Research Article

A Note on the Lebesgue-Radon-Nikodym Theorem with respect to Weighted p -adic Invariant Integral on $\mathbb{Z}_p$

Joohee Jeong and Seog-Hoon Rim

Department of Mathematics Education, Kyungpook National University, Daegu 702701, Republic of Korea

Correspondence should be addressed to Seog-Hoon Rim, shrim@knu.ac.kr

Received 1 October 2011; Accepted 11 January 2012

Academic Editor: Bevan Thompson

Copyright © 2012 J. Jeong and S.-H. Rim. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We will give the Lebesgue-Radon-Nikodym theorem with respect to weighted p -adic q -measure on $\mathbb{Z}_p$. In special case, $q = 1$, we can derive the same result as Kim, 2012; Kim et al., 2011.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p -adic integers, the field of p -adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p -adic norm $|\cdot|_p$ is defined by $|x|_p = p^{-v_p(x)} = p^{-r}$ for $x = p^r(s/t)$ where s and t are integers with $(p, s) = (p, t) = 1$ and $r \in \mathbb{Q}$ (see [1–12]).

When one speaks of q -extension, q can be regarded as an indeterminate, a complex $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|q - 1|_p < p^{-1/(p-1)}$ and we use the notations of q -numbers as follows:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \quad (1.1)$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. The fermionic invariant measure on $\mathbb{Z}_p$ is defined by Kim as follows:

$$\mu_{-1}(a + p^n \mathbb{Z}_p) = (-1)^a, \quad (1.2)$$

where

$$a + p^n \mathbb{Z}_p = \{x \in \mathbb{Z}_p \mid x \equiv a \pmod{p^n}\}, \quad (1.3)$$

and $a \in \mathbb{Z}$ with $0 \leq a < p^n$ (see [2–4, 7]).

From (1.2), the fermionic p -adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x, \quad (1.4)$$

where $f \in C(\mathbb{Z}_p)$ (see [2–4, 6–9]).

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) fermionic measure $\mathbb{Z}_p$ satisfying

$$\left| \mu_{-1}(a + p^n \mathbb{Z}_p) - \mu_{-1}(a + p^{n+1} \mathbb{Z}_p) \right|_p \leq \delta_n, \quad (1.5)$$

(see [4, 5, 10]), where $\delta_n \rightarrow 0$, a is a element of $\mathbb{Z}_p$, and δ_n is independent of a (for strongly fermionic measure, δ_n is replaced by Cp^{-n} , where C is a positive constant).

Let $f(x)$ be a function defined on $\mathbb{Z}_p$. The fermionic integral of f with respect to a weakly fermionic measure μ_{-1} is

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} f(x) \mu_{-1}(x + p^n \mathbb{Z}_p), \quad (1.6)$$

if the limit exists.

If μ_{-1} is a weakly fermionic measure on $\mathbb{Z}_p$, then we can define the Radon-Nikodym derivative of μ_{-1} with respect to the Haar measure on $\mathbb{Z}_p$ as follows:

$$f_{\mu_{-1}}(x) = \lim_{n \rightarrow \infty} \mu_{-1}(x + p^n \mathbb{Z}_p) \quad (1.7)$$

(see [4, 11]). Note that $f_{\mu_{-1}}$ is only a continuous function on $\mathbb{Z}_p$. Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, let us define $\mu_{-1,f}$ as follows:

$$\mu_{-1,f}(x + p^n \mathbb{Z}_p) = \int_{x+p^n \mathbb{Z}_p} f(x) d\mu_{-1}(x) \quad (1.8)$$

(see [4, 11]), where the integral is the fermionic p -adic invariant integral. From (1.8), we can easily note that $\mu_{-1,f}$ is a strongly fermionic measure on $\mathbb{Z}_p$. Since then

$$\begin{aligned} \left| \mu_{-1,f}(x + p^n \mathbb{Z}_p) - \mu_{-1,f}(x + p^{n+1} \mathbb{Z}_p) \right|_p &= \left| \sum_{x=0}^{p^n-1} f(x)(-1)^x - \sum_{x=0}^{p^n} f(x)(-1)^x \right|_p \\ &= \left| \frac{f(p^n)}{p^n} \right|_p |p^n| \leq Cp^{-n}, \end{aligned} \quad (1.9)$$

where C is positive constant.

