Research Article

Asymptotic Behavior of Bifurcation Curve for Sine-Gordon-Type Differential Equation

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We consider the nonlinear eigenvalue problems for the equation

\[-u''(t) + \sin u(t) = \lambda u(t), \quad t \in I = (0,1),\]

\[u(t) > 0, \quad t \in I,\]

\[u(0) = u(1) = 0,\]

where \(\lambda > 0\) is a parameter. This problem comes from sine-Gordon equation and has been investigated from a viewpoint of bifurcation theory in \(L^{\infty}\)-framework. Indeed, by using implicit function theorem, it has been shown in [1] that for \(\xi > 0\), there exists a continuous function \(\lambda = \lambda(\xi)\) such that \((u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+\) satisfies (1.1)–(1.3) with \(\|u_\xi\|_{\infty} = \xi\). Moreover, the solution set of of (1.1)–(1.3) is given by \(\Gamma := \{(u_\xi, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+; \xi > 0\}\).
Furthermore, it is well known that \( u_\xi(t) \sim \xi \sin \pi t \) for \( \xi \gg 1 \) and \( 0 < \xi \ll 1 \). Therefore, we have

\[
\lambda(\xi) \rightarrow \pi^2 \quad (\xi \rightarrow \infty),
\]

\[
\lambda(\xi) \rightarrow \pi^2 + 1 \quad (\xi \rightarrow 0).
\]

Equations (1.1)–(1.3) are the special case of the following semilinear equation:

\[
-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I,
\]

\[
u(t) > 0, \quad t \in I,
\]

\[
u(0) = u(1) = 0.
\]

The structures of the global behavior of the bifurcation curves of (1.6)–(1.8) have been studied by many authors in \( L^\infty \)-framework. We refer to [2–6] and the references therein. In particular, if \( f(u)/u \) is strictly increasing as \( u \rightarrow \infty \), then we know from [3] that \( \lambda(\xi) \) is also strictly increasing for \( \xi > 0 \) and the asymptotic behavior of \( \lambda(\xi) \) as \( \xi \rightarrow \infty \) is mainly determined by \( f(\xi)/\xi \). For example, if \( f(u) = u^p \) (\( p > 1 \)) in (1.6), then as \( \xi \rightarrow \infty \) (cf. [7]),

\[
\lambda(\xi) = \xi^{p-1} + O\left( e^{-\delta \sqrt{\xi}} \right),
\]

where \( \delta > 0 \) is a constant. However, since \((\sin u)/u\) is not strictly increasing but oscillating as a function of \( u \geq 0 \), it is interesting to study whether the oscillation property of \( \sin u \) has effect on the asymptotic shape of \( \lambda(\xi) \) for \( \xi > 0 \) or not.

Motivated by this, we first establish the precise asymptotic formula for \( \lambda(\xi) \) as \( \xi \rightarrow \infty \).

**Theorem 1.1.** As \( \xi \rightarrow \infty \),

\[
\lambda(\xi) = \pi^2 + 2\sqrt{\frac{2}{\pi}} \xi^{-3/2} \cos\left( \frac{\xi - 3}{4} \pi \right)
\]

\[
+ 2\sqrt{\frac{2}{\pi}} \xi^{-5/2} \left\{ -\frac{3}{8} \sin\left( \frac{\xi - 3}{4} \pi \right) - \frac{1}{\sqrt{2} \pi^2} \cos\left( 2\xi - \frac{1}{4} \pi \right) \right\}
\]

\[
+ \frac{1}{\pi^2} \cos \xi \cos\left( \frac{\xi - 1}{4} \pi \right) + o(\xi^{-5/2}).
\]

The local behavior of \( \lambda(\xi) \) as \( \xi \rightarrow 0 \) can be obtained formally by the method in [8]. However, it seems rather hard task to obtain the higher terms of the asymptotic expansion of \( \lambda(\xi) \), since it is necessary to solve the equations derived from the asymptotic expansion of \( \lambda(\xi) \) step by step.

