], Basarir and Altundag studied \( \Delta-N_\theta \)-asymptotically equivalent sequences in [19]. Using the idea of Sember and Raphael, Fridy and Orhan introduced lacunary statistical convergence (see [20, 21]).

A sequence \( (\alpha_k) \) of points in \( \mathbb{R} \) is called \( N_\theta \)-convergent to an element \( \ell \) of \( \mathbb{R} \) if

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |\alpha_k - \ell| = 0, \tag{3.1}
\]
and it is denoted by \( N_\theta \)-lim \( a_k = \ell \). This defines a method of sequential convergence, that is, \( G(\alpha) := N_\theta \)-lim \( a_k \). Any convergent sequence is \( N_\theta \)-convergent, but the converse is not always true. Throughout the paper \( N_\theta \) will denote the set of \( N_\theta \) convergent sequences of points in \( R \).

For example, limit of the sequence of the ratios of Fibonacci numbers converge, to the golden mean. This property ensures the regularity of lacunary sequential method obtained via the sequence of Fibonacci numbers; that is, \( \theta = (k_r) \) is the lacunary sequence defined by writing \( k_0 = 0 \) and \( k_r = F_{r+2} \) where \( (F_r) \) is the Fibonacci sequence, that is, \( F_1 = 1 \), \( F_2 = 1 \), and \( F_r = F_{r-1} + F_{r-2} \) for \( r \geq 3 \).

Now we modify the definition of \( G \)-sequential compactness to the special case, \( G = N_\theta \) [16] as in the following: a subset \( A \) of \( R \) is called \( N_\theta \)-sequentially compact if whenever \( (a_n) \) is a sequence of points in \( A \) there is an \( N_\theta \)-convergent subsequence \( z = (z_k) = (a_n) \) of \( (a_n) \) whose \( N_\theta \)-limit is in \( A \).

Adopting the technique in the proof of the necessity of Theorem 6 in [22], we see that the sequential method \( N_\theta \) is subsequential. It follows from [16, Corollary 5, page 597] that a subset \( A \) of \( R \) is sequentially compact if and only if it is \( N_\theta \)-sequentially compact. A subset \( A \) of \( R \) is closed and bounded if and only if it is \( N_\theta \)-sequentially compact. A subset of \( R \) is \( G \)-sequentially compact if and only if it is \( N_\theta \)-sequentially compact for any regular subsequential method \( G \).

In connection with \( N_\theta \)-convergent sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on \( R \):

\[
\begin{align*}
(N_\theta) &: (a_n) \in N_\theta \implies (f(a_n)) \in N_\theta, \\
(N_{\theta c}) &: (a_n) \in N_\theta \implies (f(a_n)) \in c, \\
(c) &: (a_n) \in c \implies (f(a_n)) \in c, \\
(cN_\theta) &: (a_n) \in c \implies (f(a_n)) \in N_\theta.
\end{align*}
\]  

(3.2)

We see that \( (N_\theta) \) is \( N_\theta \)-sequential continuity of \( f \), and \( (c) \) is the ordinary continuity of \( f \). It is easy to see that \( (N_{\theta c}) \) implies \( (N_\theta) \), and \( (N_\theta) \) does not imply \( (N_{\theta c}) \); \( (N_\theta) \) implies \( (cN_\theta) \), and \( (cN_\theta) \) does not imply \( (N_\theta) \); \( (N_{\theta c}) \) implies \( (c) \), and \( (c) \) does not imply \( (N_{\theta c}) \); and \( (c) \) is equivalent to \( (cN_\theta) \).

If a function \( f \) is \( N_\theta \)-sequentially continuous at a point \( a_0 \), then it is continuous at \( a_0 \). If a function \( f \) is \( N_\theta \)-sequentially continuous on a subset \( A \) of \( R \), then it is statistically continuous on \( A \). We obtain from [16, Theorem 7, page 597] that \( N_\theta \)-sequentially continuous image of any \( N_\theta \)-sequentially compact subset of \( R \) is \( N_\theta \)-sequentially compact.

