Research Article

Periodic Solutions of a Type of Liénard Higher Order Delay Functional Differential Equation with Complex Deviating Argument

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The author has studied the existence of periodic solutions of a type of higher order delay functional differential equations with neutral type by using the theory of coincidence degree, and some new sufficient conditions for the existence of periodic solutions have been obtained.

1. Introduction and Lemma

With the rapid development of modern science and technology, functional differential equation with time delay has been widely applied in many areas such as bioengineering, systems analysis, and dynamics. Functional differential equation with complex deviating argument is an important type of the above function. Because the property of the solution to this kind of equation is impossibly estimated, so the literature on the functional differential equation with complex argument is relatively rare [1]. In recent years, with the maturity of the theory of nonlinear functional analysis and algebraic topology, we have the powerful tools of the study on the functional differential equation with complex deviating argument, so it is possible to study the above equation. Furthermore, the study on the periodic solutions of functional differential equation is always one of the most important subject that people concerned for its widespread use. Many results of the study of Duffing-typed functional differential equation and Liénard-typed functional differential equation have been obtained, for example, the literatures [2–18]. Hitherto, the literature of the discussion of higher order functional differential equations has not been found a lot [19]. In this paper I have studied and derived some sufficient conditions that guarantee the existence of periodic solutions for

a type of higher order functional differential equations with complex deviating argument as the following:

$$\sum_{i=1}^m a_i x^{(i)}(t) + f(x(t))\dot{x}(t) + \beta(t)g(x(x(t))) = p(t) \quad (a_i \neq 0), \quad (*)$$

and some new results have been obtained.

In order to establish the existence of T -periodic solutions of $(*)$, we make some preparations.

Definition 1.1. Let X, Y are Banach spaces, and let Ω be an open and bounded subset in X , and let $L : \text{Dom}(L) \subseteq X \rightarrow Y$ be linear mapping; the mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \text{codim Im } L < +\infty$ and $\text{Im } L$ is closed in Y .

Definition 1.2. Let $P : X \rightarrow \ker(L)$, let $Q : Y \rightarrow Y/\text{Im}(L)$ be projectors, and let $N : \overline{\Omega} \rightarrow Y$ be nonlinear mapping; the mapping N will be called L -compact on $\overline{\Omega}$ if $QN : \overline{\Omega} \rightarrow Y/\text{Im}(L)$ and $(L|_{\ker(P)})^{-1}(I - Q)N : \overline{\Omega} \rightarrow X$ are compact.

Lemma 1.3 (see [20]). Let X, Y be Banach spaces; $L : D_L \subset X \rightarrow Y$ is a Fredholm mapping of index zero $P : X \rightarrow X$; $Q : X \rightarrow Y$ are continuous mapping projectors; Ω is an open bounded set in X ; $N : \overline{\Omega} \times [0, 1] \rightarrow Y$ is L -Compact on $\overline{\Omega}$, furthermore suppose that:

- (a) $Lx \neq \lambda N(x, \lambda)$, for all $x \in D_L \cap \partial\Omega$, $\lambda \in (0, 1)$;
- (b) $QN(x, 0) \neq 0$, for all $x \in \ker(L) \cap \partial\Omega$;
- (c) $\deg(QN(x, 0), \ker(L) \cap \Omega, 0) \neq 0$,

then the equation $Lx = N(x, 1)$ has at least one solution on $\overline{\Omega}$, where $\deg$ is Brouwer degree.

2. Main Results and Proof of Theorems

Theorem 2.1. Suppose that f, β, g, p are continuous for their variables, respectively, $p(t+T) = p(t)$, $\beta(t+T) = \beta(t) > 0$, $\int_0^T p(t)dt = 0$, and furthermore suppose that

- (a) $\exists A > 0$, for all $x \in \mathbb{R}$, when $|x| > A$, such that $xg(x) > 0$;
- (b) $\exists M > 0$, for all $x \in \mathbb{R}$, such that $|g(x)| \leq M$;
- (c) $f_1 = \sup_{x \in \mathbb{R}} |f(x)| < (a_m - k(T^{m-1} - T^{m-2} - \dots - T))/T^{m-1}$,

where $k = \max\{|a_i|\}$, $i = 1, 2, \dots, m-1$ and $a_m > k(T^{m-1} + T^{m-2} + \dots + T)$, then $(*)$ has at least one T -periodic solution.

