Research Article
Common Fixed Point Theorems for a Class of Twice Power Type Contraction Maps in G-Metric Spaces

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1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been in the center of rigorous research activity. In 2006, a new structure of generalized metric space was introduced by Mustafa and Sims [1] as an appropriate notion of generalized metric space called G-metric space. Abbas and Rhoades [2] initiated the study of common fixed point in generalized metric space. Recently, many fixed point theorems for certain contractive conditions have been established in G-metric spaces, and for more details one can refer to [3–27]. Fixed point problems have also been considered in partially ordered G-metric spaces [28–31], cone metric spaces [32], and generalized cone metric spaces [33].

In 2006, Gu and He [34] introduced a class of twice power type contractive condition in metric space, proving some common fixed point theorems for four self-maps with twice power type Φ-contractive condition.

In this paper, motivated and inspired by the above results, we introduce a new twice power type contractive condition in G-metric space, and we prove some new common fixed point theorems in complete G-metric spaces. Our results obtained in this paper differ from other comparable results already known.
Throughout the paper, we mean by \( \mathbb{N} \) the set of all natural numbers. Consistent with Mustafa and Sims [1], the following definitions and results will be needed in the sequel.

**Definition 1.1 (see [1]).** Let \( X \) be a nonempty set, and let \( G : X \times X \times X \to R^1 \) be a function satisfying the following axioms:

1. (G1) \( G(x, y, z) = 0 \) if \( x = y = z \);
2. (G2) \( 0 < G(x, x, y) \), for all \( x, y \in X \) with \( x \neq y \);
3. (G3) \( G(x, x, y) \leq G(x, y, z) \), for all \( x, y, z \in X \) with \( z \neq y \);
4. (G4) \( G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \) (symmetry in all three variables);
5. (G5) \( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \) (rectangle inequality);

then the function \( G \) is called a generalized metric, or, more specifically, a \( G \)-metric on \( X \) and the pair \( (X, G) \) are called a \( G \)-metric space.

**Definition 1.2 (see [1]).** Let \( (X, G) \) be a \( G \)-metric space, and let \( \{x_n\} \) be a sequence of points in \( X \), a point \( x \) in \( X \) is said to be the limit of the sequence \( \{x_n\} \) if \( \lim_{n,m \to \infty} G(x_n, x_m, x_l) = 0 \), and one says that sequence \( \{x_n\} \) is \( G \)-convergent to \( x \).

Thus, if \( x_n \to x \) in a \( G \)-metric space \( (X, G) \), then for any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( G(x_n, x_m, x_l) < \varepsilon \), for all \( n, m \geq N \).

**Proposition 1.3 (see [1]).** Let \( (X, G) \) be a \( G \)-metric space, then the followings are equivalent.

1. \( \{x_n\} \) is \( G \)-convergent to \( x \).
2. \( G(x_n, x_n, x) \to 0 \) as \( n \to \infty \).
3. \( G(x_n, x, x) \to 0 \) as \( n \to \infty \).
4. \( G(x_n, x_m, x) \to 0 \) as \( n, m \to \infty \).

**Definition 1.4 (see [1]).** Let \( (X, G) \) be a \( G \)-metric space. A sequence \( \{x_n\} \) is called \( G \)-Cauchy sequence if for each \( \varepsilon > 0 \) there exists a positive integer \( N \in \mathbb{N} \) such that \( G(x_n, x_m, x_l) < \varepsilon \) for all \( n, m, l \geq N \); that is, if \( G(x_n, x_m, x_l) \to 0 \) as \( n, m, l \to \infty \).

**Definition 1.5 (see [1]).** A \( G \)-metric space \( (X, G) \) is said to be \( G \)-complete if every \( G \)-Cauchy sequence in \( (X, G) \) is \( G \)-convergent in \( X \).

**Proposition 1.6 (see [1]).** Let \( (X, G) \) be a \( G \)-metric space. Then the following are equivalent.

1. The sequence \( \{x_n\} \) is \( G \)-Cauchy.
2. For every \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that \( G(x_n, x_m, x_m) < \varepsilon \), for all \( n, m \geq k \).

**Proposition 1.7 (see [1]).** Let \( (X, G) \) be a \( G \)-metric space. Then the function \( G(x, y, z) \) is jointly continuous in all three of its variables.

**Definition 1.8 (see [1]).** Let \( (X, G) \) and \( (X', G') \) be \( G \)-metric space, and \( f : (X, G) \to (X', G') \) be a function. Then \( f \) is said to be \( G \)-continuous at a point \( a \in X \) if and only if for every \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( x, y \in X \) and \( G(a, x, y) < \delta \) implies \( G'(f(a), f(x), f(y)) < \varepsilon \). A function \( f \) is \( G \)-continuous at \( X \) if and only if it is \( G \)-continuous at all \( a \in X \).
Proposition 1.9 (see [1]). Let \((X, G)\) and \((X', G')\) be \(G\)-metric space. Then \(f : X \to X'\) is \(G\)-continuous at \(x \in X\) if and only if it is \(G\)-sequentially continuous at \(x\); that is, whenever \(\{x_n\}\) is \(G\)-convergent to \(x\), \(\{f(x_n)\}\) is \(G\)-convergent to \(f(x)\).

Proposition 1.10 (see, [1]). Let \((X, G)\) be a \(G\)-metric space. Then, for any \(x, y, z, a \in X\) it follows that:

(i) if \(G(x, y, z) = 0\), then \(x = y = z\);
(ii) \(G(x, y, z) \leq G(x, x, y) + G(x, x, z)\);
(iii) \(G(x, y, y) \leq 2G(y, x, x)\);
(iv) \(G(x, y, z) \leq G(x, a, z) + G(a, y, z)\);
(v) \(G(x, y, z) \leq (2/3)(G(x, y, a) + G(x, a, z) + G(a, y, z))\);
(vi) \(G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))\).

