Research Article

Monostable-Type Travelling Wave Solutions of the Diffusive FitzHugh-Nagumo-Type System in $\mathbf{R}^N$

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Received 30 March 2012; Accepted 25 May 2012

Academic Editor: Norimichi Hirano

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This paper is concerned with monostable-type travelling wave solutions of the diffusive FitzHugh-Nagumo-type system (FHN) in $\mathbf{R}^N$ for the two components u and v . By solving v in terms of u , this system can be reduced to a nonlocal single equation for u . When the diffusion coefficients in the system are equal, we construct travelling wave solutions for the non-local equation by the method of super- and subsolutions developed by Morita and Ninomiya (2008). Moreover, we propose a condition for γ , which is similar to the condition Reinecke and Sweers (1999) used to transform (FHN) into a quasimonotone system.

1. Introduction

In the present paper, we are concerned with the diffusive FitzHugh-Nagumo-type system (FHN) in $\mathbf{R}^N$ that is,

$$\begin{aligned} u_t &= u_{\xi\xi} + \Delta_y u + f(y, u) - v, \\ v_t &= dv_{\xi\xi} + \Delta_y v + \delta(u - \gamma v), \end{aligned} \quad (1.1)$$

where $(\xi, y) \in \mathbf{R}^N = \mathbf{R}^1 \times \mathbf{R}^{N-1}$, $N \geq 2$, $\delta, \gamma > 0$ and $d \geq 0$. A typical example of $f(y, u)$ is $f(y, u) = u(1 - u)(u - \beta)$ for $0 < \beta < 1/2$. Throughout the paper we assume that f is a C^2 function in u and f, f_u , and f_{uu} are bounded in $\{(y, u) \mid y \in \Omega_y, |u| \leq K\}$ for some large constant $K > 0$. In addition, f satisfies (H1)–(H5).

FHN derived from the Hodgkin-Huxley model is a typical model for excitable media. In many fields, such as physics, chemistry, and biology, FHN has become one of the frequently

used-reaction diffusion systems to describe interesting phenomena. The solutions of interest here are traveling wave solutions. Let $x = \xi - ct$, then travelling wave solutions of (1.1) satisfy

$$u_{xx} + cu_x + \Delta_y u + f(y, u) - v = 0, \quad (1.2)$$

$$dv_{xx} + cv_x + \Delta_y v + \delta(u - \gamma v) = 0. \quad (1.3)$$

Over the past decades, this system has been extensively studied. For instance, as $N = 1$, under different assumptions, Systems (1.2) and (1.3) admit standing pulses in [1–3], infinitely many periodic solutions in [3], fronts, back waves in [4, 5] and travelling pulses in [5]. For the higher dimension case $N \geq 2$, symmetric standing waves were established by Reinecke and Sweers [6] and Wei and Winter [7].

As $\gamma \rightarrow \infty$, if the solutions are assumed to be bounded, (1.2) and (1.3) tend to the single equation

$$u_{xx} + cu_x + \Delta_y u + f(y, u) = 0. \quad (1.4)$$

Let $f(y, u)$ be a C^2 function $g(u)$ which has the property that for some $\theta \in (0, 1)$ $g(0) = g(\theta) = g(1) = 0$, $g_u(0) < 0$, $g_u(\theta) > 0$, $g_u(1) < 0$, $g < 0$ on $(0, \theta)$ and $g > 0$ on $(\theta, 1)$. In addition to the planar waves, (1.4) admits other types of solutions, including travelling curved fronts ($N = 2$), conical shapes and pyramidal shapes ($N \geq 3$) in [8–11]. Moreover, Hamel and Roquejoffre [12] established travelling wave solutions of (1.4) in $\mathbf{R}^2$ which connect one unstable periodic solution at $x \rightarrow \infty$ ($-\infty$) and one stable constant solution at $x \rightarrow -\infty$ (∞). On the other hand, travelling wave solutions of (1.4) in $\mathbf{R}^N$ connecting a unstable one-peak solution at $x \rightarrow \infty$ ($-\infty$) and a stable constant solution $x \rightarrow -\infty$ (∞) were obtained by Morita and Ninomiya [13].

