Research Article

Variational Homotopy Perturbation Method for Solving Fractional Initial Boundary Value Problems

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A variational homotopy perturbation method (VHPM) which is based on variational iteration method and homotopy perturbation method is applied to solve the approximate solution of the fractional initial boundary value problems. The nonlinear terms can be easily handled by the use of He's polynomials. It is observed that the variational iteration method is very efficient and easier to implement; illustrative examples are included to demonstrate the high accuracy and fast convergence of this new algorithm.

1. Introduction

Recently, it has turned out that many phenomena in engineering, physics, chemistry, and other sciences [1–3] can be described very successfully by models using mathematical tools for fractional calculus. The importance of obtaining the exact and approximate solutions of fractional nonlinear equations in physics and mathematics is still a significant problem that needs new methods to discover exact and approximate solutions. But these nonlinear fractional differential equations are difficult to get their exact solutions [4–7]. So, some semi-analytical techniques have also largely been used to solve these equations, such as, Adomian decomposition method [8, 9], variational iteration method [10–12], differential transform method [13, 14], Laplace decomposition method [15, 16], and homotopy perturbation method [17–23]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian polynomials, the Lagrange multiplier, divergent results, and huge computational work.

Variational homotopy perturbation method [24–26] has a very simple solution procedure and absorbs all of the positive features of variational iteration and homotopy

perturbation methods and is highly compatible with the diversity of the physical problems. In this work, we will use variational homotopy perturbation method to solve fractional partial differential equations with initial and boundary conditions. The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. This paper considers the effectiveness of the variational homotopy perturbation method in solving fractional partial equations.

2. Description of the VHPM

To illustrate the basic idea of this method [24, 25], we consider a general fractional nonlinear nonhomogeneous partial differential equation with initial conditions of the following form:

$$\begin{aligned} D_t^\alpha u(x, t) + Ru(x, t) + Nu(x, t) &= g(x, t), \\ u(x, 0) &= h(x), \quad u_t(x, 0) = f(x), \end{aligned} \quad (2.1)$$

where $g(x, t)$ is the inhomogeneous term, N represents the general nonlinear differential operator, R is the linear differential operator, and $D_t^\alpha u(x, t)$ is the Caputo fractional derivative of function $u(x, t)$ which is defined as

$${}_0D_t^\alpha u(x, t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{u^{(n)}(x, \tau) d\tau}{(t - \tau)^{\alpha+1-n}}, \quad (n - 1 < \text{Re}(\alpha) \leq n, n \in \mathbb{N}), \quad (2.2)$$

where $\Gamma(\cdot)$ denotes the Gamma function. The properties of fractional derivative can be found in [1, 2]. According to variational iteration method [10, 11], we can construct a correction functional as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\xi) \left(\frac{\partial^m u_n}{\partial \xi^m}(x, \xi) + Ru_n(x, \xi) + N\tilde{u}_n(x, \xi) - g(x, \xi) \right) d\xi, \quad (2.3)$$

where the values of the natural number m can be 1 and 2 corresponding to $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$, respectively, and λ is a Lagrange multiplier [11], which can be identified optimally via variational iteration method, for $m = 1$ and $\lambda = -1$ and for $m = 2$ and $\lambda = \xi - t$. The subscripts n denote the n th approximation, and $\tilde{u}_n$ is considered as a restricted variation. That is, $\tilde{u}_n \delta = 0$, (2.3) is called a correction functional. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation $u_{n+1}, n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0 ; consequently, the solution is given by $u = \lim_{n \rightarrow \infty} u_n$. In the homotopy perturbation method, the basic assumption is that the solutions can be written as a power series in p :

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \quad (2.4)$$

and the nonlinear term can be decomposed as

$$Nu(x, t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u), \quad (2.5)$$

where $p \in [0, 1]$ is an embedding parameter. $\mathcal{H}_n(u)$ is He's polynomials [26] that can be generated by

$$\mathcal{H}_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots \quad (2.6)$$