In [23] a nonempty subset \( A \) of \( R \) is called \( G \)-sequentially connected if there are no nonempty and disjoint \( G \)-sequentially closed subsets \( U \) and \( V \) such that \( A \subseteq U \cup V \), and \( A \cap U \) and \( A \cap V \) are nonempty. As far as \( G \)-sequentially connectedness is concerned, we see that \( N_\theta \)-sequentially continuous image of any \( N_\theta \)-sequentially connected subset of \( R \) is \( N_\theta \)-sequentially connected, so \( N_\theta \)-sequentially continuous image of any interval is an interval. Furthermore it can be easily seen that a subset of \( R \) is \( N_\theta \)-sequentially connected if and only if it is connected in the ordinary sense, and so it is an interval.

Definition 3.1. A sequence \( (a_n) \) of points in \( R \) is called \( N_\theta \)-quasi-Cauchy if \( (\Delta a_n) \) is \( N_\theta \)-convergent to 0. \( \Delta N_\theta^0 \) will denote the set of all \( N_\theta \)-quasi-Cauchy sequences of points in \( R \).
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We note that $N_\theta$-quasi-Cauchy sequences were studied in [19] in a different point of view.

Now we give the definition of $N_\theta$-ward compactness.

**Definition 3.2.** A subset $A$ of $\mathbb{R}$ is called $N_\theta$-ward compact if whenever $(\alpha_n)$ is a sequence of points in $A$, there is an $N_\theta$-quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of $(\alpha_n)$.

**Theorem 3.3.** A subset $A$ of $\mathbb{R}$ is bounded if and only if it is $N_\theta$-ward compact.

**Proof.** Let $A$ be any bounded subset of $\mathbb{R}$ and let $(\alpha_n)$ be any sequence of points in $A$. $(\alpha_n)$ is also a sequence of points in $\overline{A}$ where $\overline{A}$ denotes the closure of $A$. As $\overline{A}$ is sequentially compact, there is a convergent subsequence $(\alpha_{n_k})$ of $(\alpha_n)$ (no matter the limit is in $A$ or not). This subsequence is $N_\theta$-convergent since $N_\theta$-method is regular. Hence $(\alpha_{n_k})$ is $N_\theta$-quasi-Cauchy. Thus (a) implies (b). To prove that (b) implies (a), suppose that $A$ is unbounded. If it is unbounded above, then one can construct a sequence $(\alpha_n)$ of numbers in $A$ such that $\alpha_{n+1} > 1 + \alpha_n$ for each positive integer $n$. Then the sequence $(\alpha_n)$ does not have any $N_\theta$-quasi-Cauchy subsequence, so $A$ is not $N_\theta$-ward compact. If $A$ is bounded above and unbounded below, then similarly we obtain that $A$ is not $N_\theta$-ward compact. This completes the proof of the theorem. 

It easily follows from the preceding theorem that a closed subset of $\mathbb{R}$ is $N_\theta$-ward compact if and only if it is $N_\theta$-sequentially compact and a closed subset of $\mathbb{R}$ is $N_\theta$-ward compact if and only if it is statistically ward compact.

A sequence $\alpha = (\alpha_n)$ is $\delta$-quasi-Cauchy if $\lim_{k \to \infty} \Delta^2 \alpha_n = 0$ where $\Delta^2 \alpha_n = \alpha_{n+2} - 2\alpha_{n+1} + \alpha_n$ [3]. A subset $A$ of $\mathbb{R}$ is called $\delta$-ward compact if whenever $\alpha = (\alpha_n)$ is a sequence of points in $A$, there is a subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of $\alpha$ with $\lim_{k \to \infty} \Delta^2 z_k = 0$. It follows from the previous theorem that any $N_\theta$-ward compact subset of $\mathbb{R}$ is $\delta$-ward compact.

We see that for any regular subsequential method $G$ defined on $\mathbb{R}$, if a subset $A$ of $\mathbb{R}$ is $G$-sequentially compact, then it is $N_\theta$-ward compact. But the converse is not always true.

Now we give the definition of $N_\theta$-ward continuity in the following.