The purpose of this paper is to derive a Lebesgue-Radon-Nikodym type theorem with respect to the fermionic weighted p -adic q -measure on $\mathbb{Z}_p$.

2. The Lebesgue-Radon-Nikodym Theorem with Respect to the Weighted p -adic q -Measure

For any positive integer a and n with $a < p^n$ and $f \in \text{UD}(\mathbb{Z}_p)$, we define $\tilde{\mu}_{f,-q}$, weighted fermionic measure on $\mathbb{Z}_p$ as follows:

$$\tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) = \int_{a+p^n \mathbb{Z}_p} q^x f(x) d\mu_{-1}(x), \quad (2.1)$$

where the integral is the fermionic p -adic invariant integral on $\mathbb{Z}_p$.

From (2.1), we note that

$$\begin{aligned} \tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) &= \lim_{m \rightarrow \infty} \sum_{x=0}^{p^m-1} f(a + p^n x) (-1)^{a+p^n x} q^{a+p^n x} \\ &= (-1)^a q^a \lim_{m \rightarrow \infty} \sum_{x=0}^{p^{m-n}-1} f(a + p^n x) (-1)^x q^{p^n x} \\ &= (-1)^a \int_{\mathbb{Z}_p} f(a + p^n x) q^{a+p^n x} d\mu_{-1}(x). \end{aligned} \quad (2.2)$$

By (2.2), we get

$$\tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) = (-1)^a \int_{\mathbb{Z}_p} f(a + p^n x) q^{a+p^n x} d\mu_{-1}(x). \quad (2.3)$$

Thus, by (2.3), we have

$$\tilde{\mu}_{\alpha f + \beta g, -q} = \alpha \tilde{\mu}_{f, -q} + \beta \tilde{\mu}_{g, -q}, \quad (2.4)$$

where $f, g \in \text{UD}(\mathbb{Z}_p)$ and α, β are positive constants.

By (2.1), (2.2), (2.3), and (2.4), we get

$$|\tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p)|_p \leq \|f_q\|_\infty, \quad (2.5)$$

where $\|f_q\|_\infty = \sup_{x \in \mathbb{Z}_p} |f(x)q^x|_p$.

Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q -polynomial. Now we show that $\tilde{\mu}_{P,-q}$ is a strongly weighted fermionic p -adic invariant measure on $\mathbb{Z}_p$. Without a loss of generality, it is enough to prove the statement for $P(x) = [x]_q^k$.

For $a \in \mathbb{Z}$ with $0 \leq a < p^n$, we have

$$\tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p) = \lim_{m \rightarrow \infty} (-q)^a \sum_{i=0}^{p^{m-n}-1} q^{p^n i} [a + ip^n]_q^k (-1)^i. \quad (2.6)$$

Note that

$$q^{p^n i} = \sum_{l=0}^i \binom{i}{l} [p^n]_q^l (q-1)^l, \quad (2.7)$$

and

$$[a + ip^n]_q^k = \left([a]_q + q^a [p^n]_q [i]_{q^{p^n}} \right)^k. \quad (2.8)$$

By (2.6) and (2.8), we easily get

$$\begin{aligned} \tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p) &\equiv (-1)^a q^a [a]_q^k \pmod{[p^n]_q} \\ &\equiv (-1)^a P(a) q^a \pmod{[p^n]_q}. \end{aligned} \quad (2.9)$$

