Research Article

Asymptotic Behavior of Bifurcation Curve for Sine-Gordon-Type Differential Equation

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We consider the nonlinear eigenvalue problems for the equation $-u''(t) + \sin u(t) = \lambda u(t)$, $u(t) > 0$, $t \in I =: (0, 1)$, $u(0) = u(1) = 0$, where $\lambda > 0$ is a parameter. It is known that for a given $\xi > 0$, there exists a unique solution pair $(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+$ with $\|u_\xi\|_\infty = \xi$. We establish the precise asymptotic formulas for bifurcation curve $\lambda(\xi)$ as $\xi \rightarrow \infty$ and $\xi \rightarrow 0$ to see how the oscillation property of $\sin u$ has effect on the behavior of $\lambda(\xi)$. We also establish the precise asymptotic formula for bifurcation curve $\lambda(\alpha)$ ($\alpha = \|u_\lambda\|_2$) to show the difference between $\lambda(\xi)$ and $\lambda(\alpha)$.

1. Introduction

We consider the following nonlinear eigenvalue problem:

$$-u''(t) + \sin u(t) = \lambda u(t), \quad t \in I =: (0, 1), \quad (1.1)$$

$$u(t) > 0, \quad t \in I, \quad (1.2)$$

$$u(0) = u(1) = 0, \quad (1.3)$$

where $\lambda > 0$ is a parameter. This problem comes from sine-Gordon equation and has been investigated from a view point of bifurcation theory in L^∞ -framework. Indeed, by using implicit function theorem, it has been shown in [1] that for $\xi > 0$, there exists a continuous function $\lambda = \lambda(\xi)$ such that $(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3) with $\|u_\xi\|_\infty = \xi$. Moreover, the solution set of (1.1)–(1.3) is given by $\Gamma := \{(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+; \xi > 0\}$.

Furthermore, it is well known that $u_\xi(t) \sim \xi \sin \pi t$ for $\xi \gg 1$ and $0 < \xi \ll 1$. Therefore, we have

$$\lambda(\xi) \longrightarrow \pi^2 \quad (\xi \longrightarrow \infty), \quad (1.4)$$

$$\lambda(\xi) \longrightarrow \pi^2 + 1 \quad (\xi \longrightarrow 0). \quad (1.5)$$

Equations (1.1)–(1.3) are the special case of the following semilinear equation:

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I, \quad (1.6)$$

$$u(t) > 0, \quad t \in I, \quad (1.7)$$

$$u(0) = u(1) = 0. \quad (1.8)$$

The structures of the global behavior of the bifurcation curves of (1.6)–(1.8) have been studied by many authors in L^∞ -framework. We refer to [2–6] and the references therein. In particular, if $f(u)/u$ is strictly increasing as $u \rightarrow \infty$, then we know from [3] that $\lambda(\xi)$ is also strictly increasing for $\xi > 0$ and the asymptotic behavior of $\lambda(\xi)$ as $\xi \rightarrow \infty$ is mainly determined by $f(\xi)/\xi$. For example, if $f(u) = u^p$ ($p > 1$) in (1.6), then as $\xi \rightarrow \infty$ (cf. [7]),

$$\lambda(\xi) = \xi^{p-1} + O\left(e^{-\delta\sqrt{\xi}}\right), \quad (1.9)$$

where $\delta > 0$ is a constant. However, since $(\sin u)/u$ is not strictly increasing but oscillating as a function of $u \geq 0$, it is interesting to study whether the oscillation property of $\sin u$ has effect on the asymptotic shape of $\lambda(\xi)$ for $\xi > 0$ or not.

Motivated by this, we first establish the precise asymptotic formula for $\lambda(\xi)$ as $\xi \rightarrow \infty$.

Theorem 1.1. As $\xi \rightarrow \infty$,

$$\begin{aligned} \lambda(\xi) = & \pi^2 + 2\sqrt{\frac{2}{\pi}}\xi^{-3/2}\cos\left(\xi - \frac{3}{4}\pi\right) \\ & + 2\sqrt{\frac{2}{\pi}}\xi^{-5/2}\left\{-\frac{3}{8}\sin\left(\xi - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2}\cos\left(2\xi - \frac{1}{4}\pi\right) \right. \\ & \left. + \frac{1}{\pi^2}\cos\xi\cos\left(\xi - \frac{1}{4}\pi\right)\right\} + o\left(\xi^{-5/2}\right). \end{aligned} \quad (1.10)$$

The local behavior of $\lambda(\xi)$ as $\xi \rightarrow 0$ can be obtained formally by the method in [8]. However, it seems rather hard task to obtain the higher terms of the asymptotic expansion of $\lambda(\xi)$, since it is necessary to solve the equations derived from the asymptotic expansion of $\lambda(\xi)$ step by step.

Here, we introduce a simpler way on how to obtain the asymptotic expansion formula for $\lambda(\xi)$ as $\xi \rightarrow 0$.