2. Main Results

Theorem 2.1. Let \((X, G)\) be a complete \(G\)-metric space. Suppose the three self-mappings \(T, S, R : X \to X\) satisfy the following condition:

\[
G^2(Tx, Sy, Rz) \leq \alpha G(x, Tx, Tx)G(y, Sy, Sy) + \beta G(y, Sy, Sy)G(z, Rz, Rz) + \gamma G(x, Tx, Tx)G(z, Rz, Rz),
\]

for all \(x, y, z \in X\), where \(\alpha, \beta, \gamma\) are nonnegative real numbers and \(\alpha + \beta + \gamma < 1\). Then \(T, S,\) and \(R\) have a unique common fixed point (say \(u\)) and \(T, S, R\) are all \(G\)-continuous at \(u\).

Proof. We will proceed in two steps.
Step 1. We prove any fixed point of \(T\) is a fixed point of \(S\) and \(R\) and conversely. Assume that \(p \in X\) is such that \(Tp = p\). However, by (2.1), we have

\[
G^2(Tp, Sp, Rp) \leq \alpha G(p,Tp,Tp)G(p,Sp,Sp) + \beta G(p,Sp,Sp)G(p,Rp,Rp) + \gamma G(p,Tp,Tp)G(p,Rp,Rp)
\]

\[
= \alpha G(p,p,p)G(p,Sp,Sp) + \beta G(p,Sp,Sp)G(p,Rp,Rp) + \gamma G(p,p,p)G(p,Rp,Rp)
\]

\[
= \beta G(p,Sp,Sp)G(p,Rp,Rp).
\]

Now we discuss the above inequality in three cases.
Case (i). If \(p \neq Sp\) and \(p \neq Rp\), then, by (G3), we have

\[
G(p, Sp, Sp) \leq G(p, Sp, Rp), \quad G(p, Rp, Rp) \leq G(p, Sp, Rp).
\]

So, the above inequality becomes

\[
G^2(p, Sp, Rp) = G^2(Tp, Sp, Rp) \leq \beta G^2(p, Sp, Rp).
\]
Since \( G^2(p, Sp, Rp) > 0 \), hence we have \( \beta \geq 1 \); however, it contradicts with \( 0 \leq \beta \leq \alpha + \beta + \gamma < 1 \), so we get \( p = Sp = Rp \).

**Case (ii).** If \( p = Rp \), then we have

\[
G^2(p, Sp, Rp) = G^2(Tp, Sp, Rp) \leq \beta G(p, Sp, Sp)G(p, Rp, Rp) = 0. \tag{2.5}
\]

Hence we have \( G^2(p, Sp, Rp) = 0 \) and so \( p = Sp = Rp \).

**Case (iii).** If \( p = Sp \), we can also get \( G^2(p, Sp, Rp) = 0 \). Hence we have \( p = Sp = Rp \). Therefore \( p \) is a common fixed point of \( T, S \) and \( R \).

The same conclusion holds if \( p = Sp \) or \( p = Rp \).

**Step 2.** We prove that \( T, S \), and \( R \) have a unique common fixed point.

Let \( x_0 \in X \) be an arbitrary point, and define the sequence \( \{x_n\} \) by \( x_{3n+1} = Tx_{3n}, \ x_{3n+2} = Sx_{3n+1}, \ x_{3n+3} = Rx_{3n+2}, \ n \in \mathbb{N} \). If \( x_n = x_{n+1} \), for some \( n \), with \( n = 3m \), then \( p = x_{3m} \) is a fixed point of \( T \) and, by the first step, \( p \) is a common fixed point of \( S, T, \) and \( R \). The same holds if \( n = 3m + 1 \) or \( n = 3m + 2 \). Without loss of generality, we can assume that \( x_n \neq x_{n+1} \), for all \( n \in \mathbb{N} \).

Next, we prove sequence \( \{x_n\} \) is a G-Cauchy sequence. In fact, by (2.1) and (G3), we have

\[
G^2(x_{3n+1}, x_{3n+2}, x_{3n+3}) = G^2(Tx_{3n}, Sx_{3n+1}, Rx_{3n+2})
\]
\[
\quad \leq \alpha G(x_{3n}, Tx_{3n}, Tx_{3n})G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1})
\quad + \beta G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2})
\quad + \gamma G(x_{3n}, Tx_{3n}, Tx_{3n})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2})
\]
\[
= \alpha G(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+1}, x_{3n+2}, x_{3n+2})
\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+3})
\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+2}, x_{3n+3}, x_{3n+3})
\]
\[
\leq \alpha G(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+1}, x_{3n+2}, x_{3n+2})
\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+3})
\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+2}, x_{3n+3}, x_{3n+3}). \tag{2.6}
\]

Which gives that

\[
G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha + \gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{2.7}
\]

It follows that

\[
(1 - \beta)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha + \gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}). \tag{2.8}
\]
From $0 \leq \beta < 1$ we know that $1 - \beta > 0$. Then, we have

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \frac{\alpha + \gamma}{1 - \beta} G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

(2.9)

On the other hand, by using (2.1) and (G3), we have

$$G^2(x_{3n+2}, x_{3n+3}, x_{3n+4}) = G^2(Tx_{3n+3}, Sx_{3n+1}, Rx_{3n+2})$$

$$\leq aG(x_{3n+1}, Tx_{3n+3}, Tx_{3n+3})G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1})$$

$$+ \beta G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2})$$

$$+ \gamma G(x_{3n+1}, Tx_{3n+3}, Tx_{3n+3})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2})$$

$$= aG(x_{3n+1}, x_{3n+4}, x_{3n+4})G(x_{3n+1}, x_{3n+2}, x_{3n+2})$$

$$+ \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+3})$$

$$+ \gamma G(x_{3n+3}, x_{3n+4}, x_{3n+4})G(x_{3n+2}, x_{3n+3}, x_{3n+3})$$

$$\leq aG(x_{3n+2}, x_{3n+3}, x_{3n+4})G(x_{3n+1}, x_{3n+2}, x_{3n+2})$$

$$+ \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+3})$$

$$+ \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4})G(x_{3n+2}, x_{3n+3}, x_{3n+4}).$$

(2.10)

Which implies that

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4}).$$

(2.11)

It follows that

$$(1 - \gamma)G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta)G(x_{3n+1}, x_{3n+2}, x_{3n+3}).$$

(2.12)

Form the condition $0 \leq \gamma \leq \alpha + \beta + \gamma < 1$, we know that $1 - \gamma > 0$. Therefore, we have

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq \frac{\alpha + \beta}{1 - \gamma} G(x_{3n+1}, x_{3n+2}, x_{3n+3}).$$

