Research Article

A Classification of a Totally Umbilical Slant Submanifold of Cosymplectic Manifolds

Siraj Uddin,¹ Cenap Ozel,² and Viqar Azam Khan³

¹ Institute of Mathematical Sciences, Faculty of Science, University of Malaya,
50603 Kuala Lumpur, Malaysia

² Department of Mathematics, Abant Izzet Baysal University, 14268 Bolu, Turkey

³ Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

Correspondence should be addressed to Siraj Uddin, siraj.ch@gmail.com

Received 27 September 2011; Revised 26 December 2011; Accepted 26 December 2011

Academic Editor: Natig Atakishiyev

Copyright © 2012 Siraj Uddin et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study slant submanifolds of a cosymplectic manifold. It is shown that a totally umbilical slant submanifold M of a cosymplectic manifold $\overline{M}$ is either an anti-invariant submanifold or a 1-dimensional submanifold. We show that every totally umbilical proper slant submanifold of a cosymplectic manifold is totally geodesic.

1. Introduction

The study of slant submanifolds in complex spaces was initiated by Chen as a natural generalization of both holomorphic and totally real submanifolds [1, 2]. Since then, many research papers have appeared concerning the existence of these submanifolds as well as on the geometry of the existent slant submanifolds in different known spaces (cf. [3, 4]). The slant submanifolds of an almost contact metric manifold were defined and studied by Lotta [4]. Later on, these submanifolds were studied by Cabrerizo et al. in the setting of Sasakian manifolds [3].

Recently, Şahin proved that a totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic [5]. Our aim in the present paper is to investigate slant submanifolds in contact manifolds. Thus, we study slant submanifolds of a cosymplectic manifold. We have shown that a totally umbilical slant submanifold M of a cosymplectic manifold $\overline{M}$ is either an anti-invariant submanifold or the $\dim M = 1$ or the mean curvature vector $H \in \Gamma(\mu)$, and then we have obtained an interesting result for a totally umbilical proper slant submanifold of a cosymplectic manifold.

2. Preliminaries

Let $\overline{M}$ be a $(2n + 1)$ -dimensional manifold with $(1, 1)$ tensor field ϕ satisfying [6]:

$$\phi^2 = -I + \eta \otimes \xi, \quad (2.1)$$

where I is the identity transformation, ξ a vector field, and η a 1-form on $\overline{M}$ satisfying $\phi\xi = \eta \circ \phi = 0$ and $\eta(\xi) = 1$. Then $\overline{M}$ is said to have an almost contact structure. There always exists a Riemannian metric g on $\overline{M}$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

for all vector fields X, Y , on $\overline{M}$. From (2.2), it is easy to observe that

$$g(\phi X, Y) + g(X, \phi Y) = 0. \quad (2.3)$$

The fundamental 2-form Φ is defined as: $\Phi(X, Y) = g(X, \phi Y)$. If $[\phi, \phi] + d\eta \otimes \xi = 0$, then the almost contact structure is said to be normal, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. If $\Phi = d\eta$, the almost contact structure is a contact structure. A normal almost contact structure such that Φ is closed and $d\eta = 0$ is called *cosymplectic structure*. It is well known [7] that the cosymplectic structure is characterized by

$$(\overline{\nabla}_X \phi)Y = 0, \quad (\overline{\nabla}_X \eta)Y = 0, \quad (2.4)$$

for all vector fields X, Y , on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of g . From the formula $\overline{\nabla}_X \phi = 0$, it follows that $\overline{\nabla}_X \xi = 0$.

Let M be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric g and let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M , respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$ -module of smooth sections of a vector bundle TM over M , then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.5)$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.6)$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively for the immersion of M into $\overline{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \quad (2.7)$$

where g denotes the Riemannian metric on $\overline{M}$ as well as the one induced on M [8].