Theorem 1.2. *Let an arbitrary integer $N > 0$ be fixed. Then as $\xi \rightarrow 0$,*

$$\lambda = \pi^2 + 1 - \frac{1}{8}\xi^2 + \frac{1}{192}\left(1 + \frac{1}{8\pi^2}\right)\xi^4 + \sum_{n=3}^N a_n \xi^{2n} + o(\xi^{2N}), \quad (1.11)$$

where $\{a_n\}$ ($n = 3, 4, \dots$) are the constants determined inductively.

Next, since (1.1)–(1.3) is regarded as an eigenvalue problem, we focus our attention on studying the structure of the solution set in L^2 -framework. Suppose that $f(u) = u^p$ ($p > 1$) in (1.6). Then we know from [9] that, for a given $\alpha > 0$, there exists a unique solution pair $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ of (1.6)–(1.8) satisfying $\|u_\alpha\|_2 = \alpha$. Furthermore, $\lambda(\alpha)$ is an increasing function of $\alpha > 0$ and as $\alpha \rightarrow \infty$,

$$\lambda(\alpha) = \alpha^{p-1} + C_0 \alpha^{(p-1)/2} + O(1). \quad (1.12)$$

We see from (1.9) and (1.12) the difference between the asymptotic formulas for $\lambda(\xi)$ and $\lambda(\alpha)$ when $f(u) = u^p$ in (1.6). We refer to [4, 7, 9] for the works in this direction.

Motivated by this, it seems interesting to compare the asymptotic behavior of $\lambda(\alpha)$ and $\lambda(\xi)$ of (1.1)–(1.3) when $\xi \gg 1$ and $\alpha \gg 1$.

Now we consider (1.1)–(1.3) in L^2 -framework. Let $\alpha > 0$ be a given constant. Assume that there exists a solution pair $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfying $\|u_\alpha\|_2 = \alpha$. Then, it is natural to expect that for $t \in \bar{I}$, as $\alpha \rightarrow \infty$,

$$\frac{u_\alpha(t)}{\alpha} \longrightarrow \sqrt{2} \sin \pi t. \quad (1.13)$$

Therefore, we expect that $\|u_\alpha\|_\infty \sim \sqrt{2} \|u_\alpha\|_2$ for $\alpha \gg 1$. To obtain the existence, we apply the variational method to our situation, namely, we consider the constrained minimization problem associated with (1.1)–(1.3). Let

$$M_\alpha := \left\{ v \in H_0^1(I) : \|v\|_2 = \alpha \right\}, \quad (1.14)$$

where $\|v\|_2$ is the usual L^2 -norm of v , $\alpha > 0$ is a parameter, and $H_0^1(I)$ is the usual real Sobolev space. Then consider the following minimizing problem, which depends on $\alpha > 0$:

$$\text{Minimize } K(v) := \frac{1}{2} \|v'\|_2^2 + \int_I (1 - \cos v(t)) dt \quad \text{under the constraint } v \in M_\alpha. \quad (1.15)$$

Let

$$\beta(\alpha) := \min_{v \in M_\alpha} K(v). \quad (1.16)$$

Then by Lagrange multiplier theorem, for a given $\alpha > 0$, there exists a pair $(u_\alpha, \lambda(\alpha)) \in M_\alpha \times \mathbb{R}_+$ which satisfies (1.1)–(1.3) with $K(u_\alpha) = \beta(\alpha)$. Here, $\lambda(\alpha)$, which is called the *variational*

eigenvalue, is the Lagrange multiplier. By this variational framework, we parameterize the solution (u, λ) of (1.1)–(1.3) by α , that is, $(u, \lambda) = (u_\alpha, \lambda(\alpha)) \in M_\alpha \times \mathbb{R}_+$. Then we know from the arguments in [10, 11] that $\lambda(\alpha)$ is continuous function for $0 < \alpha \ll 1$ and $\alpha \gg 1$. Our next aim is to study precisely the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \rightarrow \infty$.

Theorem 1.3. *As $\alpha \rightarrow \infty$*

$$\begin{aligned} \lambda(\alpha) = & \pi^2 + 2^{3/4} \pi^{-1/2} \alpha^{-3/2} \cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \\ & - \pi^{-3} \alpha^{-2} \sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \\ & + 2^{1/4} \pi^{-1/2} \alpha^{-5/2} \left\{ -\frac{3}{8} \sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2} \cos\left(2\sqrt{2}\alpha - \frac{1}{4}\pi\right) \right. \\ & \quad \left. + \frac{1}{\pi^2} \cos\left(\sqrt{2}\alpha\right) \cos\left(\sqrt{2}\alpha - \frac{1}{4}\pi\right) - \frac{1}{4} \pi^{-5} \cos^3\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \right\} \\ & + o\left(\alpha^{-5/2}\right). \end{aligned} \tag{1.17}$$

By Theorems 1.1 and 1.3, we clearly understand the difference between $\lambda(\xi)$ and $\lambda(\alpha)$.

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1.1. We prove Theorem 1.2 in Section 3. Section 4 is devoted to the proof of Theorem 1.3.