(2.13)
Again, using (2.1) and (G3), we can get

\[
G^2(x_{3n+3}, x_{3n+4}, x_{3n+5}) = G^2(Tx_{3n+3}, Sx_{3n+4}, Rx_{3n+2})
\leq \alpha G(x_{3n+3}, Tx_{3n+3}, Tx_{3n+3})G(x_{3n+3}, Sx_{3n+4}, Sx_{3n+4})
+ \beta G(x_{3n+3}, Sx_{3n+4}, Sx_{3n+4})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2})
+ \gamma G(x_{3n+3}, Tx_{3n+3}, Tx_{3n+3})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2})
= \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+4})G(x_{3n+4}, x_{3n+5}, x_{3n+5})
+ \beta G(x_{3n+4}, x_{3n+5}, x_{3n+5})G(x_{3n+2}, x_{3n+3}, x_{3n+3})
+ \gamma G(x_{3n+3}, x_{3n+4}, x_{3n+4})G(x_{3n+2}, x_{3n+3}, x_{3n+3})
\leq \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+4})G(x_{3n+3}, x_{3n+4}, x_{3n+5})
+ \beta G(x_{3n+3}, x_{3n+4}, x_{3n+5})G(x_{3n+2}, x_{3n+3}, x_{3n+4})
+ \gamma G(x_{3n+5}, x_{3n+3}, x_{3n+4})G(x_{3n+2}, x_{3n+3}, x_{3n+4}).
\tag{2.14}
\]

Which implies that

\[
G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+5}) + (\beta + \gamma)G(x_{3n+2}, x_{3n+3}, x_{3n+3}).
\tag{2.15}
\]

It follows that

\[
(1 - \alpha)G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq (\beta + \gamma)G(x_{3n+2}, x_{3n+3}, x_{3n+3}).
\tag{2.16}
\]

By the condition \(0 \leq \alpha \leq \alpha + \beta + \gamma < 1\), we know that \(1 - \alpha > 0\). Hence, we have

\[
G(x_{3n+1}, x_{3n+4}, x_{3n+5}) \leq \frac{\beta + \gamma}{1 - \alpha}G(x_{3n+2}, x_{3n+3}, x_{3n+4}).
\tag{2.17}
\]

Let \(q = \max\{ (\alpha + \gamma)/(1 - \beta), (\alpha + \beta)/(1 - \gamma), (\beta + \gamma)/(1 - \alpha) \} \), then from \(0 \leq \alpha + \beta + \gamma < 1\) we know that \(0 \leq q < 1\). Combining (2.9), (2.13), and (2.17), we have

\[
G(x_n, x_{n+1}, x_{n+2}) \leq qG(x_{n-1}, x_n, x_{n+1}) \leq \cdots \leq q^nG(x_0, x_1, x_2).
\tag{2.18}
\]

Thus, by (G3) and (G5), for every \(m, n \in \mathbb{N}\), \(m > n\), noting that \(0 \leq q < 1\), we have

\[
G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m),
\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m)
\leq \left(q^n + q^{n+1} + \cdots + q^{m-1}\right)G(x_0, x_1, x_2)
\leq \frac{q^n}{1 - q}G(x_0, x_1, x_2).
\tag{2.19}
\]
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Which implies that \( G(x_n, x_m, x_m) \to 0, \) as \( n, m \to \infty. \) Thus \( \{x_n\} \) is a \( G \)-Cauchy sequence. Due to the completeness of \((X, G)\), there exists \( u \in X \), such that \( \{x_n\} \) is \( G \)-convergent to \( u \).

Next we prove \( u \) is a common fixed point of \( T, S, \) and \( R \). By using (2.1), we have

\[
G^2(Tu, x_{3n+2}, x_{3n+3}) = G^2(Tu, Sx_{3n+1}, Rx_{3n+2}) \\
\leq aG(u, Tu, Tu)G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1}) \\
+ \beta G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
+ \gamma G(u, Tu, Tu)G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
= aG(u, Tu, Tu)G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
+ \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2})G(x_{3n+3}, x_{3n+3}, x_{3n+3}) \\
+ \gamma G(u, Tu, Tu)G(x_{3n+2}, x_{3n+3}, x_{3n+3}).
\]

Letting \( n \to \infty \), and using the fact that \( G \) is continuous on its variables, we can get

\[
G^2(Tu, u, u) = 0.
\]

Which gives that \( Tu = u \), that is \( u \) is a fixed point of \( T \). By using (2.1) again, we have

\[
G^2(x_{3n+1}, Su, x_{3n+3}) = G^2(Tx_{3n}, Su, Rx_{3n+2}) \\
\leq aG(x_{3n}, x_{3n+1}, x_{3n+1})G(u, Su, Su) \\
+ \beta G(u, Su, Su)G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
+ \gamma G(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+2}, x_{3n+3}, x_{3n+3}).
\]

Letting \( n \to \infty \) at both sides, for \( G \) is continuous on its variables, it follows that

\[
G^2(u, Su, u) = 0.
\]

Therefore, \( Su = u \); that is, \( u \) is a fixed point of \( S \). Similarly, by (2.1), we can also get

\[
G^2(x_{3n+1}, x_{3n+2}, Ru) = G^2(Tx_{3n}, Sx_{3n+1}, Ru) \\
\leq aG(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
+ \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2})G(u, Ru, Ru) \\
+ \gamma G(x_{3n}, x_{3n+1}, x_{3n+1})G(u, Ru, Ru).
\]

On taking \( n \to \infty \) at both sides, since \( G \) is continuous on its variables, we get that

\[
G^2(u, u, Ru) = 0.
\]
Which gives that \( u = Ru \), therefore, \( u \) is fixed point of \( R \). Consequently, we have \( u = Tu = Su = Ru \), and \( u \) is a common fixed point of \( T, S \) and \( R \). Suppose \( v \) is another common fixed point of \( T, S \) and \( R \), and we have \( v = Tv = Sv = Rv \), then by (2.1), we have

\[
G^2(u, u, v) = G^2(Tu, Su, Rv)
\]

\[
\leq \alpha G(u, Tu, Tu)G(u, Su, Su) + \beta G(u, Su, Su)G(v, Rv, Rv) + \gamma G(u, Tu, Tu)G(v, Rv, Rv)
\]

\[
= \alpha G(u, u, u)G(u, u, u) + \beta G(u, u, u)G(v, v, v)
\]

\[
= 0.
\]

(2.26)

Which implies that \( G^2(u, u, v) = 0 \), hence, \( u = v \). Then we know the common fixed point of \( T, S, \) and \( R \) is unique.