In this paper, we use the method of super- and subsolutions developed in [13]. Due to technical restriction, we assume $d = 1$. Since (1.3) is linear, v can be solved formally in terms of u . With v expressed in terms of u , Systems (1.2) and (1.3) are reduced to the non-local equation

$$\mathcal{F}[u] := u_{xx} + cu_x + \Delta_y u + f(y, u) - B_c[u] = 0, \quad (1.5)$$

where we denote v by $B_c[u] := \delta(-\partial^2/\partial x^2 - c(\partial/\partial x) - \Delta_y + \delta\gamma)^{-1}u$. It is readily seen that if u is independent of x , then by the uniqueness theorem $B_c[u] = \delta(-\Delta_y + \delta\gamma)^{-1}u$. As $x \rightarrow \pm\infty$, the asymptotic behaviors of travelling wave solutions of (1.5) formally satisfy

$$\Delta_y u + f(y, u) - B_c[u] = 0, \quad (1.6)$$

where $B_c[u] = \delta(-\Delta_y + \delta\gamma)^{-1}u$. Our main purpose is to look for monostable-type travelling wave solutions $u(x, y)$ which connect a stable solution of (1.6) as $x \rightarrow -\infty$ (∞) and a unstable one as $x \rightarrow \infty$ ($-\infty$). Without loss of generality, we may assume that $u(+\infty, y)$ is an unstable solution. Throughout this paper, the following hypotheses are assumed.

(H1) There are two solutions $u_{\pm}(y)$ of (1.6) satisfying $u_{-}(y) \geq u_{+}(y)$. Moreover, there exist an eigenvalue $\mu > 0$ and its corresponding eigenfunction $\phi(y) > 0$ with $\max_{\{y \in \mathbb{R}^{N-1}\}} \phi(y) = 1$ and $\lim_{|y| \rightarrow \infty} \phi(y) = 0$ such that

$$\Delta_y \phi + f_u(y, u_{+}) \phi - B_c[\phi] = \mu \phi. \quad (1.7)$$

(H2) There exists no other solution $u(y)$ of (1.6) with the property $u_{-}(y) \geq u(y) \geq u_{+}(y)$.

(H3) $u_{-}(y) \geq u_{+}(y) + \epsilon \phi(y)$ for some $\epsilon > 0$.

(H4) For all small $\eta > 0$, there exists solutions $u_{+}^{\eta}(y)$ satisfying $\lim_{\eta \rightarrow 0} u_{+}^{\eta}(y) = u_{+}(y)$,

$$\begin{aligned} \Delta_y u_{+}^{\eta} + f(y, u_{+}^{\eta}) - B_c[u_{+}^{\eta}] + \eta &= 0, \\ u_{+}^{\eta}(y) &\geq u_{+}(y) + \frac{\eta}{M}, \end{aligned} \quad (1.8)$$

for some constant $M > 0$.

(H5)

$$\Delta_y \psi_i - (K_1 + \sqrt{\delta}) \psi_i \leq 0, \quad i = 1, 2, 3, \quad (1.9)$$

where $K_1 = -\min_{\{u_{-}(y) \geq u \geq u_{+}(y), y \in \mathbb{R}^{N-1}\}} f_u(y, u) > 0$, $\psi_1 = \phi$, $\psi_2 = u_{-} - u_{+}$ and $\psi_3 = u_{+}^{\eta} - u_{+}$.

To simplify the proof of the main theorem in this paper, we modify the nonlinear term $f(y, u)$ such that the minimum and maximum of $f_u(y, u)$ in $\{u(y) \in \mathbb{R}, y \in \mathbb{R}^{N-1}\}$ are the same as those in $\{u_{-}(y) \geq u \geq u_{+}(y), y \in \mathbb{R}^{N-1}\}$. For convenience, we still denote $f(y, u)$ for the new modification of f . Let $K^{*} := \max_{\{u_{-}(y) \geq u \geq u_{+}(y), y \in \mathbb{R}^{N-1}\}} f_u(y, u) > 0$ and let $K_2 > 0$ satisfy $K_2 + \delta/(\delta\gamma + K_2) = K^{*}$. We state the main theorem as follows.

Theorem 1.1. Assume $\gamma \geq 2/\sqrt{\delta} + (K_1 + \mu)/\delta$ and (H1)–(H5) hold. Then there exists $c^{*} = \max\{2\sqrt{\mu}, 2\sqrt{K_2}\} > 0$ such that for all $c \geq c^{*}$, Systems (1.2) and (1.3) admit a pair of smooth solutions (u^{*}, v^{*}) which satisfy $u_x^{*} \leq 0$, $v_x^{*} \leq 0$ and the boundary conditions $(u^{*}, v^{*})(\pm\infty, y) = (u_{\pm}(y), v_{\pm}(y))$, where $v_{\pm}(y) = B_c[u_{\pm}(y)]$.