2. Proof of Theorem 1.1

In what follows, C denotes various positive constants independent of $\xi \gg 1$. We write $\lambda = \lambda(\xi)$ for simplicity. We know from [1] that if $(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3), then

$$u_\xi(t) = u_\xi(1-t), \quad 0 \leq t \leq 1, \tag{2.1}$$

$$u_\xi\left(\frac{1}{2}\right) = \max_{0 \leq t \leq 1} u_\xi(t) = \xi, \tag{2.2}$$

$$u'_\xi(t) > 0, \quad 0 \leq t < \frac{1}{2}. \tag{2.3}$$

By (1.1), for $t \in \bar{I}$,

$$\left[u''_\xi(t) + \lambda u_\xi(t) - \sin u_\xi(t) \right] u'_\xi(t) = 0. \tag{2.4}$$

This implies that for $t \in \bar{I}$,

$$\frac{d}{dt} \left[\frac{1}{2} u'_\xi(t)^2 + \frac{1}{2} \lambda u_\xi(t)^2 + \cos u_\xi(t) \right] = 0. \tag{2.5}$$

By this, (2.2) and putting $t = 1/2$, we obtain

$$\frac{1}{2}u'_\xi(t)^2 + \frac{1}{2}\lambda u_\xi(t)^2 + \cos u_\xi(t) \equiv \text{constant} = \frac{1}{2}\lambda\xi^2 + \cos \xi. \quad (2.6)$$

By this and (2.3), for $0 \leq t \leq 1/2$,

$$u'_\xi(t) = \sqrt{\lambda(\xi^2 - u_\xi(t)^2) + 2(\cos \xi - \cos u_\xi(t))}. \quad (2.7)$$

Then by putting $s = u_\xi(t)/\xi$, we obtain

$$\begin{aligned} \frac{1}{2} &= \int_0^{1/2} dt = \int_0^{1/2} \frac{u'_\xi(t)}{\sqrt{\lambda(\xi^2 - u_\xi(t)^2) + 2(\cos \xi - \cos u_\xi(t))}} dt \\ &= \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(\cos \xi - \cos \xi s)/(\lambda\xi^2)}} ds \\ &= \frac{1}{\sqrt{\lambda}} \left\{ \int_0^1 \frac{1}{\sqrt{1 - s^2}} ds + \left(\int_0^1 \frac{1}{\sqrt{1 - s^2 + B}} ds - \int_0^1 \frac{1}{\sqrt{1 - s^2}} ds \right) \right\} \\ &= \frac{1}{\sqrt{\lambda}} \left(\frac{\pi}{2} + V \right), \end{aligned} \quad (2.8)$$

where

$$V := - \int_0^1 \frac{B}{\sqrt{1 - s^2 + B}\sqrt{1 - s^2}(\sqrt{1 - s^2 + B} + \sqrt{1 - s^2})} ds, \quad (2.9)$$

$$B := \frac{2}{\lambda\xi^2}(\cos \xi - \cos \xi s). \quad (2.10)$$

We put

$$V_1 = - \frac{1}{\lambda\xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{(1 - s^2)^{3/2}} ds, \quad (2.11)$$

$$V_2 = V - V_1. \quad (2.12)$$

Lemma 2.1. For $\xi \gg 1$

$$\begin{aligned} V_1 &= \sqrt{\frac{\pi}{2}} \frac{1}{\lambda\xi^{3/2}} \left[\left(1 + \frac{15}{128\xi^2}(1 + o(1)) \right) \cos \left(\xi - \frac{3}{4}\pi \right) \right. \\ &\quad \left. - \frac{3}{8\xi}(1 + o(1)) \sin \left(\xi - \frac{3}{4}\pi \right) \right]. \end{aligned} \quad (2.13)$$

Proof. By putting $s = \sin \theta$ in (2.11), integration by parts and l'Hopital's rule,

$$\begin{aligned}
 -V_1 &= \frac{1}{\lambda \xi^2} \int_0^{\pi/2} \frac{1}{\cos^2 \theta} (\cos \xi - \cos(\xi \sin \theta)) d\theta \\
 &= \frac{1}{\lambda \xi^2} \int_0^{\pi/2} (\tan \theta)' (\cos \xi - \cos(\xi \sin \theta)) d\theta \\
 &= \frac{1}{\lambda \xi^2} \lim_{t \rightarrow \pi/2} [\tan t (\cos \xi - \cos(\xi \sin t))]_0^t \\
 &\quad - \frac{1}{\lambda \xi} \int_0^{\pi/2} \tan \theta \cos \theta \sin(\xi \sin \theta) d\theta \\
 &= -\frac{1}{\lambda \xi} \int_0^{\pi/2} \sin \theta \sin(\xi \sin \theta) d\theta.
 \end{aligned} \tag{2.14}$$

By [12, page 962],

$$\int_0^{\pi/2} \sin \theta \sin(\xi \sin \theta) d\theta = \frac{\pi}{2} J_1(\xi), \tag{2.15}$$

where $J_1(\xi)$ is Bessel function of the first kind. For $\xi \gg 1$, by [12, page 972], we have

$$\begin{aligned}
 J_1(\xi) &= \sqrt{\frac{2}{\pi \xi}} \left[\left(1 + \frac{15}{128 \xi^2} (1 + o(1)) \right) \cos \left(\xi - \frac{3}{4} \pi \right) \right. \\
 &\quad \left. - \frac{3}{8 \xi} (1 + o(1)) \sin \left(\xi - \frac{3}{4} \pi \right) \right].
 \end{aligned} \tag{2.16}$$