Here, we introduce a simpler way on how to obtain the asymptotic expansion formula for \( \lambda(\xi) \) as \( \xi \rightarrow 0 \).
Let an arbitrary integer \( N > 0 \) be fixed. Then as \( \xi \rightarrow 0 \),

\[
\lambda = \frac{\pi^2}{1 - \frac{1}{8} \xi^2} + \frac{1}{192} \left( 1 + \frac{1}{8 \pi^2} \right) \xi^4 + \sum_{n=3}^{N} a_n \xi^{2n} + o(\xi^{2N}),
\]

(1.11)

where \( \{ a_n \} \ (n = 3, 4, \ldots) \) are the constants determined inductively.

Next, since (1.1)–(1.3) is regarded as an eigenvalue problem, we focus our attention on studying the structure of the solution set in \( L^2 \)-framework. Suppose that \( f(u) = u^p(p > 1) \) in (1.6). Then we know from [9] that, for a given \( \alpha > 0 \), there exists a unique solution pair \((u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+ \) of (1.6)–(1.8) satisfying \( \|u_\alpha\|_2 = \alpha \). Furthermore, \( \lambda(\alpha) \) is an increasing function of \( \alpha > 0 \) and as \( \alpha \rightarrow \infty \),

\[
\lambda(\alpha) = \alpha^{p-1} + C_0 \alpha^{(p-1)/2} + O(1).
\]

(1.12)

We see from (1.9) and (1.12) the difference between the asymptotic formulas for \( \lambda(\xi) \) and \( \lambda(\alpha) \) when \( f(u) = u^p \) in (1.6). We refer to [4, 7, 9] for the works in this direction.

Motivated by this, it seems interesting to compare the asymptotic behavior of \( \lambda(\alpha) \) and \( \lambda(\xi) \) of (1.1)–(1.3) when \( \xi \gg 1 \) and \( \alpha \gg 1 \).

Now we consider (1.1)–(1.3) in \( L^2 \)-framework. Let \( \alpha > 0 \) be a given constant. Assume that there exists a solution pair \((u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+ \) satisfying \( \|u_\alpha\|_2 = \alpha \). Then, it is natural to expect that for \( t \in \bar{I} \), as \( \alpha \rightarrow \infty \),

\[
\frac{u_\alpha(t)}{\alpha} \rightarrow \sqrt{2} \sin \pi t.
\]

(1.13)

Therefore, we expect that \( \|u_\alpha\|_\infty \sim \sqrt{2} \|u_\alpha\|_2 \) for \( \alpha \gg 1 \). To obtain the existence, we apply the variational method to our situation, namely, we consider the constrained minimization problem associated with (1.1)–(1.3). Let

\[
M_\alpha := \left\{ v \in H_0^1(I) : \|v\|_2 = \alpha \right\},
\]

(1.14)

where \( \|v\|_2 \) is the usual \( L^2 \)-norm of \( v \), \( \alpha > 0 \) is a parameter, and \( H_0^1(I) \) is the usual real Sobolev space. Then consider the following minimizing problem, which depends on \( \alpha > 0 \):

Minimize \( K(v) := \frac{1}{2} \|v\|_2^2 + \int_I (1 - \cos v(t)) \, dt \) under the constraint \( v \in M_\alpha \).

(1.15)

Let

\[
\beta(\alpha) := \min_{v \in M_\alpha} K(v).
\]

(1.16)

Then by Lagrange multiplier theorem, for a given \( \alpha > 0 \), there exists a pair \((u_\alpha, \lambda(\alpha)) \in M_\alpha \times \mathbb{R}_+ \) which satisfies (1.1)–(1.3) with \( K(u_\alpha) = \beta(\alpha) \). Here, \( \lambda(\alpha) \), which is called the \textit{variational
is the Lagrange multiplier. By this variational framework, we parameterize the solution \((u, \lambda)\) of (1.1)–(1.3) by \(\alpha\), that is, \((u, \lambda) = (u_{\alpha}, \lambda(\alpha))\) \(\in M_{\alpha} \times \mathbb{R}_+\). Then we know from the arguments in [10, 11] that \(\lambda(\alpha)\) is continuous function for \(0 < \alpha < 1\) and \(\alpha \gg 1\). Our next aim is to study precisely the asymptotic behavior of \(\lambda(\alpha)\) as \(\alpha \to \infty\).