Remark 1.2. In (H1), when the inequality $u_{-}(y) \geq u_{+}(y)$ is reversed, that is, $u_{-}(y) \leq u_{+}(y)$, a result similar to Theorem 1.1 can be proved except that the inequalities $u_x^{*} \leq 0$ and $v_x^{*} \leq 0$ in Theorem 1.1 need to be replaced by $u_x^{*} \geq 0$ and $v_x^{*} \geq 0$, respectively.

Remark 1.3. In fact, (H5) can be weakened to the following assumption:

$$\Delta_y \psi_i - M_i \psi_i \leq 0, \quad \text{for some constants } M_i > 0. \quad (1.10)$$

This condition holds if $\Delta_y \psi_i$ does not decay faster than ψ_i as $|y| \rightarrow \infty$. In this case, if we choose $\gamma \geq 1/\sqrt{\delta} + (K_3 + \mu)/\delta$, where $K_3 = \max\{M_1, M_2, M_3, K_1 + \sqrt{\delta}\}$, then a similar result can be proved.

It is not easy to find an example which satisfies assumptions (H1)–(H5) even for the case $f(y, u) = u(1 - u)(u - \beta)$ since the stability of the radially symmetric solutions obtained in [6, 7] has not yet been studied. However, we believe that for $\gamma \gg 1$ the structure of System (1.2) and (1.3) are similar to that of (1.5). Accordingly, we extend the result of Theorem 2.1 in [13] to the one in Theorem 1.1.

2. Proof of the Main Theorem

To prove the Theorem 1.1, we use the super- and subsolutions constructed in [13]. By considering the following equation, we construct subsolutions of $\mathcal{F}[u]$. Let $w(x)$ satisfy

$$\begin{aligned} w_{xx} + cw_x + \mu w - w^2 &= 0, \\ w(-\infty) &= \mu, \quad w(\infty) = 0. \end{aligned} \quad (2.1)$$

For all $c \geq 2\sqrt{\mu}$, the above boundary value problem admits a unique solution $w(x)$ (up to a translation) which is strictly increasing in x . Subsolutions of $\mathcal{F}[u]$ are established as follows.

Lemma 2.1. *Let $\underline{U}(x, y) = u_+(y) + \sigma\phi(y)w(x)$. Then there exists $\sigma_1 > 0$ such that $\mathcal{F}[\underline{U}] \geq 0$ for all $0 < \sigma \leq \sigma_1$ and $c \geq 2\sqrt{\mu}$.*

Proof. Let $V := wB_c[\phi] - B_c[\phi w] \geq 0$, then $V \geq 0$. Indeed, it is easy to see that $B_c[\phi] \geq 0$ by the maximum principle and $\phi > 0$. A straightforward calculation gives

$$V_{xx} + cV_x + \Delta_y V - \delta\gamma V = -w(\mu - w)B_c[\phi] \leq 0. \quad (2.2)$$

Using the maximum principle, we obtain $V \geq 0$. Therefore by (H1)

$$\begin{aligned} \mathcal{F}[\underline{U}] &= \sigma\phi(w_{xx} + cw_x) + (\Delta_y u_+ - B_c[u_+]) + \sigma w\Delta_y \phi + f(y, u_+ + \sigma\phi w) - \sigma B_c[\phi w] \\ &= \sigma\phi(w_{xx} + cw_x + \mu w) + f(y, u_+ + \sigma\phi w) - f(y, u_+) - f_u(y, u_+)\sigma\phi w + \sigma V \\ &\geq \sigma\phi w^2 + G, \end{aligned} \quad (2.3)$$

where $G = f(y, u_+ + \sigma\phi w) - f(y, u_+) - f_u(y, u_+)\sigma\phi w$.