By this, (2.14) and (2.15), we obtain (2.13). Thus, the proof is complete. $\square$

Remark 2.2. Taking (1.4) into account, (2.13) is written as

$$\begin{aligned}
 V_1 &= 2^{-1/2} \pi^{-3/2} \xi^{-3/2} (1 + o(1)) \left[\left(1 + \frac{15}{128 \xi^2} (1 + o(1)) \right) \cos \left(\xi - \frac{3}{4} \pi \right) \right. \\
 &\quad \left. - \frac{3}{8 \xi} (1 + o(1)) \sin \left(\xi - \frac{3}{4} \pi \right) \right].
 \end{aligned} \tag{2.17}$$

After we obtain (2.31) later, then (2.13) will be improved in the form (2.32).

Lemma 2.3. For $\xi \gg 1$,

$$\begin{aligned}
 V_2 &= -2^{-1/2} \pi^{-7/2} (1 + o(1)) \xi^{-5/2} \left\{ \frac{1}{\sqrt{2}} \cos \left(2\xi - \frac{1}{4} \pi \right) - \cos \xi \cos \left(\xi - \frac{1}{4} \pi \right) \right\} \\
 &\quad + o(\xi^{-5/2}).
 \end{aligned} \tag{2.18}$$

Proof. For $\xi \gg 1$ and $0 \leq s \leq 1$, by mean value theorem,

$$|B| \leq C\xi^{-1}(1-s) \leq C\xi^{-1}(1-s^2). \quad (2.19)$$

By this and Lebesgue's convergence theorem, we have

$$\begin{aligned} V_2 &= -\frac{2}{\lambda\xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1-s^2}} \\ &\quad \times \left(\frac{1}{\sqrt{1-s^2+B}(\sqrt{1-s^2+B}+\sqrt{1-s^2})} - \frac{1}{\sqrt{1-s^2}(2\sqrt{1-s^2})} \right) ds \\ &= -(1+o(1)) \frac{2}{\lambda\xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1-s^2}} \\ &\quad \times \frac{2(1-s^2) - (1-s^2+B+\sqrt{1-s^2}\sqrt{1-s^2+B})}{\sqrt{1-s^2+B}(\sqrt{1-s^2+B}+\sqrt{1-s^2})\sqrt{1-s^2}2\sqrt{1-s^2}} ds \\ &= -(1+o(1)) \frac{2}{\lambda\xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1-s^2}} \cdot \frac{1-s^2-B-\sqrt{1-s^2}\sqrt{1-s^2+B}}{4(1-s^2)^2} ds \\ &= -(1+o(1)) \frac{1}{2\lambda\xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1-s^2}} \cdot \frac{(1-s^2-B)^2 - (1-s^2)(1-s^2+B)}{(1-s^2)^2[(1-s^2-B)+\sqrt{1-s^2}\sqrt{1-s^2+B}]} ds \\ &= \frac{3}{4}(1+o(1)) \frac{1}{\lambda\xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1-s^2}} \cdot \frac{(1-s^2)B}{(1-s^2)^3} ds \\ &= \frac{3}{2}(1+o(1)) \frac{1}{\lambda^2\xi^4} \int_0^1 \frac{(\cos \xi - \cos \xi s)^2}{(1-s^2)^{5/2}} ds \\ &= \frac{3}{2}(1+o(1)) \frac{1}{\lambda^2\xi^4} \int_0^{\pi/2} \frac{(\cos \xi - \cos(\xi \sin \theta))^2}{\cos^4 \theta} d\theta \\ &= \frac{3}{2}(1+o(1)) \frac{1}{\lambda^2\xi^4} V_3, \end{aligned} \quad (2.20)$$

where

$$V_3 := \int_0^{\pi/2} \frac{(\cos \xi - \cos(\xi \sin \theta))^2}{\cos^4 \theta} d\theta. \quad (2.21)$$

We know

$$\int \frac{1}{\cos^4 \theta} d\theta = \frac{1}{3} \sin \theta \left(\frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right). \quad (2.22)$$

Taking (2.22) into account and integration by parts in V_3 , we obtain that

$$\begin{aligned} V_3 &= \lim_{\theta \rightarrow \pi/2} \left[\frac{1}{3} \sin \theta \left(\frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right) (\cos \xi - \cos(\xi \sin \theta))^2 \right]_0^\theta \\ &\quad - \frac{2}{3} \xi \int_0^{\pi/2} \sin \theta \left(\frac{1}{\cos^2 \theta} + 2 \right) (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) d\theta \\ &:= \frac{1}{3} V_4 - \frac{2}{3} \xi (V_5 + V_6), \end{aligned} \quad (2.23)$$

where

$$V_4 := \lim_{\theta \rightarrow \pi/2} \sin \theta \left(\frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right) (\cos \xi - \cos(\xi \sin \theta))^2 \quad (2.24)$$

$$V_5 := \int_0^{\pi/2} \frac{\sin \theta}{\cos^2 \theta} (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) d\theta, \quad (2.25)$$

$$V_6 := 2 \int_0^{\pi/2} \sin \theta (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) d\theta. \quad (2.26)$$