**Definition 3.4.** A function defined on a subset $A$ of $\mathbb{R}$ is called $N_\theta$-ward continuous if it preserves $N_\theta$-quasi-Cauchy sequences; that is, $(f(\alpha_n))$ is an $N_\theta$-quasi-Cauchy sequence whenever $(\alpha_n)$ is.

Sum of two $N_\theta$-ward continuous functions is $N_\theta$-ward continuous, but product of $N_\theta$-ward continuous functions need not be $N_\theta$-ward continuous.

In connection with $N_\theta$-quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$:

\begin{align*}
(\delta N_\theta) : (\alpha_n) \in \Delta N_\theta^0 \implies (f(\alpha_n)) \in \Delta N_\theta^0, \\
(\delta N_\theta c) : (\alpha_n) \in \Delta N_\theta^0 \implies (f(\alpha_n)) \in c, \\
(c) : (\alpha_n) \in c \implies (f(\alpha_n)) \in c, \\
(c \delta N_\theta) : (\alpha_n) \in c \implies (f(\alpha_n)) \in \Delta N_\theta^0, \\
(N_\theta) : (\alpha_n) \in N_\theta \implies (f(\alpha_n)) \in N_\theta^0. 
\end{align*}

(3.3)
We see that ($\delta N_\theta$) is $N_\theta$-ward continuity of $f$, $(N_\theta)$ is $N_\theta$-sequential continuity of $f$, and (c) is the ordinary continuity of $f$. It is easy to see that ($\delta N_\theta c$) implies ($\delta N_\theta$), and ($\delta N_\theta$) does not imply ($\delta N_\theta c$); ($\delta N_\theta$) implies ($\delta N_\theta c$), and ($\delta N_\theta$) does not imply ($\delta N_\theta$); ($\delta N_\theta c$) implies (c), and (c) does not imply ($\delta N_\theta c$); ($N_\theta$) clearly implies (c) as we have seen in Section 3.

Now we give the implication that ($\delta N_\theta$) implies ($N_\theta$); that is, any $N_\theta$-ward continuous function is $N_\theta$-sequentially continuous.

**Theorem 3.5.** If $f$ is $N_\theta$-ward continuous on a subset $A$ of $\mathbb{R}$, then it is $N_\theta$-sequentially continuous on $A$.

**Proof.** Assume that $f$ is an $N_\theta$-ward continuous function on a subset $A$ of $\mathbb{R}$. Let $(\alpha_n)$ be any $N_\theta$-convergent sequence with $N_\theta - \lim_{k \to \infty} \alpha_k = \alpha_0$. Then the sequence

$$ (\alpha_1, \alpha_0, \alpha_2, \alpha_0, \ldots, \alpha_{n-1}, \alpha_0, \alpha_n, \alpha_0, \ldots) $$

is also $N_\theta$-convergent to $\alpha_0$. Hence it is $N_\theta$-quasi-Cauchy. As $f$ is $N_\theta$-ward continuous, the sequence

$$ (f(\alpha_1), f(\alpha_0), f(\alpha_2), f(\alpha_0), \ldots, f(\alpha_{n-1}), f(\alpha_0), f(\alpha_n), f(\alpha_0), \ldots) $$

is $N_\theta$-quasi-Cauchy. It follows from this that the sequence $(f(\alpha_n))$ $N_\theta$-converges to $f(\alpha_0)$. This completes the proof of the theorem.

The converse is not always true for the function $f(x) = x^2$ is an example since the sequence $(\sqrt{n})$ is $N_\theta$-quasi-Cauchy while $(f(\sqrt{n})) = (n)$ is not. \(\square\)

**Corollary 3.6.** If $f$ is $N_\theta$-ward continuous on a subset $A$ of $\mathbb{R}$, then it is continuous on $A$.

**Proof.** The proof immediately follows from the preceding theorem, so it is omitted. \(\square\)

**Corollary 3.7.** If $f$ is $N_\theta$-ward continuous on a subset $A$ of $\mathbb{R}$, then it is statistically continuous on $A$.