Proof of Theorem 2.1. In order to use continuation theorem to obtain T -periodic solution of $(*)$, we firstly make some required preparations. Let

$$X = \{x \in C^{m-1}(\mathbb{R}, \mathbb{R}) \mid x(t+T) = x(t)\}, \quad Y = \{y \in C(\mathbb{R}, \mathbb{R}) \mid y(t+T) = y(t)\}, \quad (2.1)$$

and the norm of X and Y is $\|x\| = \max_{0 \leq i \leq m-1} \{|x^{(i)}|_\infty\}$, $|x^{(i)}|_\infty = \max_{t \in \mathbb{R}} \{|x^{(i)}(t)|\}$, $i = 1, 2, \dots, m-1$, and $\|y\| = \max_{t \in \mathbb{R}} \{|y(t)|\}$, respectively; then the X and Y with this norm are Banach spaces.

Firstly, we study the priori bound of T -periodic solution of following equation:

$$\sum_{i=1}^m a_i x^{(i)}(t) + \lambda f(x(t))\dot{x}(t) + \lambda \beta(t)g(x(x(t))) = \lambda^2 p(t). \quad (2.2)$$

Suppose that $x = x(t) \in X$ is an arbitrary T -periodic solution of (2.2), put $x(t)$ into, (2.2) and then integrate both sides of (2.2) on $[0, T]$, so

$$\int_0^T \beta(t)g(x(x(t)))dt = 0. \quad (2.3)$$

For the continuity of β, g, x , there must exist a number $t_0 \in [0, T]$ such that

$$\beta(t_0)g(x(x(t_0))) = 0, \quad (2.4)$$

that is,

$$g(x(x(t_0))) = 0. \quad (2.5)$$

For the condition (a) of Theorem 2.1, we have

$$|x(x(t_0))| \leq A. \quad (2.6)$$

Let

$$x(t_0) = nT - t_1, \quad n \in N, \quad t_1 \in [0, T], \quad (2.7)$$

so

$$|x(t_1)| = |x(x(t_0))| \leq A. \quad (2.8)$$

In view of

$$\forall t \in [0, T], \quad x(t) = x(t_1) + \int_{t_1}^t \dot{x}(s)ds, \quad (2.9)$$

we have

$$|x(t)| = \left| x(t_1) + \int_{t_1}^t \dot{x}(s)ds \right| \leq A + \int_{t_1}^t |\dot{x}(s)|ds \leq A + \int_0^T |\dot{x}(t)|dt, \quad (2.10)$$

that is,

$$\left| x^{(0)} \right|_{\infty} = |x|_{\infty} \leq A + \int_0^T |\dot{x}(t)| dt. \quad (2.11)$$

Noting $x(t) = x(t+T)$, so there must exist the number $\xi_i \in [0, T]$ such that $x^{(i)}(\xi_i) = 0$, where $i = 1, 2, 3, \dots, m-1$.

For all $t \in [0, T]$,

$$x^{(i)}(t) = x^{(i)}(\xi_i) + \int_{\xi_i}^t x^{(i+1)}(s) ds = \int_{\xi_i}^t x^{(i+1)}(s) ds, \quad (2.12)$$

we have

$$\begin{aligned} \left| x^{(i)}(t) \right| &= \left| \int_{\xi_i}^t x^{(i+1)}(s) ds \right| \leq \int_0^T \left| x^{(i+1)}(t) \right| dt \leq T \int_0^T \left| x^{(i+2)}(t) \right| dt \\ &\leq T^2 \cdot \int_0^T \left| x^{(i+3)}(t) \right| dt \leq \dots \leq T^{m-(i+1)} \int_0^T \left| x^{(i+m-i)}(t) \right| dt = T^{m-(i+1)} \int_0^T \left| x^{(m)}(t) \right| dt, \end{aligned} \quad (2.13)$$

that is,

$$\left| x^{(i)} \right|_{\infty} \leq T^{m-(i+1)} \int_0^T \left| x^{(m)}(t) \right| dt, \quad i = 1, 2, \dots, m-1. \quad (2.14)$$