For $x \in \mathbb{Z}_p$, let $x \equiv x_n \pmod{p^n}$ and $x \equiv x_{n+1} \pmod{p^{n+1}}$, where $x_n, x_{n+1} \in \mathbb{Z}$ with $0 \leq x_n < p^n$ and $0 \leq x_{n+1} < p^{n+1}$. Then we have

$$\left| \tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p) - \tilde{\mu}_{P,-q}(a + p^{n+1} \mathbb{Z}_p) \right|_p \leq C p^{-v_p(1-q^{p^n})}, \quad (2.10)$$

where C is positive constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}_{P,-q}}(a) = \lim_{n \rightarrow \infty} \tilde{\mu}_{P,-q}(a + p^n \mathbb{Z}_p). \quad (2.11)$$

Then, by (2.9) and (2.10), we see that

$$f_{\tilde{\mu}_{P,-q}}(a) = (-1)^a q^a [a]_q^k = (-1)^a q^a P(a). \quad (2.12)$$

Since $f_{\tilde{\mu}_{p,-q}}(x)$ is continuous function on $\mathbb{Z}_p$. For $x \in \mathbb{Z}_p$, we have

$$f_{\tilde{\mu}_{p,-q}}(x) = (-1)^x q^x [x]_q^k, \quad (k \in \mathbb{Z}_+). \quad (2.13)$$

Let $g \in \text{UD}(\mathbb{Z}_p)$. Then, by (2.10), (2.12), and (2.13), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{p,-q}(x) &= \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} g(x) \tilde{\mu}_{p,-q}(x + p^n \mathbb{Z}_p) \\ &= \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} g(x) q^x [x]_q^k (-1)^x \\ &= \int_{\mathbb{Z}_p} g(x) q^x [x]_q^k d\mu_{-1}(x). \end{aligned} \quad (2.14)$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.1. *Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q -polynomial. Then $\tilde{\mu}_{p,-q}$ is a strongly weighted fermionic p -adic invariant measure on $\mathbb{Z}_p$; that is,*

$$f_{\tilde{\mu}_{p,-q}}(x) = (-1)^x q^x P(x), \quad \forall x \in \mathbb{Z}_p. \quad (2.15)$$

Furthermore, for any $g \in \text{UD}(\mathbb{Z}_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{p,-q}(x) = \int_{\mathbb{Z}_p} g(x) P(x) q^x d\mu_{-1}(x), \quad (2.16)$$

where the second integral is weighted fermionic p -adic invariant integral on $\mathbb{Z}_p$.

Let $f(x) = \sum_{n=0}^{\infty} a_{n,q} \binom{x}{n}_q$ be the Mahler q -expansion of continuous function on $\mathbb{Z}_p$, where

$$\binom{x}{n}_q = \frac{[x]_q [x-1]_q \cdots [x-n+1]_q}{[n]_q!}. \quad (2.17)$$

Then we note that $\lim_{n \rightarrow \infty} |a_{n,q}| = 0$.

Let

$$f_m(x) = \sum_{i=0}^m a_{i,q} \binom{x}{i}_q \in \mathbb{C}_p[[x]_q]. \quad (2.18)$$

Then

$$\|f - f_m\|_{\infty} \leq \sup_{n \leq m} |a_{n,q}|. \quad (2.19)$$

The function $f(x)$ can be rewritten as $f = f_m + f - f_m$. Thus, by (2.4) and (2.19), we get

$$\begin{aligned} & \left| \tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) - \tilde{\mu}_{f,-q}(a + p^{n+1} \mathbb{Z}_p) \right| \\ & \leq \max \left\{ \left| \tilde{\mu}_{f_m,-q}(a + p^n \mathbb{Z}_p) - \tilde{\mu}_{f_m,-q}(a + p^{n+1} \mathbb{Z}_p) \right|, \right. \\ & \quad \left. \left| \tilde{\mu}_{f-f_m,-q}(a + p^n \mathbb{Z}_p) - \tilde{\mu}_{f-f_m,-q}(a + p^{n+1} \mathbb{Z}_p) \right| \right\}. \end{aligned} \quad (2.20)$$