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \quad (2.8)$$

where PX is the tangential component and FX is the normal component of ϕX . Similarly for any $N \in \Gamma(T^\perp M)$, we write

$$\phi N = tN + fN, \quad (2.9)$$

where tN is the tangential component and fN is the normal component of ϕN . If we denote the orthogonal complementary distribution of $F(TM)$ in $T^\perp M$ by μ , then we have the direct sum

$$T^\perp M = F(TM) \oplus \mu. \quad (2.10)$$

We can see that μ is an invariant subbundle with respect to ϕ . Furthermore, the covariant derivatives of the tensor fields P and F are defined as

$$\begin{aligned} (\bar{\nabla}_X P)Y &= \nabla_X PY - P\nabla_X Y, \\ (\bar{\nabla}_X F)Y &= \nabla_X^\perp FY - F\nabla_X Y, \end{aligned} \quad (2.11)$$

for any $X, Y \in \Gamma(TM)$.

A submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is called *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \quad (2.12)$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is given by

$$H = \sum_{i=1}^m h(e_i, e_i), \quad (2.13)$$

where m is the dimension of M and $\{e_1, e_2, \dots, e_m\}$ is the local orthonormal frame on M . A submanifold M is said to be *totally geodesic* if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is *minimal* if $H = 0$ on M .

3. Slant Submanifolds

Throughout the section, we assume that M is a slant submanifold of a cosymplectic manifold $\bar{M}$. We always consider such submanifold tangent to the structure vector field ξ . For each

nonzero vector X tangent to M at x , we denote by $0 \leq \theta(X) \leq \pi/2$, the angle between ϕX and $T_x M$, known as the *Wirtinger angle* of X . If the Wirtinger angle $\theta(X)$ is constant, that is, independent of the choice of $x \in M$ and $X \in T_x M - \{\xi\}$, then M is said to be a *slant submanifold* [4]. In this case the constant angle θ is called *slant angle* of the slant submanifold. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant submanifold. If M is a slant submanifold of an almost contact metric manifold, then the tangent bundle TM is decomposed as

$$TM = \mathfrak{D} \oplus \langle \xi \rangle, \quad (3.1)$$

where $\langle \xi \rangle$ denotes the distribution spanned by the structure vector field ξ and $\mathfrak{D}$ is the complementary distribution of $\langle \xi \rangle$ in TM , known as the *slant distribution*.

We recall the following result for a slant submanifold.

Theorem 3.1 (see [3]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \quad (3.2)$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2 \theta$.

The following relations are straightforward consequence of (3.2):

$$g(PX, PY) = \cos^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (3.3)$$

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (3.4)$$

for any X, Y tangent to M .

Now, we prove the following.

Theorem 3.2. *Let M be a totally umbilical slant submanifold of a cosymplectic manifold $\overline{M}$. Then at least one of the following statements is true:*

- (i) M is an anti-invariant submanifold;
- (ii) M is a 1-dimensional submanifold;
- (iii) If M is a proper slant submanifold, then $H \in \Gamma(\mu)$,

where H is the mean curvature vector of the submanifold M .

Proof. Let M be a totally umbilical slant submanifold of a cosymplectic manifold $\overline{M}$, then for any $X, Y \in \Gamma(TM)$, we have

$$h(PX, PY) = g(PX, PY)H. \quad (3.5)$$

From (2.5) and (3.3), we deduce that

$$\overline{\nabla}_{PX} PX - \nabla_{PX} PX = \cos^2 \theta \{g(X, X) - \eta(X)\eta(X)\}H. \quad (3.6)$$

Using (2.8) and the fact that $\overline{M}$ is cosymplectic we obtain that

$$\phi \overline{\nabla}_{PX} X - \overline{\nabla}_{PX} FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.7)$$

Then from (2.5) and (2.6), we get

$$\phi \nabla_{PX} X + \phi h(X, PX) + A_{FX} PX - \nabla_{PX}^\perp FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.8)$$

Thus by (2.8) and (2.12), we obtain

$$P \nabla_{PX} X + F \nabla_{PX} X + g(PX, X) \phi H + A_{FX} PX - \nabla_{PX}^\perp FX - \nabla_{PX} PX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.9)$$