Let $M_1 = \min_{\{u_-(y) \geq u_+(y), y \in \mathbb{R}^{N-1}\}} f_{uu}(y, u)$. By choosing $\sigma \leq \epsilon/\mu$ and using (H3), we obtain $u_+ \leq u_+ + \sigma\phi w \leq u_+ + \epsilon\phi \leq u_-$. According to the mean value theorem, we have $G \geq 0$ if $M_1 \geq 0$ and $G \geq M_1\sigma^2\phi^2w^2$ if $M_1 < 0$. Therefore $\mathcal{F}[\underline{U}] \geq 0$ if $\sigma \leq \sigma_1$, where $\sigma_1 = \epsilon/\mu$ as $M_1 \geq 0$ and $\sigma_1 = \min\{\epsilon/\mu, -1/M_1\}$ as $M_1 < 0$. The proof is completed. $\square$

In what follows we construct supersolutions of $\mathcal{F}[u]$.

Lemma 2.2. *Let $Q(x) = e^{-((c-\sqrt{(c^2-4K_2)/2}))x}$ and $U^+(x, y) = u_+^{\eta}(y) + Q(x)$, where $K_2 > 0$ satisfies $K_2 + \delta/(\delta\gamma + K_2) = K^*$ and $c \geq 2\sqrt{K_2}$. Then $\mathcal{F}[U^+] < 0$.*

Proof. Note that $Q_{xx} + cQ_x + K_2Q = 0$ and $0 < B_c[Q] < \infty$. Indeed, by the uniqueness theorem we have $B_c[Q(x)] = \delta(-\partial^2/\partial x^2 - c(\partial/\partial x) + \delta\gamma)^{-1}Q$ and

$$B_c[Q] = \frac{\delta}{\sqrt{c^2 + 4\gamma\delta}} \int_{-\infty}^{+\infty} e^{-(\sqrt{c^2 + 4\gamma\delta}/2)|x-\xi| + (c/2)(\xi-x)} Q(\xi) d\xi = \frac{\delta}{\delta\gamma + K_2} Q(x). \quad (2.4)$$

It follows from (H4) that

$$\begin{aligned} \mathcal{F}[U^+] &= (Q_{xx} + cQ_x) + \left(\Delta_y u_+^\eta - B_c[u_+^\eta] \right) + f(y, u_+^\eta + Q) - B_c[Q] \\ &= -K_2Q + f(y, u_+^\eta + Q) - f(y, u_+^\eta) - \eta - B_c[Q] \\ &= \left\{ -K_2 + f_u(y, u_+^\eta + \theta Q) - \frac{\delta}{\delta\gamma + K_2} \right\} Q - \eta \leq -\eta < 0, \end{aligned} \quad (2.5)$$

where $0 \leq \theta \leq 1$. The last second inequality is due to

$$K_2 + \frac{\delta}{\delta\gamma + K_2} = \max_{\{u_-(y) \geq u_+(y), y \in \mathbb{R}^{N-1}\}} f_u(y, u). \quad (2.6)$$

We complete the proof of the lemma. $\square$

Let

$$\mathcal{L}[u] = u_{xx} + cu_x + \Delta_y u - (K_1 + \mu + \sqrt{\delta})u, \quad (2.7)$$

where $K_1 = -\min_{\{u_-(y) \geq u_+(y), y \in \mathbb{R}^{N-1}\}} f_u(y, u) > 0$.

To show the existences of travelling wave solutions of (1.6), we use the following iteration process:

$$\begin{aligned} u_n(x, y) &= \mathcal{L}^{-1} \left(-f(u_{n-1}) + B_c[u_{n-1}] - (K_1 + \mu + \sqrt{\delta})u_{n-1} \right), \quad n = 1, 2, \dots, \\ u_0(x, y) &= \underline{U}. \end{aligned} \quad (2.8)$$

In the following lemma, we assert that the supersolutions of $\mathcal{F}$ are greater than or equal to the subsolutions of $\mathcal{F}$. Moreover, we show that both $U^+ - \underline{U}$ and $u_- - \underline{U}$ are supersolutions of $\mathcal{L}$, which is useful in the proof of iteration process.