From Theorem 2.1, we note that

$$\left| \tilde{\mu}_{f-f_m,-q}(a + p^n \mathbb{Z}_p) \right|_p \leq \|f - f_m\|_\infty \leq C_1 p^{-2v_p(1-q^{p^n})}, \quad (2.21)$$

where C_1 are positive constants. For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$. So, we see that

$$\begin{aligned} & \left| \tilde{\mu}_{f_m,-q}(a + p^n \mathbb{Z}_p) - \tilde{\mu}_{f_m,-q}(a + p^{n+1} \mathbb{Z}_p) \right|_p \\ & = \left| \frac{f_m([p^n]_q) q^{p^n}}{[p^n]_q^2} \right|_p \left| [p^n]_q^2 \right|_p \leq \|f_m q^x\|_\infty \left| [p^n]_q^2 \right|_p \leq C_2 p^{-2v_p(1-q^{p^n})}, \end{aligned} \quad (2.22)$$

where C_2 is a positive constant.

By (2.21), we get

$$\begin{aligned} & \left| (-1)^a f(a) q^a - \tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) \right|_p \\ & \leq \max \left\{ \left| q^a f(a) - f_m(a) q^a \right|_p, \left| q^a f_m(a) - \tilde{\mu}_{f_m,-q}(a + p^n \mathbb{Z}_p) \right|_p, \left| \tilde{\mu}_{f-f_m,-q}(a + p^n \mathbb{Z}_p) \right|_p \right\} \\ & \leq \max \left\{ \left| f(a) - f_m(a) \right|_p, \left| f_m(a) - \tilde{\mu}_{f_m,-q}(a + p^n \mathbb{Z}_p) \right|_p, \|f - f_m\|_\infty \right\}. \end{aligned} \quad (2.23)$$

Let us assume that we fix $\epsilon > 0$ and fix m such that $\|f - f_m\| < \epsilon$. Then we have

$$\left| (-q)^a f(a) - \tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) \right|_p \leq \epsilon, \quad \text{for } n \gg 0. \quad (2.24)$$

Thus, by (2.24), we have

$$f_{\tilde{\mu}_{f,-q}}(a) = \lim_{n \rightarrow \infty} \tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) = (-1)^a q^a f(a). \quad (2.25)$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \leq p^{-n}$. Then we get

$$\begin{aligned} \tilde{\mu}_{f,-q}(a + p^n \mathbb{Z}_p) &= \tilde{\mu}_{f_m,-q}(a + p^n \mathbb{Z}_p) + \tilde{\mu}_{f-f_m,-q}(a + p^n \mathbb{Z}_p) \\ &= (-1)^a q^a f(a) \pmod{[p^n]_q^2}. \end{aligned} \quad (2.26)$$

For $g \in \text{UD}(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{f,-q}(x) = \int_{\mathbb{Z}_p} f(x)g(x)q^x d\mu_{-1}(x). \quad (2.27)$$

Let f be the function from $\text{UD}(\mathbb{Z}_p)$ to $\text{Lip}(\mathbb{Z}_p)$. We easily see that $q^x \mu_{-1}(x + p^n \mathbb{Z}_p)$ is a strongly weighted p -adic invariant measure on $\mathbb{Z}_p$ and

$$\left| (f_q)_{\mu_{-1}}(a) - q^a \mu_{-1}(a + p^n \mathbb{Z}_p) \right|_p \leq C_3 p^{-2v_p(1-q^{p^n})}, \quad (2.28)$$

where $f_q(x) = f(x)q^x$ and C_3 is positive constant and $n \in \mathbb{Z}_+$.

If $\tilde{\mu}_{1,-q}$ is associated with strongly weighted fermionic invariant measure on $\mathbb{Z}_p$, then we have

$$\left| \tilde{\mu}_{1,-q}(a + p^n \mathbb{Z}_p) - (f_q)_{\mu_{-1}}(a) \right|_p \leq C_4 p^{-2v_p(1-q^{p^n})}, \quad (2.29)$$

where $n > 0$ and C_4 is positive constant.