To show that \( T \) is G-continuous at \( u \), let \( \{y_n\} \) be any sequence in \( X \) such that \( \{y_n\} \) is G-convergent to \( u \). For \( n \in \mathbb{N} \), we have

\[
G^2(Ty_n, u, u) = G^2(Ty_n, Su, Ru)
\]

\[
\leq \alpha G(y_n, Ty_n, Ty_n)G(u, Su, Su) + \beta G(u, Su, Su)G(u, Ru, Ru) + \gamma G(y_n, Ty_n, Ty_n)G(u, Ru, Ru)
\]

\[
= \alpha G(y_n, y_n, y_n)G(u, u, u) + \beta G(u, u, u)G(u, u, u)
\]

\[
= 0.
\]

(2.27)

Which implies that \( \lim_{n \to \infty} G^2(Ty_n, u, u) = 0 \). Hence \( \{Ty_n\} \) is G-convergent to \( u = Tu \). So \( T \) is G-continuous at \( u \). Similarly, we can also prove that \( S, R \) are G-continuous at \( u \). Therefore, we complete the proof.

\[\square\]

**Corollary 2.2.** Let \((X, G)\) be a complete G-metric space. Suppose the three self-mappings \( T, S, R : X \to X \) satisfy the condition:

\[
G^2(T^p x, S^r y, R^r z) \leq \alpha G(x, T^p x, T^p x)G(y, S^r y, S^r y) + \beta G(y, S^r y, S^r y)G(z, R^r z, R^r z) + \gamma G(x, T^p x, T^p x)G(z, R^r z, R^r z),
\]

(2.28)

for all \( x, y, z \in X \), where \( p, s, r \in \mathbb{N} \), \( \alpha, \beta, \gamma \) are nonnegative real numbers and \( \alpha + \beta + \gamma < 1 \). Then \( T, S, \) and \( R \) have a unique common fixed point (say \( u \)) and \( T^p, S^r, R^r \) are all G-continuous at \( u \).

**Proof.** From Theorem 2.1 we know that \( T^p, S^r, R^r \) have a unique common fixed point (say \( u \)); that is, \( T^p u = u \), \( S^r u = u \), \( R^r u = u \), and \( T^p, S^r, R^r \) are G-continuous at \( u \). Since \( Tu = TT^p u = T^{p+1} u = T^p Tu \), so \( Tu \) is another fixed point of \( T^p \). Suppose \( Su = SS^r u = S^{r+1} u = g^s gu \), so \( Su \) is another
fixed point of $S^s$, and $Ru = RR'u = R^{s+1}u = R'Ru$, so $Ru$ is another fixed point of $R'$. By (G3) and the condition (2.28) in Corollary 2.2, we have

$$
G^2(Tu, S^sTu, R'Tu) = G^2(T^pTu, S^sTu, R'Tu)
\leq \alpha G(Tu, T^pTu, T^pTu)G(Tu, S^sTu, S^sTu) + \beta G(Tu, S^sTu, S^sTu)G(Tu, R'Tu, R'Tu) + \gamma G(Tu, T^pTu, T^pTu)G(Tu, R'Tu, R'Tu)
\leq \beta G(Tu, S^sTu, S^sTu)G(Tu, R'Tu, R'Tu)
\leq \beta G(Tu, S^sTu, R'Tu)G(Tu, S^sTu, R'Tu).
$$

(2.29)

Since $0 \leq \beta < 1$, we can get $G^2(Tu, S^sTu, R'Tu) = 0$. That means $Tu = T^pTu = S^sTu = R'Tu$, hence $Tu$ is another common fixed point of $T^p$, $S^s$-and $R'$. Since the common fixed point of $T^p$, $S^s$-and $R'$ is unique, we deduce that $u = Tu$. By the same argument, we can prove $u = S^sTu = Ru$. Thus, we have $u = Tu = S^sTu = Ru$. Suppose $v$ is another common fixed point of $T$, $S$, and $R$, then $v = T^pTu = S^sTv = R'Tv$, and by using the condition (2.28) in Corollary 2.2 again, we have

$$
G^2(v, u, u) = G^2(T^pTv, S^sTv, R'Tv)
\leq \alpha G(v, T^pTv, T^pTv)G(u, S^sTv, S^sTv) + \beta G(u, S^sTv, S^sTv)G(u, R'Tv, R'Tv) + \gamma G(v, T^pTv, T^pTv)G(u, R'Tv, R'Tv)
\leq \beta G(v, S^sTv, S^sTv)G(u, R'Tv, R'Tv)
\leq \beta G(v, S^sTv, R'Tv)G(u, S^sTv, R'Tv)
\leq \beta G(v, S^sTv, R'Tv)G(u, S^sTv, R'Tv).
$$

(2.30)

Which implies that $G^2(v, u, u) = 0$, hence $v = u$. So the common fixed of $T$, $S$, and $R$ is unique. It is obvious that every fixed point of $T$ is a fixed point of $S$ and $R$ and conversely.

\[\square\]

**Corollary 2.3.** Let $(X, G)$ be a complete G-metric space. Suppose the self-mapping $T : X \to X$ satisfies the following condition:

$$
G^2(Tx, Ty, Tz) \leq \alpha G(x, Tx, Tx)G(y, Ty, Ty) + \beta G(y, Ty, Ty)G(z, Tz, Tz) + \gamma G(x, Tx, Tx)G(z, Tz, Tz),
$$

(2.31)

for all $x, y, z \in X$, where $\alpha, \beta, \gamma$ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then $T$ has a unique fixed point (say $u$) and $T$ is $G$-continuous at $u$.