It is well known that any continuous function on a compact subset $A$ of $\mathbb{R}$ is uniformly continuous on $A$. It is also true for a regular subsequential method $G$ that any $N_\theta$-ward continuous function on a $G$-sequentially compact subset $A$ of $\mathbb{R}$ is also uniformly continuous on $A$ (see [6]). Furthermore, for $N_\theta$-ward continuous functions defined on an $N_\theta$-ward compact subset of $\mathbb{R}$, we have the following.

**Theorem 3.8.** Let $A$ be an $N_\theta$-ward compact subset $A$ of $\mathbb{R}$ and let $f : A \to \mathbb{R}$ be an $N_\theta$-ward continuous function on $A$. Then $f$ is uniformly continuous on $A$.

**Proof.** Suppose that $f$ is not uniformly continuous on $A$ so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there are $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For each positive integer $n$, there exist $\alpha_n$ and $\beta_n$ such that $|\alpha_n - \beta_n| < 1/n$, and $|f(\alpha_n) - f(\beta_n)| \geq \varepsilon_0$. Since $A$ is $N_\theta$-ward compact, there exists an $N_\theta$-quasi-Cauchy subsequence $(\alpha_{n_i})$ of the sequence $(\alpha_n)$. It is clear
that the corresponding subsequence \((\beta_n)\) of the sequence \((\beta_n)\) is also \(N\)-null sequences, that is,

\[
\beta_{n_k+1} - \beta_n = (\beta_{n_k+1} - \alpha_{n_k+1}) + (\alpha_{n_k+1} - \alpha_n) + (\alpha_n - \beta_n).
\] (3.6)

On the other hand, it follows from the equality \(\alpha_{n_k+1} - \beta_n = \alpha_{n_k} - \alpha_n + \alpha_n - \beta_n\) that the sequence \((\alpha_{n_k+1} - \beta_n)\) is \(N\)-convergent to 0. Hence the sequence

\[
(\alpha_n, \beta_n, \alpha_n, \beta_n, \alpha_n, \beta_n, \ldots)
\] (3.7)

is \(N\)-null sequences. Thus \(f\) does not preserve \(N\)-null sequences. This contradiction completes the proof of the theorem.

Corollary 3.9. If a function \(f\) is \(N\)-ward continuous on a bounded subset \(A\) of \(R\), then it is uniformly continuous on \(A\).

Proof. The proof follows from the preceding theorem and Theorem 3.3.

Theorem 3.10. \(N\)-ward continuous image of any \(N\)-ward compact subset of \(R\) is \(N\)-ward compact.

Proof. Assume that \(f\) is an \(N\)-ward continuous function on a subset \(A\) of \(R\) and \(E\) is an \(N\)-ward compact subset of \(A\). Let \((\beta_n)\) be any sequence of points in \(f(E)\). Write \(\beta_n = f(\alpha_n)\) where \(\alpha_n \in E\) for each positive integer \(n\). \(N\)-ward compactness of \(E\) implies that there is a subsequence \((\gamma_k) = (\alpha_{n_k})\) of \((\alpha_n)\) with \(N\)-lim \(k \to \infty \Delta \gamma_k = 0\). Write \((t_k) = (f(\gamma_k))\). As \(f\) is \(N\)-ward continuous, \((f(\gamma_k))\) is \(N\)-ward quasi-Cauchy. Thus we have obtained a subsequence \((t_k)\) of the sequence \((f(\alpha_n))\) with \(N\)-lim \(k \to \infty \Delta t_k = 0\). Thus \(f(E)\) is \(N\)-ward compact. This completes the proof of the theorem.

Corollary 3.11. \(N\)-ward continuous image of any compact subset of \(R\) is \(N\)-ward compact.

The proof follows from the preceding theorem.

Corollary 3.12. \(N\)-ward continuous image of any bounded subset of \(R\) is bounded.

The proof follows from Theorems 3.3 and 3.10.

Corollary 3.13. \(N\)-ward continuous image of a \(G\)-sequentially compact subset of \(R\) is \(N\)-ward compact for any regular subsequential method \(G\).

For a further study, we suggest to investigate \(N\)-null sequences of fuzzy points and \(N\)-ward continuity for the fuzzy functions (see [24] for the definitions and related concepts in fuzzy setting). However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work.
References

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