The variational homotopy perturbation method is obtained by the elegant coupling of correction functional (2.3) of variational iteration method with He's polynomials and is given by

$$\begin{aligned} & \sum_{n=0}^{\infty} p^n u_n(x, t) \\ &= u_0(x, t) + p \int_0^t \lambda(\xi) \left(\sum_{n=0}^{\infty} p^n \frac{\partial^\alpha u_n}{\partial \xi^\alpha}(x, \xi) + \sum_{n=0}^{\infty} p^n Ru(x, \xi) + \sum_{n=0}^{\infty} p^n \mathcal{H}_n(u) - g(x, \xi) \right) d\xi. \end{aligned} \quad (2.7)$$

Comparisons of like powers of p give solutions of various orders.

This method does not resort to linearization or assumptions of weak nonlinearity, the solutions generated in the form of general solution, and it is more realistic compared to the method of simplifying the physical problems.

3. Approximate Solutions of Fractional Equations

In order to assess the advantages and the accuracy of the variational homotopy perturbation method for fractional equations, we have applied it to the following several problems.

Case 1. Consider the one-dimensional fractional initial boundary value problem which describes the heat-like models [24]:

$$D_t^\alpha u(x, t) = \frac{1}{2} x^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad (3.1)$$

with boundary conditions

$$\begin{aligned} u(0, t) &= 0, & u(1, t) &= e^t, \\ u(x, 0) &= x^2, \end{aligned} \quad (3.2)$$

where $0 < \alpha \leq 1$, and the correct functional is given as

$$u_{n+1}(x, t) = x^2 - \int_0^t \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha}(x, \xi) - \frac{1}{2} x^2 u_{nxx} \right) d\xi. \quad (3.3)$$

Applying the modified variational iteration method, one has

$$\begin{aligned} \sum_{n=0}^{\infty} p^n u_n(x, t) = x^2 - p \int_0^t & \left(\left(\frac{\partial^\alpha u_0}{\partial \xi^\alpha} + p \frac{\partial^\alpha u_1}{\partial \xi^\alpha} + p^2 \frac{\partial^\alpha u_2}{\partial \xi^\alpha} + \cdots \right) \right. \\ & \left. - \frac{1}{2} \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_2}{\partial x^2} + \cdots \right) \right) d\xi. \end{aligned} \quad (3.4)$$

Comparing the coefficient of like powers of p ,

$$\begin{aligned} p^0 : u_0(x, t) &= x^2, \\ p^1 : u_1(x, t) &= x^2(1 + t), \\ p^2 : u_2(x, t) &= x^2(1 + t) + tx^2 + \frac{t^2 x^2}{2} - \frac{t^{2-\alpha} x^2}{\Gamma(3-\alpha)}, \\ p^3 : u_2(x, t) &= x^2 + 3tx^2 + \frac{3t^2 x^2}{2} + \frac{t^3 x^2}{6} + \frac{t^{3-2\alpha} x^2}{\Gamma(4-2\alpha)} - \frac{3t^{2-\alpha} x^2}{\Gamma(3-\alpha)} - \frac{2t^{3-\beta} x^2}{\Gamma(4-\beta)}, \\ &\vdots \end{aligned} \quad (3.5)$$

and so on, in this manner the rest of component of the solution can be obtained. If we take $\alpha = 1$, the first few components of the solution of (3.1) are as follows:

$$\begin{aligned} u_0(x, t) &= x^2, \\ u_1(x, t) &= x^2(1 + t), \\ p^2 : u_2(x, t) &= x^2 \left(1 + t + \frac{t^2}{2!} \right), \\ p^3 : u_2(x, t) &= x^2 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \right), \\ &\vdots \end{aligned} \quad (3.6)$$

The solution of (3.1) in series form is given by

$$u(x, t) = x^2 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right). \quad (3.7)$$

The series solution in a closed form is given by $u(x, t) = x^2 e^t$, which was given in [24].