Lemma 2.3. Assume $\gamma \geq 2/\sqrt{\delta} + (K_1 + \mu)/\delta$ and let $\bar{U} := \min\{U^+(x, y), u_-(y)\}$. Then for all $\eta > 0$ there exists $\sigma_2 > 0$ depending on η such that for all $0 < \sigma \leq \sigma_2$ one has

$$\bar{U} \geq \underline{U}, \quad \mathcal{L}[U^+ - \underline{U}] \leq 0, \quad \mathcal{L}[u_- - \underline{U}] \leq 0. \quad (2.9)$$

Proof. For the case $\bar{U} = u_-(y)$ we take $\sigma \leq \epsilon/\mu$, then

$$\bar{U} - \underline{U} = u_-(y) - u_+(y) - \sigma\phi(y)w(x) \geq u_-(y) - u_+(y) - \epsilon\phi(y) \geq 0. \quad (2.10)$$

The last inequality holds by (H3). On the other hand,

$$\mathcal{L}[u_- - \underline{U}] = \Delta_y(u_- - u_+) - (K_1 + \mu + \sqrt{\delta})(u_- - u_+) + A, \quad (2.11)$$

where $A = -\sigma\phi(w_{xx} + cw_x) + (K_1 + \mu + \sqrt{\delta})\sigma\phi w - \sigma w\Delta_y\phi$. According to (H5), $|A| \leq \sigma C\phi$ for some positive constant $C = C(\mu, \delta, K_1)$. By choosing $\sigma \leq \epsilon\mu/C$, we obtain

$$\begin{aligned} \mathcal{L}[u_- - \underline{U}] &\leq \Delta_y(u_- - u_+) - (K_1 + \sqrt{\delta})(u_- - u_+) - \mu(u_- - u_+) + \sigma C\phi \\ &\leq -\epsilon\mu\phi + \sigma C\phi \leq 0, \end{aligned} \quad (2.12)$$

which holds due to assumptions (H3) and (H5).

For the case $\bar{U} = u_+^\eta(y) + Q(x)$, given $\eta > 0$ we choose $\sigma \leq \eta/\mu M$ and use assumption (H4), then

$$\bar{U} - \underline{U} = u_+^\eta(y) + Q(x) - u_+(y) - \sigma\phi(y)w(x) \geq \frac{\eta}{M} - \sigma\mu \geq 0. \quad (2.13)$$

Moreover,

$$\begin{aligned} \mathcal{L}[U^+ - \underline{U}] &= \Delta_y(u_+^\eta - u_+) - (K_1 + \mu + \sqrt{\delta})(u_+^\eta - u_+) + A + Q_{xx} + Q_x \\ &\quad - (K_1 + \mu + \sqrt{\delta})Q. \end{aligned} \quad (2.14)$$

It is readily seen that $Q_{xx} + Q_x - (K_1 + \mu + \sqrt{\delta})Q \leq 0$. By (H4) and (H5),

$$\mathcal{L}[U^+ - \underline{U}] \leq -\frac{\eta\mu}{M} + \sigma C \leq 0 \quad \text{if } \sigma \leq \frac{\eta\mu}{MC}. \quad (2.15)$$

Setting $\sigma_2 = \min\{\epsilon/\mu, \epsilon\mu/C, \eta/\mu M, \eta\mu/MC\}$, the lemma holds. $\square$

To generalize the result of Theorem 2.1 in [13], the nonlocal term of (1.5) needs to be better estimated. More precisely, we point wisely control $B_c[u]$ by the local term u such that the iterative sequence u_n is comparable with u_{n-1} .

Lemma 2.4. *Let $u \in C^2(\mathbf{R}^N)$ be nonnegative and solve $u_{xx} + cu_x + \Delta_y u - au \leq 0$ for some constant a . Assume $\gamma \geq a/\delta + 1/b$ for some b . Then $bu - B_c[u] \geq 0$.*

Proof. Let $v = B_c[u]$ and $U = bu - v$. Then $v \geq 0$ because of $u \geq 0$ and the maximum principle. Our main purpose is to claim $U \geq 0$. By the assumption of u and the definition of v , we have

$$U_{xx} + cU_x + \Delta_y U - \frac{ab + \delta}{b}U \leq -\left(\delta\gamma - a - \frac{\delta}{b}\right)v \leq 0. \quad (2.16)$$

The last inequality follows from the hypothesis of γ and the nonnegativity of v . By the maximum principle, $U \geq 0$. $\square$

As γ becomes large, we claim that the iterative sequence u_n is increasing.