Then by l'Hopital's rule,

$$\begin{aligned} V_4 &= \lim_{\theta \rightarrow \pi/2} \sin \theta \left(\frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right) (\cos \xi - \cos(\xi \sin \theta))^2 \\ &= \lim_{\theta \rightarrow \pi/2} \frac{(1 + 2 \cos^2 \theta) (\cos \theta - \cos(\xi \sin \theta))^2}{\cos^3 \theta} \\ &= \lim_{\theta \rightarrow \pi/2} \frac{(\cos \xi - \cos(\xi \sin \theta))^2}{\cos^3 \theta} \\ &= \lim_{\theta \rightarrow \pi/2} - \frac{2 \xi (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta)}{3 \cos \theta \sin \theta} \\ &= \lim_{\theta \rightarrow \pi/2} - \frac{2 \xi \sin \xi (\cos \xi - \cos(\xi \sin \theta))}{3 \cos \theta \sin \theta} \\ &= \lim_{\theta \rightarrow \pi/2} - \frac{2 \xi^2 \sin \xi \sin(\xi \sin \theta) \cos \theta}{3 \cos(2\theta)} = 0. \end{aligned} \quad (2.27)$$

We next calculate V_5 . We know from [12, pages 442 and 972] that for $z \gg 1$,

$$\begin{aligned} \int_0^{\pi/2} \cos(z \cos \theta) d\theta &= \frac{\pi}{2} J_0(z) \\ &= \sqrt{\frac{\pi}{2}} (1 + o(1)) z^{-1/2} \cos\left(z - \frac{1}{4}\pi\right), \end{aligned} \quad (2.28)$$

where $J_0(z)$ is Bessel function. Integration by parts in (2.25), applying the l'Hopital's rule, putting $\theta = \pi/2 - \eta$ and taking (2.28) into account, we obtain

$$\begin{aligned}
 V_5 &= \lim_{\theta \rightarrow \pi/2} \left[\frac{1}{\cos \theta} (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) \right]_0^\theta \\
 &\quad - \xi \int_0^{\pi/2} \left(\sin^2(\xi \sin \theta) + \cos \xi \cos(\xi \sin \theta) - \cos^2(\xi \sin \theta) \right) d\theta \\
 &= \xi \int_0^{\pi/2} \cos(2\xi \sin \theta) d\theta - \xi \cos \xi \int_0^{\pi/2} \cos(\xi \sin \theta) d\theta \\
 &= \xi \int_0^{\pi/2} \cos(2\xi \cos \eta) d\eta - \xi \cos \xi \int_0^{\pi/2} \cos(\xi \cos \eta) d\eta \\
 &= \sqrt{\frac{\pi}{2}} \xi^{1/2} (1 + o(1)) \left(\frac{1}{\sqrt{2}} \cos \left(2\xi - \frac{1}{4}\pi \right) - \cos \xi \cos \left(\xi - \frac{1}{4}\pi \right) \right).
 \end{aligned} \tag{2.29}$$

Clearly,

$$V_6 = O(1). \tag{2.30}$$

By (1.4), (2.20), (2.23), (2.27), (2.29), and (2.30), we obtain (2.18). Thus the proof is complete. $\square$

Proof of Theorem 1.1. By (2.8), Lemmas 2.1 and 2.3,

$$\lambda = \pi^2 + 4\pi V + 4V^2 = \pi^2 + 4\pi V_1 + O\left(\xi^{-5/2}\right) = \pi^2 + O\left(\xi^{-3/2}\right). \tag{2.31}$$

By this and Lemma 2.1,

$$\begin{aligned}
 V_1 &= \sqrt{\frac{\pi}{2}} \frac{1}{\xi^{3/2}} \left(\pi^2 + O(\xi^{-3/2}) \right)^{-1} \\
 &\quad \times \left(\cos \left(\xi - \frac{3}{4}\pi \right) - \frac{3}{8} (1 + o(1)) \xi^{-1} \sin \left(\xi - \frac{3}{4}\pi \right) + O(\xi^{-2}) \right) \\
 &= 2^{-1/2} \pi^{-3/2} \xi^{-3/2} \left(\cos \left(\xi - \frac{3}{4}\pi \right) - \frac{3}{8} \xi^{-1} \sin \left(\xi - \frac{3}{4}\pi \right) \right) + o(\xi^{-5/2}).
 \end{aligned} \tag{2.32}$$