Combining (2.11), (2.14), we get

$$\left| x^{(0)} \right|_{\infty} = |x|_{\infty} \leq A + T^{m-1} \int_0^T \left| x^{(m)}(t) \right| dt. \quad (2.15)$$

By (2.2), we get

$$\begin{aligned} \int_0^T |a_m x^{(m)}(t)| dt &\leq \int_0^T |\lambda f(x(t)) \dot{x}(t)| dt + \int_0^T |\lambda \beta(t) g(x(x(t)))| dt + \int_0^T |\lambda^2 p(t)| dt \\ &\quad + \int_0^T |a_1 \dot{x}(t)| dt + \int_0^T |a_2 \ddot{x}(t)| dt + \dots + \int_0^T |a_{m-3} x^{(m-3)}(t)| dt \\ &\quad + \int_0^T |a_{m-2} x^{(m-2)}(t)| dt + \dots + \int_0^T |a_{m-1} x^{(m-1)}(t)| dt, \end{aligned} \quad (2.16)$$

where $\beta_1 = \max_{t \in \mathbb{R}} \beta(t)$, $p_1 = \max_{t \in \mathbb{R}} \{|p(t)|\}$, and $k = \max\{|a_i|\}$, $i = 1, 2, 3, \dots, m-1$.

Noting (2.14) and the conditions (b), (c) of Theorem 2.1, we have

$$\begin{aligned} \int_0^T |a_m x^{(m)}(t)| dt &\leq f_1 T \cdot T^{m-2} \int_0^T |x^{(m)}(t)| dt + \beta_1 T M + p_1 T \\ &\quad + kT \cdot T^{m-(1+1)} \int_0^T |x^{(m)}(t)| dt + kT \cdot T^{m-(2+1)} \int_0^T |x^{(m)}(t)| dt \\ &\quad + \cdots + kT \cdot T^{m-(m-1+1)} \int_0^T |x^{(m)}(t)| dt, \end{aligned} \quad (2.17)$$

so

$$a_m \int_0^T |x^{(m)}(t)| dt \leq (kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1}) \int_0^T |x^{(m)}(t)| dt + \beta_1 T M + p_1 T, \quad (2.18)$$

where $a_m > T^{m-1} \sum_{i=1}^m f_i + kT^{m-1} + kT^{m-2} + \cdots + kT$.

Let

$$\frac{\beta_1 T M + p_1 T}{a_0 - (kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1})} \triangleq A_1, \quad (2.19)$$

that is,

$$\int_0^T |x^{(m)}(t)| dt \leq A_1. \quad (2.20)$$

Noting (2.14), (2.15), and (2.20), we have

$$\begin{aligned} |x^{(0)}|_\infty &= |x|_\infty \leq A + T^{m-1} A_1 \triangleq \omega_0, \\ |x^{(i)}|_\infty &\leq T^{m-(i+1)} A_1 \triangleq \omega_i, \quad i = 1, 2, \dots, m-1. \end{aligned} \quad (2.21)$$

Let $\omega = \max_{0 \leq i \leq m} \{\omega_i + 1\}$, and let $\Omega = \{x \mid x \in X : \|x\| < \omega\}$; then Ω is an open and bounded set in X .

Let

$$L : D_L \subset X \longrightarrow Y : x \longrightarrow Lx = \sum_{i=1}^m a_i x^{(i)}(t), \quad (2.22)$$

$$N : X \times I \longrightarrow Y : x \longrightarrow N(x, \lambda) = -f(x(t))\dot{x}(t) - \beta(t)g(x(x(t))) + \lambda p(t);$$

then the corresponding equation of $Lx = \lambda N(x, \lambda)$ is (2.2).