Case 2. Consider the following homogeneous fractional coupled Burger's equation [26]:

$$D_t^\alpha u - u_{xx} - 2uu_x + (uv)_x = 0, \quad (3.8)$$

$$D_t^\beta v - v_{xx} - 2vv_x + (uv)_x = 0, \quad (3.9)$$

with initial conditions

$$u(x, 0) = \sin x, \quad (3.10)$$

$$v(x, 0) = \sin x, \quad (3.11)$$

where $0 < \alpha, \beta \leq 1$. The correction functional for the previous fractional coupled system is given by

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left(\frac{\partial^\alpha u_n}{\partial \xi^\alpha} - \frac{\partial^2 u_n}{\partial x^2} - 2u_n(u_n)_x + (u_n v_n)_x \right) d\xi, \quad (3.12)$$

$$v_{n+1}(x, t) = v_n(x, t) - \int_0^t \left(\frac{\partial^\beta v_n}{\partial \xi^\beta} - \frac{\partial^2 v_n}{\partial x^2} - 2v_n(v_n)_x + (u_n v_n)_x \right) d\xi. \quad (3.13)$$

Applying the variational homotopy perturbation method using He's polynomials, we get

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \cdots &= u_0(x, t) - p \int_0^t \left(\left(\frac{\partial^\alpha u_0}{\partial \xi^\alpha} + p \frac{\partial^\alpha u_1}{\partial \xi^\alpha} + \cdots \right) - \left(\frac{\partial^2 u_0}{\partial x^2} \right) + p \frac{\partial^2 u_1}{\partial x^2} + \cdots \right) d\xi, \\ &+ p \int_0^t \left(u_0 u_{0x} + p(u_0 u_{1x} + u_1 u_{0x}) + p^2(u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} + \cdots) \right) \\ &- \left((u_0 v_0)_{xx} + p(u_1 v_0 + u_0 v_1)_{xx} + p^2(u_0 v_2 + u_1 v_1 + u_0 v_2)_{xx} \right) d\xi, \\ v_0 + pv_1 + p^2v_2 + \cdots &= v_0(x, t) - p \int_0^t \left(\left(\frac{\partial^\beta v_0}{\partial \xi^\beta} + p \frac{\partial^\beta v_1}{\partial \xi^\beta} + \cdots \right) - \left(\frac{\partial^2 v_0}{\partial x^2} \right) + p \frac{\partial^2 v_1}{\partial x^2} + \cdots \right) d\xi \\ &+ p \int_0^t \left(v_0 v_{0x} + p(v_0 v_{1x} + v_1 v_{0x}) + p^2(v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x} + \cdots) \right) \\ &- \left((u_0 v_0)_{xx} + p(u_1 v_0 + u_0 v_1)_{xx} + p^2(u_0 v_2 + u_1 v_1 + u_0 v_2)_{xx} \right) d\xi. \end{aligned} \quad (3.14)$$

Comparing the coefficient of like powers of p , one has

$$u_0(x, 0) = \sin x,$$

$$v_0(x, 0) = \sin x,$$

$$u_1(x, 0) = \sin x - t \sin x,$$

$$\begin{aligned}
v_1(x, 0) &= \sin x - t \sin x, \\
u_2(x, 0) &= \sin x - 2t \sin x + \frac{t^2 \sin x}{2} + \frac{t^{2-\alpha} \sin x}{\Gamma(3-\alpha)}, \\
v_2(x, 0) &= \sin x - 2t \sin x + \frac{t^2 \sin x}{2} + \frac{t^{2-\beta} \sin x}{\Gamma(3-\beta)}, \\
u_3(x, 0) &= \sin x - 3t \sin x + \frac{3t^2 \sin x}{2} - \frac{t^3 \sin x}{6} + \frac{t^{3-2\alpha} \sin x}{\Gamma(4-2\alpha)} + \frac{3t^{2-\alpha} \sin x}{\Gamma(3-\alpha)} - \frac{2t^{3-\alpha} \sin x}{\Gamma(4-\alpha)}, \\
v_3(x, 0) &= \sin x - 3t \sin x + \frac{3t^2 \sin x}{2} - \frac{t^3 \sin x}{6} + \frac{t^{3-2\beta} \sin x}{\Gamma(4-2\beta)} + \frac{3t^{2-\beta} \sin x}{\Gamma(3-\beta)} - \frac{2t^{3-\beta} \sin x}{\Gamma(4-\beta)}.
\end{aligned} \tag{3.15}$$