Equating the normal components, we get

$$F \nabla_{PX} X - \nabla_{PX}^\perp FX = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} H. \quad (3.10)$$

On the other hand, from (3.4), we have

$$g(FX, FX) = \sin^2 \theta \{ g(X, X) - \eta(X) \eta(X) \}, \quad (3.11)$$

for any $X \in \Gamma(TM)$. Taking the covariant derivative of the above equation with respect to PX , we obtain

$$2g(\overline{\nabla}_{PX} FX, FX) = 2\sin^2 \theta g(\overline{\nabla}_{PX} X, X) - 2\sin^2 \theta \eta(X) g(\overline{\nabla}_{PX} X, \xi) - 2\sin^2 \theta \eta(X) g(X, \overline{\nabla}_{PX} \xi). \quad (3.12)$$

Using the property of metric connection $\overline{\nabla}$, the last two terms of the right-hand side are cancelling each other, thus we have

$$g(\overline{\nabla}_{PX} FX, FX) = \sin^2 \theta g(\overline{\nabla}_{PX} X, X). \quad (3.13)$$

Then by (2.5) and (2.6), we derive

$$g(\nabla_{PX}^\perp FX, FX) = \sin^2 \theta g(\nabla_{PX} X, X). \quad (3.14)$$

Now, taking the inner product in (3.10) with FX , for any $X \in \Gamma(TM)$, then

$$g(F \nabla_{PX} X, FX) - g(\nabla_{PX}^\perp FX, FX) = \cos^2 \theta \left\{ \|X\|^2 - \eta^2(X) \right\} g(H, FX). \quad (3.15)$$

Then from (3.4) and (3.14), we obtain

$$-\sin^2\theta\eta(X)\eta(\nabla_{PX}X) = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}g(H, FX), \quad (3.16)$$

or

$$-\sin^2\theta\eta(X)g(\nabla_{PX}X, \xi) = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}g(H, FX). \quad (3.17)$$

Using (2.5), we derive

$$-\sin^2\theta\eta(X)g(\bar{\nabla}_{PX}X, \xi) = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}g(H, FX). \quad (3.18)$$

Since $\bar{\nabla}$ is the metric connection, then the above equation can be written as

$$\sin^2\theta\eta(X)g(X, \bar{\nabla}_{PX}\xi) = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}g(H, FX). \quad (3.19)$$

As $\bar{M}$ is cosymplectic thus using the fact that $\bar{\nabla}_{PX}\xi = 0$, the left hand side of the above equation vanishes identically, then

$$\cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}g(H, FX) = 0. \quad (3.20)$$

Thus from (3.20), it follows that either $\theta = \pi/2$ or $X = \xi$ or $H \in \Gamma(\mu)$, where μ is the invariant normal subbundle orthogonal to FTM . This completes the proof. $\square$

Theorem 3.3. *Every totally umbilical proper slant submanifold M of a cosymplectic manifold $\bar{M}$ is totally geodesic, provided $\nabla_X^\perp H \in \Gamma(\mu)$, for any $X \in TM$.*

Proof. As $\bar{M}$ is cosymplectic, then we have

$$\bar{\nabla}_U\phi V = \phi\bar{\nabla}_U V, \quad (3.21)$$

for any $U, V \in \Gamma(T\bar{M})$. Using this fact and formulae (2.5) and (2.8) we obtain that

$$\bar{\nabla}_X PY + \bar{\nabla}_X FY = P\nabla_X Y + F\nabla_X Y + \phi h(X, Y), \quad (3.22)$$

for any $X, Y \in \Gamma(TM)$. Then from (2.5), (2.6) and (2.12), we get

$$\nabla_X PY + h(X, PY) - A_{FY}X + \nabla_X^\perp FY = P\nabla_X Y + F\nabla_X Y + g(X, Y)\phi H. \quad (3.23)$$