**Theorem 1.3.** As \(\alpha \to \infty\)

\[
\lambda(\alpha) = \pi^2 + 2^{3/4}\pi^{-1/2}\alpha^{-3/2}\cos(\sqrt{2}\alpha - \frac{3}{4}\pi) \\
- \pi^{-3}\alpha^{-2}\sin(\sqrt{2}\alpha - \frac{3}{4}\pi)\cos(\sqrt{2}\alpha - \frac{3}{4}\pi) \\
+ 2^{1/4}\pi^{-1/2}\alpha^{-5/2}\left\{-\frac{3}{8}\sin(\sqrt{2}\alpha - \frac{3}{4}\pi) - \frac{1}{\sqrt{2}\pi^2}\cos(2\sqrt{2}\alpha - \frac{1}{4}\pi) \\
+ \frac{1}{\pi^2}\cos(\sqrt{2}\alpha)\cos(\sqrt{2}\alpha - \frac{1}{4}\pi) - \frac{1}{4}\pi^{-5}\cos^3(\sqrt{2}\alpha - \frac{3}{4}\pi)\right\} \\
+ o(\alpha^{-5/2}).
\]

(1.17)

By Theorems 1.1 and 1.3, we clearly understand the difference between \(\lambda(\xi)\) and \(\lambda(\alpha)\).

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1.1. We prove Theorem 1.2 in Section 3. Section 4 is devoted to the proof of Theorem 1.3.

**2. Proof of Theorem 1.1**

In what follows, \(C\) denotes various positive constants independent of \(\xi \gg 1\). We write \(\lambda = \lambda(\xi)\) for simplicity. We know from [1] that if \((u_{\xi}, \lambda(\xi)) \in C^2(\bar{I}) \times \mathbb{R}_+\) satisfies (1.1)–(1.3), then

\[
u_{\xi}(t) = u_{\xi}(1-t), \quad 0 \leq t \leq 1,
\]

(2.1)

\[
u_{\xi}\left(\frac{1}{2}\right) = \max_{0 \leq t \leq 1} u_{\xi}(t) = \xi,
\]

(2.2)

\[
u_{\xi}'(t) > 0, \quad 0 \leq t < \frac{1}{2}.
\]

(2.3)

By (1.1), for \(t \in \bar{I}\),

\[
\left[u_{\xi}'(t) + \lambda u_{\xi}(t) - \sin u_{\xi}(t)\right]u_{\xi}'(t) = 0.
\]

(2.4)

This implies that for \(t \in \bar{I}\),

\[
\frac{d}{dt}\left[\frac{1}{2}u_{\xi}'(t)^2 + \frac{1}{2}\lambda u_{\xi}(t)^2 + \cos u_{\xi}(t)\right] = 0.
\]

(2.5)
By this, (2.2) and putting $t = 1/2$, we obtain
\[ \frac{1}{2} u'_s(t)^2 + \frac{1}{2} \lambda u_s(t)^2 + \cos u(t) \equiv \text{constant} = \frac{1}{2} \lambda \xi^2 + \cos \xi. \] (2.6)

By this and (2.3), for $0 \leq t \leq 1/2$,
\[ u'_s(t) = \sqrt{\lambda \left( \xi^2 - u_s(t)^2 \right) + 2 \left( \cos \xi - \cos u(t) \right)}. \] (2.7)