**Proof.** Let $T = S = R$ in Theorem 2.1, we can get this conclusion holds.

\[\square\]
Corollary 2.4. Let \((X, G)\) be a complete \(G\)-metric space. Suppose the self-mapping \(T : X \to X\) satisfies the following condition:

\[
G^2(Tp x, Tp y, Tp z) \leq \alpha G(x, Tp x, Tp x)G(y, Tp y, Tp y) + \beta G(y, Tp y, Tp y)G(z, Tp z, Tp z) \\
+ \gamma G(x, Tp x, Tp x)G(z, Tp z, Tp z).
\]  

(2.32)

for all \(x, y, z \in X\), where \(\alpha, \beta, \gamma\) are nonnegative real numbers and \(\alpha + \beta + \gamma < 1\). Then \(T\) has a unique fixed point (say \(u\)) and \(T^p\) is \(G\)-continuous at \(u\).

**Proof.** Let \(T = S = R, p = s = r\) in Corollary 2.2, we can get this conclusion holds. \(\square\)

Theorem 2.5. Let \((X, G)\) be a complete \(G\)-metric space, and let \(T, S, R : X \to X\) be three self-mappings in \(X\), which satisfy the following condition.

\[
G^2(T x, S y, R z) \leq \alpha G(x, T x, S y)G(y, S y, R z) + \beta G(y, S y, R z)G(z, R z, T x) \\
+ \gamma G(x, T x, S y)G(z, R z, T x).
\]  

(2.33)

for all \(x, y, z \in X\), \(\alpha, \beta, \gamma\) are nonnegative real numbers and \(\alpha + \beta + \gamma < 1\). Then \(T, S\) and \(R\) have a unique common fixed point (say \(u\)) and \(T, S, R\) are all \(G\)-continuous at \(u\).

**Proof.** We will proceed in two steps.

Step 1. We prove any fixed point of \(T\) is a fixed point of \(S\) and \(R\) and conversely. Assume that \(p \in X\) is such that \(Tp = p\). Now we prove that \(p = Sp\) and \(p = Rp\). If it is not the case, then for \(p \neq Sp\) and \(p \neq Rp\), by (2.33) and (G3) we have

\[
G^2(Tp, Sp, Rp) \leq \alpha G(p, Tp, Sp)G(p, Sp, Rp) + \beta G(p, Sp, Rp)G(p, Rp, Tp) \\
+ \gamma G(p, Tp, Sp)G(p, Rp, Tp) \\
= \alpha G(p, p, Sp)G(p, Sp, Rp) + \beta G(p, Sp, Rp)G(p, Rp, p) \\
+ \gamma G(p, p, Sp)G(p, Rp, p) \\
\leq \alpha G(p, Rp, Sp)G(p, Sp, Rp) + \beta G(p, Sp, Rp)G(p, Rp, Sp) \\
+ \gamma G(p, Rp, Sp)G(p, Rp, Sp) \\
= (\alpha + \beta + \gamma)G^2(p, Rp, Sp).
\]  

(2.34)

It follows that

\[
G^2(p, Sp, Rp) = G^2(Tp, Sp, Rp) \leq (\alpha + \beta + \gamma)G^2(p, Sp, Rp).
\]  

(2.35)

Since \(G^2(p, Sp, Rp) > 0\), hence we have \(\alpha + \beta + \gamma \geq 1\), however it contradicts with the condition \(0 \leq \alpha + \beta + \gamma < 1\), so we can have \(p = Sp = Rp\), hence \(p\) is a common fixed point of \(T, S,\) and \(R\).

Analogously, following the similar arguments to those given above, we can obtain a contradiction for \(p \neq Sp\) and \(p = Rp\) or \(p = Sp\) and \(p \neq Rp\). Hence in all the cases, we conclude that \(p = Sp = Rp\). The same conclusion holds if \(p = Sp\) or \(p = Rp\).
Step 2. We prove that $T$, $S$ and $R$ have a unique common fixed point. Let $x_0 \in X$ be an arbitrary point, and define the sequence $\{x_n\}$ by $x_{3n+1} = Tx_{3n}$, $x_{3n+2} = Sx_{3n+1}$, $x_{3n+3} = Rx_{3n+2}$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$, for some $n$, with $n = 3m$, then $p = x_{3m}$ is a fixed point of $T$ and, by the first step, $p$ is a common fixed point of $S$, $T$, and $R$. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Without loss of generality, we can assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. We first prove the sequence $\{x_n\}$ is a $G$-Cauchy sequence. In fact, by using (2.33) and (G3), we have

$$G^2(x_{3n+1}, x_{3n+2}, x_{3n+3}) = G^2(Tx_{3n}, Sx_{3n+1}, Rx_{3n+2})$$

$$\leq aG(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+1}, x_{3n+2}, x_{3n+3})$$

$$+ \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+1})$$

$$+ \gamma G(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+1}).$$

Which gives that

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \frac{\alpha + \gamma}{1 - \beta} G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

(2.36)

From $0 \leq \beta < 1$, we know that $1 - \beta > 0$. Then, we have

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \frac{\alpha + \gamma}{1 - \beta} G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

(2.37)

On the other hand, by using (2.33) and (G3), we have

$$G^2(x_{3n+2}, x_{3n+3}, x_{3n+4}) = G^2(Tx_{3n+3}, Sx_{3n+1}, Rx_{3n+2})$$

$$\leq aG(x_{3n+3}, x_{3n+4}, x_{3n+2})G(x_{3n+1}, x_{3n+2}, x_{3n+3})$$

$$+ \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+4})$$

$$+ \gamma G(x_{3n+3}, x_{3n+4}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+4}).$$

Which implies that

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta) G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4}).$$

(2.38)

It follows that

$$(1 - \gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta) G(x_{3n+1}, x_{3n+2}, x_{3n+3}).$$

(2.39)
Since $0 \leq \gamma < 1$, we know that $1 - \gamma > 0$. So, we have

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq \frac{a + \beta}{1 - \gamma} G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \quad (2.43)$$

Again, using (2.33) and (G3), we can get

$$G^2(x_{3n+3}, x_{3n+4}, x_{3n+5}) = G^2(Tx_{3n+3}, Sx_{3n+4}, Rx_{3n+2})$$

$$\leq aG(x_{3n+3}, x_{3n+4}, x_{3n+5})G(x_{3n+4}, x_{3n+5}, x_{3n+3})$$

$$+ \beta G(x_{3n+4}, x_{3n+5}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+4})$$

$$+ \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4})G(x_{3n+3}, x_{3n+4}, x_{3n+5}). \quad (2.44)$$