Now, we define projection operators as follows;

$$\begin{aligned} P : X &\longrightarrow \ker(L) : x \longrightarrow Px = \frac{1}{T} \int_0^T x(t) dt, \\ Q : Y &\longrightarrow \frac{Y}{\operatorname{Im}(L)} : y \longrightarrow Qy = \frac{1}{T} \int_0^T y(t) dt. \end{aligned} \quad (2.23)$$

Obviously, P, Q are continuous operators, $\operatorname{Im}(P) = \mathbb{R} = \ker(L)$, $\ker(Q) = \operatorname{Im}(L)$, and it is easy to prove that L is a Fredholm mapping of index zero and is L -Compact on $\bar{\Omega}$.

From the above discussion and the construction of Ω , we know that for all $x \in D_L \cap \partial\Omega$, $\lambda \in (0, 1)$, $Lx \neq \lambda N(x, \lambda)$, therefore the condition (a) of Lemma 1.3 holds.

For arbitrary $x \in \ker(L) \cap \partial\Omega$, $\|x\| = \omega$, by the definition of Q, N , we have

$$\begin{aligned} QN(x, 0) &= \frac{1}{T} \int_0^T [-f(x(t))\dot{x}(t) - \beta(t)g(x(x(t)))] dt \\ &= -\frac{1}{T} \int_0^T \beta(t)g(x(x(t))) dt, \end{aligned} \quad (2.24)$$

so

$$\begin{aligned} xQN(x, 0) &= -\frac{1}{T} x \int_0^T \beta(t)g(x(x(t))) dt \\ &= -\frac{1}{T} xg(x) \int_0^T \beta(t) dt \neq 0, \end{aligned} \quad (2.25)$$

therefore the condition (b) of Lemma 1.3 holds.

Making a transformation.

$$H(x, \mu) = -\mu x + (1 - \mu)QN(x, 0), \quad \forall x \in \partial\Omega \cap \ker(L), \mu \in [0, 1], \quad (2.26)$$

we have

$$\begin{aligned} xH(x, \mu) &= -\mu x^2 + x(1 - \mu)QN(x, 0) \\ &= -\mu x^2 - (1 - \mu) \frac{1}{T} g(x)x \int_0^T \beta(t) dt < 0. \end{aligned} \quad (2.27)$$

So $xH(x, \mu) \neq 0$, that is, $H(x, \mu) \neq 0$ is a homotopy, $\deg(QN(x, 0), \ker(L) \cap \Omega, 0) = \deg(-I, \ker(L) \cap \Omega, 0) = \deg(-I, \mathbb{R} \cap \Omega, 0) \neq 0$, where I is an identity mapping, and the condition (c) of Lemma 1.3 holds.

From above all, the requirements of Lemma 1.3 are all satisfied, so $(*)$ has at least one T -periodic solution under the condition of Theorem 2.1, so the proof of Theorem 2.1 is completed. $\square$

Remark 2.2. In Theorem 2.1, if $\beta(t) < 0$ and the condition (a) of Theorem 2.1 is when $|x| > A$, $xg(x) < 0$, and the rest are unchangeable, then $(*)$ has at least one T -periodic solution.

If the $g(x)$ is not a bounded function, we have the following theorem.

Theorem 2.3. Suppose that f, β, g, p are continuous for their variables, respectively, $p(t+T) = p(t)$, $\beta(t+T) = \beta(t) > 0$, $\int_0^T p(t)dt = 0$, and furthermore suppose following:

- (a) $\exists A > 0$, for all $x \in \mathbb{R}$, when $|x| > A$, such that $xg(x) > 0$;
- (b) $\exists M > 0$, for all $x \in \mathbb{R}$, such that $|g(x)| \leq M|x| + c$;
- (c) $f_1 = \sup_{x \in \mathbb{R}} |f(x)| < (a_m - kT^{m-1} - kT^{m-2} - \dots - kT - \beta_1 T^m) / T^{m-1}$,

where $k = \max\{|a_i|\}$, $i = 1, 2, \dots, m-1$, and $a_m > kT^{m-1} + kT^{m-2} + \dots + kT + \beta_1 T^m$, then $(*)$ has at least one T -periodic solution.