If we take $\alpha = \beta = 1$, the series solutions in closed-form are given by

$$\begin{aligned}
u(x, t) &= \sin x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \right) = \exp(-t) \sin x, \\
v(x, t) &= \sin x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \right) = \exp(-t) \sin x,
\end{aligned} \tag{3.16}$$

which is in full agreement with the result given in [26].

Figure 1 depicts the third-order approximate solutions $u(x, t)$ or $v(x, t)$ for (3.8)–(3.11) by using the variational homotopy perturbation method when choosing $x = 0.8$. From the figure, it is clear to see the time evolution of fractional Burger equation and we know that the approximate solution of the model is continuous with the fractional parameter α . Figure 2 shows the third-order approximate solutions $u(x, t)$ or $v(x, t)$ for (3.8)–(3.11) when $t = 0.8$, and we also know that the solution of the fractional nonlinear equation changes with the parameters α .

Case 3. In this case, we consider the space fractional backward Kolmogorov equation as follows [27]:

$$\frac{\partial u(x, t)}{\partial t} = \frac{(x+1)\partial^\beta u}{\partial x^\beta} + \frac{(x^2 e^t)\partial^{2\beta} u}{\partial x^{2\beta}}, \tag{3.17}$$

subject to the initial condition:

$$u(x, 0) = x + 1, \tag{3.18}$$

where $0 < \beta \leq 1$. Space fractional derivative is also the Caputo definition with respect to x and is defined as

$$D_x^\beta u(x, t) = \frac{1}{\Gamma(m-\beta)} \int_0^x \frac{u^{(m)}(\xi, t) d\xi}{(x-\xi)^{\beta+1-m}} \quad (m-1 < \text{Re}(\beta) \leq m, m \in \mathbb{N}). \tag{3.19}$$

Figure 1: The surface of third-order approximate solution $u(x, t)$ of (3.8) and (3.9) when $x = 0.8$.

Figure 2: The surface of third-order approximate solution $u(x, t)$ of (3.8) and (3.9) when $t = 0.8$.

The correction functional for the previous space fractional system is given by

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n(x, \xi)}{\partial \xi} - \frac{(x+1)\partial^\beta u_n}{\partial x^\beta} - \frac{(x^2 e^\xi)\partial^{2\beta} u_n}{\partial x^{2\beta}} \right) d\xi. \quad (3.20)$$

Applying the modified variational iteration method, one has

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = x + 1 - \int_0^t \left(\frac{\partial \sum_{n=0}^{\infty} p^n u_n(x, t)}{\partial \xi} - \frac{(x+1)\partial^\beta \sum_{n=0}^{\infty} p^n u_n(x, t)}{\partial x^\beta} - \frac{(x^2 e^\xi)\partial^{2\beta} \sum_{n=0}^{\infty} p^n u_n(x, t)}{\partial x^{2\beta}} \right) d\xi. \quad (3.21)$$