Which implies that

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq aG(x_{3n+3}, x_{3n+4}, x_{3n+5}) + (\beta + \gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \quad (2.45)$$

It follows that

$$(1 - a)G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq (\beta + \gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \quad (2.46)$$

Since $0 \leq a \leq a + \beta + \gamma < 1$, we know that $1 - a > 0$. So we have

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq \frac{\beta + \gamma}{1 - a} G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \quad (2.47)$$

Let $q = \max\{(a + \gamma)/(1 - \beta), (\alpha + \beta)/(1 - \gamma), (\beta + \gamma)/(1 - \alpha)\}$, then $0 \leq q < 1$, and by combining (2.39), (2.43), and (2.47), we have

$$G(x_n, x_{n+1}, x_{n+2}) \leq qG(x_{n-1}, x_n, x_{n+1}) \leq \cdots \leq q^n G(x_0, x_1, x_2). \quad (2.48)$$

Thus, by (G3) and (G5), for every $m, n \in \mathbb{N}$, if $m > n$, noting that $0 \leq q < 1$, we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m)$$

$$\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_{m+1})$$

$$\leq \left(q^n + q^{n+1} + \cdots + q^{m-1}\right) G(x_0, x_1, x_2)$$

$$\leq \frac{q^n}{1 - q} G(x_0, x_1, x_2). \quad (2.49)$$

Which implies that $G(x_n, x_m, x_m) \to 0$, as $n, m \to \infty$. Thus $\{x_n\}$ is a $G$-Cauchy sequence. Due to the completeness of $(X, G)$, there exists $u \in X$, such that $\{x_n\}$ is $G$-convergent to $u$. 
Now we prove \( u \) is a common fixed point of \( T, S, \) and \( R \). By using (2.33), we have

\[
G^2(Tu, x_{3n+2}, x_{3n+3}) = G^2(Tu, Sx_{3n+1}, Rx_{3n+2}) \\
\leq aG(u, Tu, Sx_{3n+1})G(x_{3n+1}, Sx_{3n+1}, Rx_{3n+2}) + \beta G(x_{3n+1}, Sx_{3n+1}, Rx_{3n+2})G(x_{3n+2}, Rx_{3n+2}, Tu) \\
+ \gamma G(u, Tu, Sx_{3n+1})G(x_{3n+2}, Rx_{3n+2}, Tu) \\
= aG(u, Tu, x_{3n+2})G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, Tu) \\
+ \gamma G(u, Tu, x_{3n+2})G(x_{3n+2}, x_{3n+3}, Tu). \\
\tag{2.50}
\]

Letting \( n \to \infty \), and using the fact that \( G \) is continuous on its variables and \( \gamma < 1 \), we can get

\[
G^2(Tu, u, u) \leq \gamma G^2(u, u, Tu). \\
\tag{2.51}
\]

Which gives that \( Tu = u \), hence \( u \) is a fixed point of \( T \). By using (2.33) again, we have

\[
G^2(x_{3n+1}, Su, x_{3n+3}) = G^2(Tx_{3n}, Su, Rx_{3n+2}) \\
\leq aG(x_{3n}, x_{3n+1}, Su)G(u, Su, x_{3n+3}) + \beta G(u, Su, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+1}) \\
+ \gamma G(x_{3n}, x_{3n+1}, Su)G(x_{3n+2}, x_{3n+3}, x_{3n+1}). \\
\tag{2.52}
\]

Letting \( n \to \infty \) at both sides, for \( G \) is continuous in its variables, it follows that

\[
G^2(u, Su, u) \leq aG^2(u, Su, u). \\
\tag{2.53}
\]

For \( 0 \leq a < 1 \), Therefore, we can get \( G^2(u, Su, u) = 0 \), hence \( Su = u \), hence \( u \) is a fixed point of \( S \). Similarly, by (2.33), we can also get

\[
G^2(x_{3n+1}, x_{3n+2}, Ru) = G^2(Tx_{3n}, Sx_{3n+1}, Ru) \\
\leq aG(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+1}, x_{3n+2}, Ru) + \beta G(x_{3n+1}, x_{3n+2}, Ru)G(u, Ru, x_{3n+1}) \\
+ \gamma G(x_{3n}, x_{3n+1}, x_{3n+2})G(u, Ru, x_{3n+1}). \\
\tag{2.54}
\]

On taking \( n \to \infty \) at both sides, since \( G \) is continuous in its variables, we get that

\[
G^2(u, u, Ru) \leq \beta G^2(u, u, Ru). \\
\tag{2.55}
\]
Since $0 \leq \beta < 1$, so we get $G^2(u, u, Ru) = 0$, hence $u = Ru$, therefore, $u$ is a fixed point of $R$. Consequently, we have $u = Tu = Su = Ru$, and $u$ is a common fixed point of $T, S$, and $R$. Suppose $v \neq u$ is another common fixed point of $T, S$, and $R$, and we have $v = Tv = Sv = Rv$, then by (2.33), we have

\[
G^2(u, u, v) = G^2(Tu, Su, Rv) \\
\leq aG(u, Tu, Su)G(u, Su, Rv) + \beta G(u, Su, Rv)G(v, Rv, Tu) \\
+ \gamma G(u, Tu, Su)G(v, Rv, Tu) \\
= aG(u, u, u)G(u, u, v) + \beta G(u, u, v)G(v, v, u) + \gamma G(u, u, u)G(v, v, u).
\]

Which gives that

\[
G^2(u, u, v) \leq \beta G(u, u, v)G(v, v, u). \tag{2.57}
\]

Hence, we can get $G(u, u, v) \leq \beta G(v, v, u)$. By using (2.33) again, we get

\[
G^2(u, v, v) = G^2(Tu, Sv, Rv) \\
\leq aG(u, Tu, Sv)G(v, Sv, Rv) + \beta G(v, Sv, Rv)G(v, Rv, Tu) \\
+ \gamma G(u, Tu, Sv)G(v, Rv, Tu) \\
= aG(u, u, v)G(v, v, v) + \beta G(v, v, v)G(v, v, u) + \gamma G(u, u, v)G(v, v, u).
\]