Proof of Theorem 2.3. Banach spaces X, Y and the mappings L, P, Q , and N are the same to Theorem 2.1, and their property are equal to Theorem 2.1, then the corresponding equation of $Lx = \lambda N(x, \lambda)$ is

$$\sum_{i=1}^m a_i x^{(i)}(t) + \lambda f(x(t)) \dot{x}(t) + \lambda \beta(t) g(x(x(t))) = \lambda^2 p(t). \quad (2.28)$$

It is similar to Theorem 2.1, there must exist a number $t_1 \in [0, T]$, such that

$$|x(t_1)| \leq A, \quad (2.29)$$

and it is easy to obtain

$$\begin{aligned} |x^{(i)}|_{\infty} &\leq T^{m-(i+1)} \int_0^T |x^{(m)}(t)| dt, \quad i = 1, 2, \dots, m-1, \\ |x^{(0)}|_{\infty} &= |x|_{\infty} \leq A + T^{m-1} \int_0^T |x^{(m)}(t)| dt. \end{aligned} \quad (2.30)$$

Noting (2.28), (2.30) and the conditions (b), (c) of Theorem 2.3, we have

$$\begin{aligned} \int_0^T |a_m x^{(m)}| dt &\leq \int_0^T |\lambda f(x(t)) \dot{x}(t)| dt + \int_0^T |\lambda \beta(t) g(x(x(t)))| dt + \int_0^T |\lambda^2 p(t)| dt \\ &\quad + kT \cdot T^{m-(1+1)} \int_0^T |x^{(m)}(t)| dt + kT \cdot T^{m-(2+1)} \int_0^T |x^{(m)}(t)| dt \\ &\quad + \dots + kT \cdot T^{m-(m-1+1)} \int_0^T |x^{(m)}(t)| dt \end{aligned}$$

$$\begin{aligned}
&\leq f_1 T \cdot T^{m-2} \int_0^T |x^m(t)| dt + \beta_1 T [M|x(x(t))| + c] + p_1 T \\
&\quad + kT \cdot T^{m-(1+1)} \int_0^T |x^m(t)| dt + kT \cdot T^{m-(2+1)} \int_0^T |x^m(t)| dt \\
&\quad + \dots + kT \cdot T^{m-(m-1+1)} \int_0^T |x^m(t)| dt.
\end{aligned} \tag{2.31}$$

So

$$\begin{aligned}
a_m \int_0^T |x^m(t)| dt &\leq (kT^{m-1} + kT^{m-2} + \dots + kT + f_1 T^{m-1}) \int_0^T |x^m(t)| dt \\
&\quad + \beta_1 T c + \beta_1 T M |x|_\alpha + p_1 T \\
&\leq (kT^{m-1} + kT^{m-2} + \dots + kT + f_1 T^{m-1}) \int_0^T |x^m(t)| dt \\
&\quad + \beta_1 T c + \beta_1 T M \left(A + T^{m-1} \int_0^T |x^{(m)}(t)| dt \right) + p_1 T \\
&\leq (kT^{m-1} + kT^{m-2} + \dots + kT + f_1 T^{m-1} + \beta_1 T^m) \int_0^T |x^m(t)| dt \\
&\quad + \beta_1 T c + \beta_1 T M A + p_1 T,
\end{aligned} \tag{2.32}$$

where $k = \max\{|a_i|\}$, $i = 1, 2, 3, \dots, m-1$, and $a_m > kT^{m-1} + kT^{m-2} + \dots + kT + f_1 T^{m-1} + \beta_1 T^m$.

Let

$$\frac{\beta_1 T c + \beta_1 T M A + p_1 T}{a_m - (kT^{m-1} + kT^{m-2} + \dots + kT + f_1 T^{m-1} + \beta_1 T^m)} \triangleq A_1, \tag{2.33}$$

that is,

$$\int_0^T |x^m(t)| dt \leq A_1. \tag{2.34}$$

Noting (2.30) and (2.34), we have

$$\begin{aligned}
|x^{(0)}|_\infty &= |x|_\infty \leq A + T^{m-1} A_1 \triangleq \omega_0, \\
|x^{(i)}|_\infty &\leq T^{m-(i+1)} A_1 \triangleq \omega_i, \quad i = 1, 2, \dots, m-1.
\end{aligned} \tag{2.35}$$

Let $\omega = \max_{0 \leq i \leq m} \{\omega_i + 1\}$, and we take $\Omega = \{x \mid x \in X : \|x\| < \omega\}$; then Ω is an open and bounded set in X .