Taking the inner product in (3.23) with ϕH and using the fact that $H \in \Gamma(\mu)$ (by Theorem 3.2), we obtain

$$g(h(X, PY), \phi H) + g(\nabla_X^\perp FY, \phi H) = g(X, Y)g(\phi H, \phi H). \quad (3.24)$$

Then from (2.2) and (2.12), we derive

$$g(X, PY)g(H, \phi H) + g(\nabla_X^\perp FY, \phi H) = g(X, Y)g(H, H). \quad (3.25)$$

That is,

$$g(\nabla_X^\perp FY, \phi H) = g(X, Y)\|H\|^2. \quad (3.26)$$

Now, we consider

$$\bar{\nabla}_X \phi H = \phi \bar{\nabla}_X H, \quad (3.27)$$

for any $X \in \Gamma(TM)$. From (2.6), we obtain

$$-A_{\phi H}X + \nabla_X^\perp \phi H = \phi(-A_HX + \nabla_X^\perp H). \quad (3.28)$$

Thus, on using (2.8), (2.9), we get

$$-A_{\phi H}X + \nabla_X^\perp \phi H = -PA_HX - FA_HX + t\nabla_X^\perp H + f\nabla_X^\perp H. \quad (3.29)$$

Taking the inner product with FY , for any $Y \in \Gamma(TM)$, then

$$g(\nabla_X^\perp \phi H, FY) = -g(FA_HX, FY) + g(f\nabla_X^\perp H, FY). \quad (3.30)$$

Since $f\nabla_X^\perp H \in \Gamma(\mu)$, then by (3.4) the above equation takes the form

$$g(\nabla_X^\perp \phi H, FY) = -\sin^2\theta\{g(A_HX, Y) - \eta(A_HX)\eta(Y)\}. \quad (3.31)$$

Using (2.6), (2.7), and (2.12), we get

$$g(\bar{\nabla}_X \phi H, FY) = -\sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}\|H\|^2. \quad (3.32)$$

The above equation can be written as

$$g(\bar{\nabla}_X FY, \phi H) = \sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}\|H\|^2. \quad (3.33)$$

Again using the fact that $H \in \Gamma(\mu)$, then by (2.6), we obtain

$$g(\nabla_X^\perp FY, \phi H) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\} \|H\|^2. \quad (3.34)$$

From (3.26) and (3.34), we derive

$$\{\cos^2 \theta g(X, Y) + \sin^2 \theta \eta(X)\eta(Y)\} \|H\|^2 = 0. \quad (3.35)$$

Thus, (3.35) implies either $H = 0$ or $\theta = \tan^{-1}(\sqrt{-g(X, Y)/\eta(X)\eta(Y)})$, which is not possible, because the slant angle $\theta \in (0, \pi/2)$. Hence, M is totally geodesic in $\overline{M}$. $\square$

Acknowledgment

The authors are grateful to the anonymous referee for his valuable comments and the first author is supported by the research Grant RG117/10AFR (University of Malaya).

References

- [1] B. Y. Chen, "Slant immersions," *Bulletin of the Australian Mathematical Society*, vol. 41, no. 1, pp. 135–147, 1990.
- [2] B.-Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, 1990.
- [3] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández, "Slant submanifolds in Sasakian manifolds," *Glasgow Mathematical Journal*, vol. 42, no. 1, pp. 125–138, 2000.
- [4] A. Lotta, "Slant submanifolds in contact geometry," *Bulletin Mathematical Society Roumanie*, vol. 39, pp. 183–198, 1996.
- [5] B. Şahin, "Every totally umbilical proper slant submanifold of a Kahler manifold is totally geodesic," *Results in Mathematics*, vol. 54, no. 1-2, pp. 167–172, 2009.
- [6] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Springer-Verlag, New York, NY, USA, 1976.
- [7] G. D. Ludden, "Submanifolds of cosymplectic manifolds," *Journal of Differential Geometry*, vol. 4, pp. 237–244, 1970.
- [8] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, World Scientific Publishing, Singapore, Singapore, 1984.