Which implies that

\[
G^2(u, v, v) \leq \gamma G(u, u, v)G(v, v, u). \tag{2.59}
\]

Hence, we can get

\[
G(u, v, v) \leq \gamma G(u, u, v). \tag{2.60}
\]

By combining $G(u, u, v) \leq \beta G(v, v, u)$, we can have

\[
G(u, v, v) \leq \gamma G(u, u, v) \leq \beta \gamma G(v, v, u). \tag{2.61}
\]

Since $v \neq u$, $G(u, v, v) > 0$, so we have that $\beta \gamma \geq 1$. Since $0 \leq \beta$, $\gamma < 1$, we know $0 \leq \beta \gamma < 1$, so it’s a contradiction. Hence, we get $u = v$. Then we know the common fixed point of $T, S$, and $R$ is unique.
To show that $T$ is $G$-continuous at $u$, let $\{y_n\}$ be any sequence in $X$ such that $\{y_n\}$ is $G$-convergent to $u$. For $n \in \mathbb{N}$, we have

$$G^2(Ty_n, u, u) = G^2(Ty_n, Su, Ru)$$

$$\leq \alpha G(y_n, Ty_n, Su)G(u, Su, Ru) + \beta G(u, Su, Ru)G(u, Ru, Ty_n)$$

$$+ \gamma G(y_n, Ty_n, Su)G(u, Ru, Ty_n)$$

$$= \alpha G(y_n, Ty_n, u)G(u, u, u) + \beta G(u, u, u)G(u, u, Ty_n)$$

$$+ \gamma G(y_n, Ty_n, u)G(u, u, Ty_n)$$

$$= \gamma G(y_n, Ty_n, u)G(u, u, Ty_n).$$

Which implies that

$$G(Ty_n, u, u) \leq \gamma G(y_n, Ty_n, u). \quad (2.63)$$

On taking $n \to \infty$ at both sides, considering $\gamma < 1$, we get $\lim_{n \to \infty} G(Ty_n, u, u) = 0$. Hence $\{Ty_n\}$ is $G$-convergent to $u = Tu$. So $T$ is $G$-continuous at $u$. Similarly, we can also prove that $S, R$ are $G$-continuous at $u$. Therefore, we complete the proof.

Now we introduce an example to support Theorem 2.5.

**Example 2.6.** Let $X = [0, 1]$, and let $(X, G)$ be a $G$-metric space defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z$ in $X$. Let $T, S, R$ be three self-mappings defined by

$$Tx = \begin{cases} 
1, & x \in \left[0, \frac{1}{2}\right] \\
\frac{6}{7}, & x \in \left(\frac{1}{2}, 1\right]
\end{cases}, \quad Sx = \begin{cases} 
\frac{7}{8}, & x \in \left[0, \frac{1}{2}\right] \\
\frac{6}{7}, & x \in \left(\frac{1}{2}, 1\right]
\end{cases}, \quad Rx = \frac{6}{7}, \quad x \in [0, 1]. \quad (2.64)
$$

Next we proof the mappings $T, S, R$ are satisfying Condition (2.33) of Theorem 2.5 with $\alpha = 1/7$, $\beta = 1/7$ and $\gamma = 4/7$.

**Case 1.** If $x, y \in [0, 1/2], \ z \in [0, 1]$, then

$$G^2(Tx, Sy, Rz) = G^2\left(1, \frac{7}{8}, \frac{6}{7}\right) = \frac{4}{49},$$

$$G(x, Tx, Sy) = G\left(x, 1, \frac{7}{8}\right) = |x - 1| + \left|x - \frac{7}{8}\right| + \frac{3}{8} + \frac{1}{8} = 1,$$

$$G(y, Sy, Rz) = G\left(y, \frac{7}{8}, \frac{6}{7}\right) = \left|y - \frac{7}{8}\right| + \left|y - \frac{6}{7}\right| + \frac{3}{8} + \frac{5}{14} + \frac{1}{56} = \frac{3}{4},$$

$$G(z, Rz, Tx) = G\left(z, \frac{6}{7}, 1\right) = \left|z - \frac{6}{7}\right| + \frac{1}{7} + |z - 1| \geq 0 + \frac{1}{7} + 0 = \frac{1}{7}. \quad (2.65)$$
Thus, we have

\[
G^2(Tx, Sy, Rz) = \frac{4}{49} \leq \alpha \cdot 1 \cdot \frac{3}{4} + \beta \cdot \frac{3}{4} \cdot \frac{1}{7} + \gamma \cdot 1 \cdot \frac{1}{7} \\
\leq aG(x, Tx)G(y, Sy, Rz) + \beta G(y, Sy, Rz)G(z, Rz, Tx) + \gamma G(x, Tx, Sy)G(z, Rz, Tx).
\]  

(2.66)

Case 2. If \( x \in [0, 1/2], \ y \in (1/2, 1], \ z \in [0, 1] \), then we can get

\[
G^2(Tx, Sy, Rz) = G^2\left(1, \frac{6}{7}, \frac{6}{7}\right) = \frac{4}{49}, \\
G(x, Tx, Sy) = G\left(x, 1, \frac{6}{7}\right) = \left|x - 1\right| + \left|x - \frac{6}{7}\right| + \frac{1}{7} \geq \frac{1}{2} + \frac{5}{14} + \frac{1}{7} = 1, \\
G(y, Sy, Rz) = G\left(y, \frac{6}{7}, \frac{6}{7}\right) = \left|y - \frac{6}{7}\right| + \left|y - \frac{6}{7}\right| \geq 0 + 0 = 0, \\
G(z, Rz, Tx) = G\left(z, \frac{6}{7}, 1\right) = \left|z - \frac{6}{7}\right| + \frac{1}{7} + \left|z - 1\right| \geq 0 + 0 = 1.
\]  

(2.67)

Thus, we have

\[
G^2(Tx, Sy, Rz) = \frac{4}{49} \leq \alpha \cdot 1 \cdot 0 + \beta \cdot 0 \cdot \frac{1}{7} + \gamma \cdot 1 \cdot \frac{1}{7} \\
\leq aG(x, Tx)G(y, Sy, Rz) + \beta G(y, Sy, Rz)G(z, Rz, Tx) + \gamma G(x, Tx, Sy)G(z, Rz, Tx).
\]  