For $n \gg 0$, we have

$$\begin{aligned} & \left| q^a \mu_{-1}(a + p^n \mathbb{Z}_p) - \tilde{\mu}_{1,-q}(a + p^n \mathbb{Z}_p) \right|_p \\ & \leq \left| q^a \mu_{-1}(a + p^n \mathbb{Z}_p) - (f_q)_{\tilde{\mu}_{-1}}(a) \right|_p + \left| (f_q)_{\tilde{\mu}_{-1}}(a) - \tilde{\mu}_{1,-q}(a + p^n \mathbb{Z}_p) \right|_p \leq K, \end{aligned} \quad (2.30)$$

where K is positive constant.

Hence, $q\mu_{-1} - \tilde{\mu}_{1,-q}$ is a weighted measure on $\mathbb{Z}_p$. Therefore, we obtain the following theorem.

Theorem 2.2. *Let $q\mu_{-1}$ be a strongly weighted p -adic invariant measure on $\mathbb{Z}_p$, and assume that the fermionic weighted Radon-Nikodym derivative $(f_q)_{\mu_{-1}}$ on $\mathbb{Z}_p$ is uniformly differentiable function. Suppose that $\tilde{\mu}_{1,-q}$ is the strongly weighted fermionic p -adic invariant measure associated with $(f_q)_{\mu_{-1}}$. Then there exists a weighted measure $\tilde{\mu}_{2,-q}$ on $\mathbb{Z}_p$ such that*

$$q^x \mu_{-1}(x + p^n \mathbb{Z}_p) = \tilde{\mu}_{1,-q}(x + p^n \mathbb{Z}_p) + \tilde{\mu}_{2,-q}(x + p^n \mathbb{Z}_p). \quad (2.31)$$

Acknowledgments

The authors would like to appreciate the referees and editor for their sincere suggestions for improving our paper.

References

- [1] A. Bayad and T. Kim, "Identities involving values of Bernstein, q -Bernoulli, and q -Euler polynomials," *Russian Journal of Mathematical Physics*, vol. 18, no. 2, pp. 133–143, 2011.
- [2] J. Choi, T. Kim, and Y. H. Kim, "A note on the q -analogues of Euler numbers and polynomials," *Honam Mathematical Journal*, vol. 33, no. 4, pp. 529–534, 2011.

- [3] T. Kim, " q -Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
- [4] T. Kim, "Lebesgue-Radon-Nikodym theorem with respect to fermionic p -adic invariant measure on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 19, 2012.
- [5] T. Kim, "Lebesgue-Radon-Nikodym theorem with respect to q -Volkenborn distribution on μ_q ," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 266–271, 2007.
- [6] T. Kim, "A note on q -Bernstein polynomials," *Russian Journal of Mathematical Physics*, vol. 18, no. 1, pp. 73–82, 2011.
- [7] T. Kim, "Note on the Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 17, no. 2, pp. 131–136, 2008.
- [8] T. Kim, "Some identities on the q -Euler polynomials of higher order and q -Stirling numbers by the fermionic p -adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.
- [9] T. Kim, "New approach to q -Euler polynomials of higher order," *Russian Journal of Mathematical Physics*, vol. 17, no. 2, pp. 218–225, 2010.
- [10] T. Kim, J. Choi, and H. Kim, "A note on the weighted Lebesgue-Radon-Nikodym theorem with respect to p -adic invariant integral on $\mathbb{Z}_p$," *Journal of Applied mathematics and Informatics*, vol. 30, no. 1-2, pp. 211–217, 2012.
- [11] T. Kim, D. V. Dolgy, S. H. Lee, and C. S. Ryoo, "Analogue of lebesgue-radon-nikodym theorem with respect to p -adic q -measure on $\mathbb{Z}_p$," *Abstract and Applied Analysis*, vol. 2011, Article ID 637634, 6 pages, 2011.
- [12] T. Kim, S. D. Kim, and D.-W. Park, "On uniform differentiability and q -Mahler expansions," *Advanced Studies in Contemporary Mathematics*, vol. 4, no. 1, pp. 35–41, 2001.