By this, (2.31) and Lemmas 2.1 and 2.3,

$$\begin{aligned}
 \lambda &= \pi^2 + 4\pi(V_1 + V_2) + O(V^2) \\
 &= \pi^2 + 4\pi \left\{ 2^{-1/2} \pi^{-3/2} \xi^{-3/2} \left(\cos \left(\xi - \frac{3}{4}\pi \right) - \frac{3}{8} \xi^{-1} \sin \left(\xi - \frac{3}{4}\pi \right) \right) \right. \\
 &\quad \left. - 2^{-1/2} \pi^{-7/2} \xi^{-5/2} \left(\frac{1}{\sqrt{2}} \cos \left(2\xi - \frac{1}{4}\pi \right) - \cos \xi \cos \left(\xi - \frac{1}{4}\pi \right) \right) \right\} \\
 &\quad + o(\xi^{-5/2}).
 \end{aligned} \tag{2.33}$$

By this, we obtain (1.10). Thus, the proof is complete. $\square$

3. Proof of Theorem 1.2

We write $\lambda = \lambda(\xi)$ for simplicity. We prove (1.11) by showing the calculation to get a_2 . The argument to obtain a_n ($n \geq 3$) is the same as that to obtain a_2 . The argument in this section is a variant used in [11, Section 2]. By (2.8) and (2.10), we have

$$\frac{1}{2} = \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{1}{\sqrt{1-s^2+B}} ds. \tag{3.1}$$

Since $0 < \xi \ll 1$, by Taylor expansion, for $0 \leq s \leq 1$, we obtain

$$\cos \xi - \cos \xi s = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \xi^{2k} (1-s^{2k}). \tag{3.2}$$

By this and (3.1),

$$\sqrt{\lambda} = 2 \int_0^1 \frac{1}{\sqrt{1-s^2}} \left(1 + \frac{2}{\lambda \xi^2} \frac{1}{1-s^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \xi^{2k} (1-s^{2k}) \right)^{-1/2} ds. \tag{3.3}$$

By using this, direct calculation gives us Theorem 1.2. For completeness, we calculate (1.11) up to the third term.

Step 1. We have

$$1 + \frac{2}{\lambda \xi^2} \frac{1}{1-s^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \xi^{2k} (1-s^{2k}) = 1 - \frac{1}{\lambda} + \frac{1}{12\lambda} \frac{1-s^4}{1-s^2} \xi^2 + o(\xi^2). \tag{3.4}$$

By (3.3), (3.4), and Taylor expansion,

$$\begin{aligned}\sqrt{\lambda} &= 2 \int_0^1 \frac{1}{\sqrt{1-s^2}} \left(1 - \frac{1}{\lambda} + \frac{1}{12\lambda} (1+s^2) \xi^2 + o(\xi^2) \right)^{-1/2} ds \\ &= \frac{2\sqrt{\lambda}}{\sqrt{\lambda-1}} \int_0^1 \frac{1}{\sqrt{1-s^2}} \left(1 - \frac{1}{24(\lambda-1)} (1+s^2) \xi^2 + o(\xi^2) \right) ds.\end{aligned}\quad (3.5)$$

By this, (1.5) and direct calculation, we obtain

$$\sqrt{\lambda-1} = \pi - \frac{1}{16\pi} \xi^2 + o(\xi^2). \quad (3.6)$$

This implies

$$\lambda = \pi^2 + 1 - \frac{1}{8} \xi^2 + o(\xi^2). \quad (3.7)$$

Step 2. Now we calculate the third term of $\lambda(\xi)$. First, we note that

$$\int_0^1 \frac{1+s^2+s^4}{\sqrt{1-s^2}} ds = \frac{15}{16} \pi, \quad \int_0^1 \frac{(1+s^2)^2}{\sqrt{1-s^2}} ds = \frac{19}{16} \pi. \quad (3.8)$$

By this, (1.5), (3.3), (3.7), Taylor expansion, and the same calculation as that to obtain (3.5),

$$\begin{aligned}\sqrt{\lambda-1} &= 2 \int_0^1 \frac{1}{\sqrt{1-s^2}} \left\{ 1 - \frac{1}{2} \left(\frac{1}{12(\lambda-1)} (1+s^2) \xi^2 \right. \right. \\ &\quad \left. \left. - \frac{1}{360(\lambda-1)} (1+s^2+s^4) \xi^4 \right) + \frac{3}{8} \frac{1}{144(\lambda-1)^2} (1+s^2)^2 \xi^4 + o(\xi^4) \right\} ds \\ &= \pi - \frac{1}{16\pi} \xi^2 \left(1 + \frac{1}{8\pi^2} \xi^2 + o(\xi^2) \right) + \frac{1}{360\pi} \frac{15}{16} \xi^4 + \frac{1}{192\pi^3} \frac{19}{16} \xi^4 + o(\xi^4) \\ &= \pi - \frac{1}{16\pi} \xi^2 + \frac{1}{384\pi} \left(1 - \frac{5}{8\pi^2} \right) \xi^4 + o(\xi^4).\end{aligned}\quad (3.9)$$

By this, we obtain (1.11) up to the third term. Thus, the proof is complete.