Similarly to Theorem 2.1, we prove easily that L is a Fredholm mapping of index zero and N is L -compact on $\overline{\Omega}$ and the conditions (a), (b), and (c) of Lemma 1.3 hold.

From above all, the requirements of Lemma 1.3 are all satisfied, so $(*)$ has at least one T -periodic solution under the condition of Theorem 2.3, so far the proof of Theorem 2.3 is completed. $\square$

Remark 2.4. In Theorem 2.3, if $\beta(t) < 0$ and the condition (a) of Theorem 2.3 is when $|x| > A$, $xg(x) < 0$, and the rest are unchangeable, then $(*)$ has at least one T -periodic solution.

If the $\int_0^T p(t)dt \neq 0$, we have the following theorem.

Theorem 2.5. Suppose that f, β, g, p are continuous for their variables, respectively, $\beta(t+T) = \beta(t) > 0$, and meet the condition (a) of Theorem 2.1 and furthermore suppose as follows:

- (a) $\lim_{|x| \rightarrow +\infty} |g(x)| = +\infty$;
- (b) $\exists a, b, c > 0$, such that $|g(x)| \leq ag(x) + b|x| + c$;
- (c) $f_1 = \sup_{x \in \mathbb{R}} |f(x)| (a_m - kT^{m-1} - kT^{m-2} - \dots - kT - f_1T^{m-1} - b\beta_1T^m)/T^{m-1}$,

where $k = \max\{|a_i|\}$, $i = 1, 2, \dots, m-1$, and $a_m > kT^{m-1} + kT^{m-2} + \dots + kT + f_1T^{m-1} + b\beta_1T^m$, then $(*)$ has at least one T -periodic solution.

Proof of Theorem 2.5. Banach spaces X, Y and the mappings L, P, Q , and N are the same to Theorem 2.1, and their property are equal to Theorem 2.1, then the corresponding equation of $Lx = \lambda N(x, \lambda)$ is

$$\sum_{i=1}^m a_i x^{(i)}(t) + \lambda f(x(t))\dot{x}(t) + \lambda \beta(t)g(x(x(t))) = \lambda^2 p(t). \quad (2.36)$$

Suppose that $x = x(t) \in X$ is an arbitrary T -periodic solution of (2.36), put $x(t)$ into (2.36), and then integrate both sides of (2.36) on $[0, T]$, so

$$\int_0^T \beta(t)g(x(x(t)))dt = \int_0^T \lambda p(t)dt. \quad (2.37)$$

For the continuity of β, g, x , there must exist a number $t_1 \in [0, T]$, such that

$$g(x(x(t_1))) = \frac{\lambda \int_0^T p(t)dt}{\int_0^T \beta(t)dt}. \quad (2.38)$$

Combing the condition (a) of Theorem 2.5, there must exist $A_1 > 0$, such that

$$|x(x(t_1))| \leq A_1. \quad (2.39)$$

Similarly to Theorem 2.1, we have

$$\left| x^{(i)} \right|_{\infty} \leq T^{m-(i+1)} \int_0^T \left| x^{(m)}(t) \right| dt, \quad i = 1, 2, \dots, m-1, \quad (2.40)$$

$$\left| x^{(0)} \right|_{\infty} = |x|_{\infty} \leq A_1 + T^{m-1} \int_0^T \left| x^{(m)}(t) \right| dt. \quad (2.41)$$