Lemma 2.5. Assume $\gamma \geq 2/\sqrt{\delta} + (K_1 + \mu)/\delta$ and $c \geq c^* = \max\{2\sqrt{\mu}, 2\sqrt{K_2}\}$, then for all $\eta > 0$ and $0 < \sigma \leq \min\{\sigma_1, \sigma_2\}$ one has $u_{n,x} \leq 0$ and

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq \bar{U}. \quad (2.17)$$

Proof. We first claim that $u_n \leq \bar{U}$ for all n . Indeed, by Lemmas 2.3 and 2.4 (take $a = K_1 + \mu + \sqrt{\delta}$ and $b = \sqrt{\delta}$) we obtain

$$\sqrt{\delta}(U^+ - u_0) - B_c[U^+ - u_0] \geq 0. \quad (2.18)$$

Therefore Lemmas 2.2 and 2.3 yield

$$\begin{aligned} \mathcal{L}[U^+ - u_1] &\leq -f(U^+) + B_c[U^+] + f(u_0) - B_c[u_0] - (K_1 + \mu + \sqrt{\delta})(U^+ - u_0) \\ &\leq \{-f_u(\theta U^+(1 - \theta)u_0) - K_1\}(U^+ - u_0) \leq 0, \end{aligned} \quad (2.19)$$

where $0 \leq \theta \leq 1$. According to the maximum principle, $U^+ - u_1 \geq 0$. It follows from the proof of $U^+ - u_1 \geq 0$ that $u_- - u_1 \geq 0$. Therefore $u_1 \leq \bar{U}$. Continuing this process, we have $u_n \leq \bar{U}$ for all n by induction.

Next observe that $\mathcal{L}[u_1 - u_0] = -\mathcal{F}[\underline{U}] \leq 0$ due to Lemma 2.1. By the maximum principle, $u_1 - u_0 \geq 0$. Applying Lemma 2.4 to $u_1 - u_0$, we have

$$\sqrt{\delta}(u_1 - u_0) - B_c[u_1 - u_0] \geq 0. \quad (2.20)$$

Therefore

$$\begin{aligned} \mathcal{L}[u_2 - u_1] &= -(f(u_1) - f(u_0)) + B_c[u_1 - u_0] - (K_1 + \mu + \sqrt{\delta})(u_1 - u_0) \\ &\leq \{-f_u(\theta u_1 + (1 - \theta)u_0) - K_1\}(u_1 - u_0) - \sqrt{\delta}(u_1 - u_0) + B_c[u_1 - u_0] \\ &\leq 0, \end{aligned} \quad (2.21)$$

where $0 \leq \theta \leq 1$. Thus $u_2 \geq u_1$. By induction, the sequence of functions $\{u_n\}$ is nondecreasing. On the other hand, observe that $u_{0,x} = \sigma\phi w_x < 0$. Therefore by (H5), we obtain

$$\begin{aligned}\mathcal{L}[-u_{0,x}] &= \sigma\phi(\mu w_x - 2w w_x) - \sigma w_x \Delta_y \phi + (K_1 + \mu + \sqrt{\delta})\sigma\phi w_x \\ &= -\sigma w_x \left\{ \Delta_y \phi - (K_1 + \sqrt{\delta})\phi + (-2\mu + 2w)\phi \right\} \leq 0.\end{aligned}\tag{2.22}$$

Using Lemma 2.4 again, we have

$$\begin{aligned}\sqrt{\delta}(-u_{0,x}) - B_c[-u_{0,x}] &\geq 0, \\ \mathcal{L}[u_{1,x}] &= -f_u(u_0)u_{0,x} + B_c[u_{0,x}] - (K_1 + \mu + \sqrt{\delta})u_{0,x} \geq 0.\end{aligned}\tag{2.23}$$

Then $u_{1,x} \leq 0$ by the maximum principle. Inducting in n , we obtain $u_{n,x} \leq 0$. $\square$

Proof of Theorem 1.1. By Lemma 2.5, we define $u^*(x, y) = \lim_{n \rightarrow \infty} u_n(x, y)$. Following the proof of Theorem 2.1 in [13], (H2) and (H3), for all $c \geq c^*$ we obtain that $u^*(x, y)$ is a smooth solution of (1.5), $u_x^* \leq 0$ and $u^*(\pm\infty, y) = u_{\pm}(y)$. Let $v^* = B_c[u^*]$, then $v_x^* = B_c[u_x^*] \leq 0$ by the maximum principle. We complete the proof of the theorem. $\square$

Acknowledgments

C.-C. Huang wishes to express his sincere gratitude to Dr. Li-Chang Hung for his careful reading of the paper and valuable suggestions for improvement, and to his supervisor, Professor Chiun-Chuan Chen for his warm encouragement and invaluable advice throughout his Ph.D. program.

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