Comparing the coefficient of like powders of P , we can obtain the following approximations:

$$\begin{aligned}
u_0 &= x + 1, \\
u_1 &= 1 + x - \frac{2x^{3-2\beta}}{\Gamma(2-2\beta)} + \frac{2e^t x^{3-2\beta}}{\Gamma(2-2\beta)} + \frac{tx^{1-\beta}}{\Gamma(2-\beta)} + \frac{tx^{2-\beta}}{\Gamma(2-\beta)}, \\
u_2 &= 1 + x + \frac{6x^{5-4\beta}}{\Gamma(4-4\beta)} - \frac{12e^t x^{5-4\beta}}{\Gamma(4-4\beta)} + \frac{6e^t x^{5-4\beta}}{\Gamma(4-4\beta)} - \frac{10x^{5-4\beta}\beta}{\Gamma(4-4\beta)} + \frac{20e^t x^{5-4\beta}\beta}{\Gamma(4-4\beta)} \\
&\quad - \frac{10e^{2t} x^{5-4\beta}\beta}{\Gamma(4-4\beta)} + \frac{4x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)} - \frac{8e^t x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)} + \frac{4e^{2t} x^{5-4\beta}\beta^2}{\Gamma(4-4\beta)} + \frac{x^{3-3\beta}\beta}{\Gamma(2-3\beta)} - \frac{e^t x^{3-3\beta}}{\Gamma(2-3\beta)} \\
&\quad + \frac{e^t tx^{3-3\beta}}{\Gamma(2-3\beta)} + \frac{2x^{4-3\beta}\beta}{\Gamma(3-3\beta)} - \frac{2e^t x^{4-3\beta}}{\Gamma(3-3\beta)} + \frac{2e^t tx^{4-3\beta}}{\Gamma(3-3\beta)} - \frac{x^{4-3\beta}\beta}{\Gamma(3-3\beta)} + \frac{e^t x^{4-3\beta}\beta}{\Gamma(3-3\beta)} \\
&\quad + \frac{tx^{2-\beta}}{\Gamma(2-\beta)} - \frac{e^t tx^{4-3\beta}\beta}{\Gamma(3-3\beta)} + \frac{t^2 x^{1-2\beta}}{2\Gamma(2-2\beta)} + \frac{t^2 x^{2-2\beta}}{2\Gamma(2-2\beta)} - \frac{x^{3-2\beta}}{\Gamma(2-2\beta)} + \frac{e^t x^{3-2\beta}}{\Gamma(2-2\beta)} \\
&\quad + \frac{tx^{1-\beta}}{\Gamma(2-\beta)} + \frac{2e^t x^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} - \frac{2x^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} - \frac{2tx^{3-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} \\
&\quad - \frac{2x^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} + \frac{2e^t x^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} - \frac{2tx^{4-3\beta}\Gamma(4-2\beta)}{\Gamma(4-3\beta)\Gamma(2-2\beta)} \\
&\quad + \frac{t^2 x^{2-2\beta}\Gamma(3-\beta)}{2\Gamma(3-2\beta)} + \frac{t^2 x^{3-2\beta}\Gamma(3-\beta)}{2\Gamma(3-2\beta)},
\end{aligned} \tag{3.22}$$

and so on; in the same manner the rest of components of the iteration formula (3.21) can be obtained using the Mathematica package. When fractional derivatives are $\beta = 1$, the exact solution of (3.17) was given in [27] using homotopy perturbation method and the approximate solution of (3.17) is

$$\begin{aligned}
u_0 &= x + 1, \\
u_1 &= (x + 1)(1 + t), \\
u_2 &= (x + 1) \left(1 + t + \frac{t^2}{2!} \right), \\
&\vdots
\end{aligned} \tag{3.23}$$

and so on. Hence, we have the closed form

$$u(x, t) = (x + 1) \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) = \exp(t)(x + 1), \tag{3.24}$$

which is the exact solution of the corresponding integer problem.

4. Conclusion

In this work, a variational homotopy perturbation method which is based on homotopy perturbation method and variational iteration method is used to solve fractional partial equations. The nonlinear terms can be easily handled by the use of He's polynomials. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the result, and the size reduction amounts to an improvement of the performance of the approach. The VHPM can be applied for some other engineering system with less computational work.

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