4. Proof of Theorem 1.3

In this section, we assume that $\alpha \gg 1$. We write $\lambda = \lambda(\alpha)$ for simplicity. We consider the solution pair $(\lambda(\alpha), u_\alpha) \in \mathbb{R}_+ \times M_\alpha$. We obtain from the same argument as that in [10, Theorem 1.2] that

$$\frac{u_\alpha(t)}{\alpha} \longrightarrow \sqrt{2} \sin \pi t \quad (4.1)$$

uniformly on $[0, 1]$ as $\alpha \rightarrow \infty$. By this, we have

$$\|u_\alpha\|_\infty = \sqrt{2}\alpha(1 + o(1)). \quad (4.2)$$

Furthermore, by [13, Lemma 2.4], we see that $\beta(\alpha)$ is continuous for $\alpha > 0$. By multiplying u_α by (1.1) and integration by parts, we obtain

$$\lambda(\alpha)\alpha^2 = \|u'_\alpha\|_2^2 + \int_0^1 u_\alpha(t) \sin u_\alpha(t) dt. \quad (4.3)$$

By this and (1.16), for $\alpha \gg 1$,

$$\lambda(\alpha)\alpha^2 = 2\beta(\alpha) + \int_0^1 u_\alpha(t) \sin u_\alpha(t) dt - 2 \int_0^1 (1 - \cos u_\alpha(t)) dt. \quad (4.4)$$

This along with (4.1) implies that $\lambda(\alpha)$ is continuous for $\alpha \gg 1$.

Lemma 4.1. For $\alpha \gg 1$,

$$\|u_\alpha\|_\infty^2 = \left(1 - \frac{2}{\sqrt{\lambda}} \left(\frac{\pi}{4} + U\right)\right)^{-1} \alpha^2, \quad (4.5)$$

where

$$U = - \int_0^1 \frac{\sqrt{1-s^2}B}{\sqrt{1-s^2+B}(\sqrt{1-s^2+B} + \sqrt{1-s^2})} ds. \quad (4.6)$$

Proof. By (2.7), (2.10), and putting $\theta = u_\alpha$ and $s = \theta / \|u_\alpha\|_\infty$,

$$\begin{aligned}
 \|u_\alpha\|_\infty^2 - \alpha^2 &= 2 \int_0^{1/2} \frac{(\|u_\alpha\|_\infty^2 - u_\alpha(t)^2) u'_\alpha(t)}{\sqrt{\lambda(\|u_\alpha\|_\infty^2 - u_\alpha(t)^2) + 2(\cos \|u_\alpha\|_\infty - \cos u_\alpha(t))}} dt \\
 &= 2 \int_0^{\|u_\alpha\|_\infty} \frac{\|u_\alpha\|_\infty^2 - \theta^2}{\sqrt{\lambda(\|u_\alpha\|_\infty^2 - \theta^2) + 2(\cos \|u_\alpha\|_\infty - \cos \theta)}} d\theta \\
 &= 2 \frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda}} \int_0^1 \frac{1 - s^2}{\sqrt{1 - s^2 + B}} ds \\
 &= 2 \frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda}} \left[\int_0^1 \sqrt{1 - s^2} ds + \int_0^1 \left(\frac{1 - s^2}{\sqrt{1 - s^2 + B}} - \sqrt{1 - s^2} \right) ds \right] \\
 &= 2 \frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda}} \left(\frac{\pi}{4} + U \right).
 \end{aligned} \tag{4.7}$$

Now, the result follows easily from (4.7). Thus, the proof is complete. $\square$

Lemma 4.2. For $\alpha \gg 1$,

$$\|u_\alpha\|_\infty = \sqrt{2}\alpha - 2^{-3/4}\pi^{-5/2}\alpha^{-1/2} \cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) + o(\alpha^{-1/2}). \tag{4.8}$$

Proof. By (2.10) and (4.6),

$$|U| \leq \frac{C}{\lambda \|u_\alpha\|_\infty^2} \left| \int_0^1 \frac{\cos \|u_\alpha\|_\infty - \cos(\|u_\alpha\|_\infty s)}{\sqrt{1 - s^2}} ds \right| \leq C(\|u_\alpha\|_\infty^{-2}). \tag{4.9}$$

By this, (2.8), Lemma 2.1, and Taylor expansion,

$$\begin{aligned}
 1 - \frac{2}{\sqrt{\lambda}} \left(\frac{\pi}{4} + U \right) &= 1 - 2(\pi + 2V)^{-1} \left(\frac{\pi}{4} + U \right) \\
 &= \frac{1}{2} - \frac{2}{\pi} \left(U - \frac{V}{2}(1 + o(1)) \right) = \frac{1}{2} + \frac{1}{\pi} V(1 + o(1)).
 \end{aligned} \tag{4.10}$$

By this, (4.5), (2.12), (2.13), (2.18), Taylor expansion, and (4.2),

$$\begin{aligned}
 \|u_\alpha\|_\infty &= \left(\frac{1}{2} + \frac{1}{\pi} V(1 + o(1)) \right)^{-1/2} \alpha \\
 &= \sqrt{2} \left(1 - \frac{1}{\pi} V(1 + o(1)) \right) \alpha \\
 &= \sqrt{2} \alpha - 2^{-3/4} \pi^{-5/2} \alpha^{-1/2} \cos \left(\sqrt{2} \alpha - \frac{3}{4} \pi \right) + o(\alpha^{-1/2}).
 \end{aligned} \tag{4.11}$$