By (2.36), (2.37), (2.39), and (2.41) and the conditions (b), (c) of Theorem 2.5, we have

$$\begin{aligned} \int_0^T \left| a_m x^{(m)} \right| dt &\leq \int_0^T \left| \lambda f(x(t)) \dot{x}(t) \right| dt + \int_0^T \left| \lambda \beta(t) g(x(x(t))) \right| dt + \int_0^T \left| \lambda^2 p(t) \right| dt \\ &\quad + kT \cdot T^{m-(1+1)} \int_0^T \left| x^{(m)}(t) \right| dt + kT \cdot T^{m-(2+1)} \int_0^T \left| x^{(m)}(t) \right| dt \\ &\quad + \dots + kT \cdot T^{m-(m-1+1)} \int_0^T \left| x^{(m)}(t) \right| dt \\ &\leq f_1 T |\dot{x}|_{\alpha} + \left(kT^{m-1} + kT^{m-2} + \dots + a_{m-1} T \right) \int_0^T \left| x^{(m)}(t) \right| dt \\ &\quad + \int_0^T a \beta(t) g(x(x(t))) dt + \int_0^T b \beta(t) [|x(x(t))| + c] dt + p_1 T \\ &\leq f_1 T T^{m-2} \int_0^T \left| x^{(m)}(t) \right| dt + \left(kT^{m-1} + kT^{m-2} + \dots + kT \right) \int_0^T \left| x^{(m)}(t) \right| dt \quad (2.42) \\ &\quad + a \beta_1 T p_1 + b \beta_1 T |x|_{\alpha} + b \beta_1 T c + p_1 T \leq f_1 T^{m-1} \int_0^T \left| x^{(m)}(t) \right| dt \\ &\quad + \left(kT^{m-1} + kT^{m-2} + \dots + kT \right) \int_0^T \left| x^{(m)}(t) \right| dt \\ &\quad + b \beta_1 T \left(A_1 + T^{m-1} \int_0^T \left| x^{(m)}(t) \right| dt \right) + a \beta_1 T p_1 + b \beta_1 T c + p_1 T \\ &\leq \left(kT^{m-1} + kT^{m-2} + \dots + kT + f_1 T^{m-1} + b \beta_1 T^m \right) \int_0^T \left| x^{(m)}(t) \right| dt \\ &\quad + b \beta_1 T A_1 + a \beta_1 T p_1 + b \beta_1 T c + p_1 T. \end{aligned}$$

So

$$\begin{aligned} &\left[a_m - \left(kT^{m-1} + kT^{m-2} + \dots + kT + f_1 T^{m-1} + b \beta_1 T^m \right) \right] \int_0^T \left| x^{(m)}(t) \right| dt \\ &\leq b \beta_1 T A_1 + a \beta_1 T p_1 + b \beta_1 T c + p_1 T. \end{aligned} \quad (2.43)$$

Let

$$\frac{b\beta_1 T A_1 + a\beta_1 T p_1 + b\beta_1 T c + p_1 T}{a_m - (kT^{m-1} + kT^{m-2} + \cdots + kT + f_1 T^{m-1} + b\beta_1 T^m)} \triangleq A_2, \quad (2.44)$$

that is,

$$\int_0^T |x^m(t)| dt \leq A_2. \quad (2.45)$$

Noting (2.40), (2.41), and (2.45), we have

$$\begin{aligned} |x^{(0)}|_\infty &= |x|_\infty \leq A + T^{m-1} A_2 \triangleq \omega_0, \\ |x^{(i)}|_\infty &\leq T^{m-(i+1)} A_2 \triangleq \omega_i, \quad i = 1, 2, \dots, m-1. \end{aligned} \quad (2.46)$$

For condition (a), there exist $M_0 > 0$ and $A_0 > 0$, such that $|x| > M_0$, $|g(x)| > A_0$; let $\omega = \max_{0 \leq i \leq m} \{\omega_i + 1, M_0\}$, and we take $\Omega = \{x \mid x \in X : \|x\| < \omega\}$; then Ω is an open and bounded set in X .

Similarly to Theorem 2.1, we prove easily that L is a Fredholm mapping of index zero and N is L -compact on $\bar{\Omega}$ and the conditions (a), (b), and (c) of Lemma 1.3 hold.

From above all, the requirements of Lemma 1.3 are all satisfied, so (*) has at least one T -periodic solution under the condition of Theorem 2.5, so the proof of Theorem 2.5 is completed. $\square$

Remark 2.6. In Theorem 2.5, if $\beta(t) < 0$ and the condition (a) of Theorem 2.1 is when $|x| > A$, $xg(x) < 0$, and the rest are unchangeable, then (*) has at least one T -periodic solution.

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