(2.68)

Case 3. If \( x \in (1/2, 1], \ y \in [0, 1/2], \ z \in [0, 1] \), then we have

\[
G^2(Tx, Sy, Rz) = G^2\left(\frac{6}{7}, \frac{6}{8}, \frac{6}{7}\right) = \frac{1}{784}, \\
G(x, Tx, Sy) = G\left(x, \frac{6}{7}, \frac{7}{8}\right) = \left|x - \frac{6}{7}\right| + \left|x - \frac{7}{8}\right| + \frac{1}{56} \geq 0 + 0 + \frac{1}{56} = \frac{1}{56}, \\
G(y, Sy, Rz) = G\left(y, \frac{7}{8}, \frac{6}{7}\right) = \left|y - \frac{7}{8}\right| + \left|y - \frac{6}{7}\right| + \frac{1}{56} \geq \frac{3}{8} + \frac{5}{14} + \frac{1}{56} = \frac{3}{4}, \\
G(z, Rz, Tx) = G\left(z, \frac{6}{7}, \frac{6}{8}\right) = \left|z - \frac{6}{7}\right| + \left|z - \frac{6}{8}\right| \geq 0 + 0 = 0.
\]  

(2.69)
Thus, we have

\[ G^2(Tx, Sy, Rz) = \frac{1}{784} \leq \alpha \cdot \frac{3}{56} + \beta \cdot \frac{3}{4} + \gamma \cdot \frac{1}{56} \]

\[ \leq \alpha G(x, Tx, Sy)G(y, Sy, Rz) + \beta G(y, Sy, Rz)G(z, Rz, Tx) + \gamma G(x, Tx, Sy)G(z, Rz, Tx). \]  

(2.70)

**Case 4.** If \( x, y \in (1/2, 1], z \in [0, 1] \), then we have

\[ G^2(Tx, Sy, Rz) = G^2(\frac{6}{7}, \frac{6}{7}, \frac{6}{7}) = 0. \]  

(2.71)

Thus, we have

\[ G^2(Tx, Sy, Rz) = 0 \]

\[ \leq \alpha G(x, Tx, Sy)G(y, Sy, Rz) + \beta G(y, Sy, Rz)G(z, Rz, Tx) + \gamma G(x, Tx, Sy)G(z, Rz, Tx). \]  

(2.72)

Then in all of the above cases, the mappings \( T, S, \) and \( R \) satisfy the contractive condition (2.33) of Theorem 2.5 with \( \alpha = 1/7, \beta = 1/7, \gamma = 4/7 \). So that all the conditions of Theorem 2.5 are satisfied. Moreover, \( 6/7 \) is the unique common fixed point for all of the three mappings \( T, S, \) and \( R \).

At last, we prove \( T, S, \) and \( R \) are all \( G \)-continuous at the common fixed point \( 6/7 \). Since \( 6/7 \in (1/2, 1], \) and let the sequence \( \{y_n\} \subset (0, 1] \) and \( y_n \to (6/7)(n \to \infty) \), then there exists \( N \in \mathbb{N} \) such that \( \{y_n\} \subset (1/2, 1] \), for all \( n > N \). Without loss of generality, we can assume that \( \{y_n\} \subset (1/2, 1] \), and so \( Ty_n = 6/7, \ Sy_n = 6/7 \) and \( Ry_n = 6/7 \). Therefore,

\[ \lim_{n \to \infty} Ty_n = \lim_{n \to \infty} Sy_n = \lim_{n \to \infty} Ry_n = \frac{6}{7}. \]  

(2.73)

Which implies that \( T, S, \) and \( R \) are all \( G \)-continuous at the common fixed point \( 6/7 \).

**Corollary 2.7.** Let \( (X, G) \) be a complete \( G \)-metric space. Suppose the three self-mappings \( T, S, R : X \to X \) satisfy the condition:

\[ G^2(T^px, S^py, R^rz) \leq \alpha G(x, T^px, S^py)G(y, S^py, R^rz) + \beta G(y, S^py, R^rz)G(z, R^rz, T^px) + \gamma G(x, T^px, S^py)G(z, R^rz, T^px), \]  

\[ (2.74) \]

for all \( x, y, z \in X \), where \( p, s, r \in \mathbb{N}, \alpha, \beta, \gamma \) are nonnegative real numbers and \( \alpha + \beta + \gamma < 1 \). Then \( T, S, \) and \( R \) have a unique common fixed point (say \( u \)) and \( T^p, S^s, R^r \) are all \( G \)-continuous at \( u \).

**Proof.** Since the proof of Corollary 2.7 is very similar to that of Corollary 2.2, so we delete it. \( \square \)
Corollary 2.8. Let \((X,G)\) be a complete \(G\)-metric space, and let \(T : X \rightarrow X\) be a self-mapping in \(X\), which satisfies the following condition:

\[
G^2(Tx, Ty, Tz) \leq \alpha G(x, Tx, Ty)G(y, Ty, Tz) + \beta G(y, Ty, Tz)G(z, Tz, Tx) + \gamma G(x, Tx, Ty)G(z, Tz, Tx).
\]

(2.75)

for all \(x, y, z \in X\), where \(\alpha, \beta, \gamma\) are nonnegative real numbers and \(\alpha + \beta + \gamma < 1\). Then \(T\) has a unique fixed point (say \(u\)) and \(T\) is \(G\)-continuous at \(u\).

Corollary 2.9. Let \((X,G)\) be a complete \(G\)-metric space, and let \(T : X \rightarrow X\) be a self-mapping in \(X\), which satisfies the following condition:

\[
G^2(T^p x, T^p y, T^p z) \leq \alpha G(x, T^p x, T^p y)G(y, T^p y, T^p z) + \beta G(y, T^p y, T^p z)G(z, T^p z, T^p x) + \gamma G(x, T^p x, T^p y)G(z, T^p z, T^p x).
\]

(2.76)

for all \(x, y, z \in X\), where \(p \in \mathbb{N}\), \(\alpha, \beta, \gamma\) are nonnegative real numbers and \(\alpha + \beta + \gamma < 1\). Then \(T\) has a unique fixed point (say \(u\)) and \(T^p\) is \(G\)-continuous at \(u\).

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References

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