Thus, the proof is complete. $\square$

Proof of Theorem 1.3. By Lemma 4.2, we put

$$\begin{aligned}
 \|u_\alpha\|_\infty &= \sqrt{2} \alpha + A \alpha^{-1/2} + o(\alpha^{-1/2}), \\
 A &= -2^{-3/4} \pi^{-5/2} \cos \left(\sqrt{2} \alpha - \frac{3}{4} \pi \right).
 \end{aligned} \tag{4.12}$$

Then substitute (4.12) for (1.10) and use Taylor expansion to obtain

$$\begin{aligned}
 \lambda &= \pi^2 + 2 \sqrt{\frac{2}{\pi}} \left(\sqrt{2} \alpha + A \alpha^{-1/2} + o(\alpha^{-1/2}) \right)^{-3/2} \cos \left(\sqrt{2} \alpha + A \alpha^{-1/2} + o(\alpha^{-1/2}) - \frac{3}{4} \pi \right) \\
 &\quad + 2 \sqrt{\frac{2}{\pi}} (\sqrt{2} \alpha)^{-5/2} \left\{ -\frac{3}{8} \sin \left(\sqrt{2} \alpha - \frac{3}{4} \pi \right) - \frac{1}{\sqrt{2} \pi^2} \cos \left(2\sqrt{2} \alpha - \frac{1}{4} \pi \right) \right. \\
 &\quad \left. + \frac{1}{\pi^2} \cos(\sqrt{2} \alpha) \cos \left(\sqrt{2} \alpha - \frac{1}{4} \pi \right) \right\} + o(\alpha^{-5/2}) \\
 &= \pi^2 + 2^{3/4} \pi^{-1/2} \alpha^{-3/2} \left(1 + \frac{1}{\sqrt{2}} A \alpha^{-3/2} + o(\alpha^{-3/2}) \right)^{-3/2} \\
 &\quad \times \left\{ \cos \left(\sqrt{2} \alpha - \frac{3}{4} \pi \right) \cos \left(A \alpha^{-1/2} (1 + o(1)) \right) \right. \\
 &\quad \left. - \sin \left(\sqrt{2} \alpha - \frac{3}{4} \pi \right) \sin \left(A \alpha^{-1/2} (1 + o(1)) \right) \right\} \\
 &\quad + 2 \sqrt{\frac{2}{\pi}} (\sqrt{2} \alpha)^{-5/2} \left\{ -\frac{3}{8} \sin \left(\sqrt{2} \alpha - \frac{3}{4} \pi \right) - \frac{1}{\sqrt{2} \pi^2} \cos \left(2\sqrt{2} \alpha - \frac{1}{4} \pi \right) \right. \\
 &\quad \left. + \frac{1}{\pi^2} \cos(\sqrt{2} \alpha) \cos \left(\sqrt{2} \alpha - \frac{1}{4} \pi \right) \right\} + o(\alpha^{-5/2})
 \end{aligned}$$

$$\begin{aligned}
&= \pi^2 + 2^{3/4} \pi^{-1/2} \alpha^{-3/2} \left(1 - \frac{3}{2\sqrt{2}} A \alpha^{-3/2} + o(\alpha^{-3/2}) \right) \\
&\quad \times \left\{ \left(1 - \frac{1}{2} A^2 \alpha^{-1} (1 + o(1)) \right) \cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \right. \\
&\quad \left. - A \alpha^{-1/2} (1 + o(1)) \sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \right\} \\
&\quad + 2^{1/4} \pi^{-1/2} \alpha^{-5/2} \left\{ -\frac{3}{8} \sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2} \cos\left(2\sqrt{2}\alpha - \frac{1}{4}\pi\right) \right. \\
&\quad \left. + \frac{1}{\pi^2} \cos(\sqrt{2}\alpha) \cos\left(\sqrt{2}\alpha - \frac{1}{4}\pi\right) \right\} + o(\alpha^{-5/2}) \\
&= \pi^2 + 2^{3/4} \pi^{-1/2} \alpha^{-3/2} \cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \\
&\quad - \pi^{-3} \alpha^{-2} \sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \cos\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \\
&\quad + 2^{1/4} \pi^{-1/2} \alpha^{-5/2} \left\{ -\frac{3}{8} \sin\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) - \frac{1}{\sqrt{2}\pi^2} \cos\left(2\sqrt{2}\alpha - \frac{1}{4}\pi\right) \right. \\
&\quad \left. + \frac{1}{\pi^2} \cos(\sqrt{2}\alpha) \cos\left(\sqrt{2}\alpha - \frac{1}{4}\pi\right) - \frac{1}{4} \pi^{-5} \cos^3\left(\sqrt{2}\alpha - \frac{3}{4}\pi\right) \right\} \\
&\quad + o(\alpha^{-5/2}).
\end{aligned} \tag{4.13}$$

Thus, we obtain (1.17) and the proof is complete. $\square$

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