Then by putting $s = u_s(t)/\xi$, we obtain
\[
\frac{1}{2} = \int_{0}^{1/2} dt = \int_{0}^{1/2} \frac{u'_s(t)}{\sqrt{\lambda \left( \xi^2 - u_s(t)^2 \right) + 2 \left( \cos \xi - \cos u(t) \right)}} dt
\]
\[ = \frac{1}{\sqrt{\lambda}} \int_{0}^{1} \frac{1}{\sqrt{1 - s^2 + 2 \left( \cos \xi - \cos \xi s \right) / (\lambda \xi^2)}} ds \] (2.8)
\[ = \frac{1}{\sqrt{\lambda}} \left[ \int_{0}^{1} \frac{1}{\sqrt{1 - s^2}} ds + \left( \int_{0}^{1} \frac{1}{\sqrt{1 - s^2 + B}} ds - \int_{0}^{1} \frac{1}{\sqrt{1 - s^2}} ds \right) \right]
\[ = \frac{1}{\sqrt{\lambda}} \left( \frac{\pi}{2} + V \right), \]

where
\[ V := -\int_{0}^{1} B \sqrt{1 - s^2 + \sqrt{1 - s^2 \left( \sqrt{1 - s^2 + B} + \sqrt{1 - s^2} \right)}} ds, \] (2.9)
\[ B := \frac{2}{\lambda \xi^2} \left( \cos \xi - \cos \xi s \right). \] (2.10)

We put
\[ V_1 = -\frac{1}{\lambda \xi^2} \int_{0}^{1} \frac{\cos \xi - \cos \xi s}{(1 - s^2)^{3/2}} ds, \] (2.11)
\[ V_2 = V - V_1. \] (2.12)

**Lemma 2.1.** For $\xi \gg 1$
\[ V_1 = \sqrt{\frac{\pi}{2}} \frac{1}{\lambda \xi^{3/2}} \left[ \left( 1 + \frac{15}{128 \xi^2} (1 + o(1)) \right) \cos \left( \xi - \frac{3}{4} \pi \right) - \frac{3}{8 \xi} (1 + o(1)) \sin \left( \xi - \frac{3}{4} \pi \right) \right]. \] (2.13)
Proof. By putting \( s = \sin \theta \) in (2.11), integration by parts and l'Hopital's rule,

\[
-V_1 = \frac{1}{\lambda^2} \int_0^{\pi/2} \frac{1}{\cos^2 \theta} (\cos \xi - \cos(\xi \sin \theta)) d\theta \\
= \frac{1}{\lambda^2} \int_0^{\pi/2} (\tan \theta)'(\cos \xi - \cos(\xi \sin \theta)) d\theta \\
= \frac{1}{\lambda^2} \lim_{\lambda \to \pi/2} \left[ \tan t(\cos \xi - \cos(\xi \sin t)) \right]_0^t \\
- \frac{1}{\lambda^2} \int_0^{\pi/2} \tan \theta \cos \theta \sin(\xi \sin \theta) d\theta \\
= -\frac{1}{\lambda^2} \int_0^{\pi/2} \sin \theta \sin(\xi \sin \theta) d\theta.
\]

By [12, page 962],

\[
\int_0^{\pi/2} \sin \theta \sin(\xi \sin \theta) d\theta = \frac{\pi}{2} J_1(\xi),
\]

where \( J_1(\xi) \) is Bessel function of the first kind. For \( \xi \gg 1 \), by [12, page 972], we have

\[
J_1(\xi) = \sqrt{\frac{2}{\pi \xi}} \left[ \left( 1 + \frac{15}{128 \xi^2} (1 + o(1)) \right) \cos \left( \xi - \frac{3}{4} \pi \right) \\
- \frac{3}{8 \xi} (1 + o(1)) \sin \left( \xi - \frac{3}{4} \pi \right) \right].
\]

By this, (2.14) and (2.15), we obtain (2.13). Thus, the proof is complete. \( \square \)

Remark 2.2. Taking (1.4) into account, (2.13) is written as

\[
V_1 = 2^{-1/2} \pi^{-3/2} \xi^{-3/2} (1 + o(1)) \left[ \left( 1 + \frac{15}{128 \xi^2} (1 + o(1)) \right) \cos \left( \xi - \frac{3}{4} \pi \right) \\
- \frac{3}{8 \xi} (1 + o(1)) \sin \left( \xi - \frac{3}{4} \pi \right) \right].
\]

After we obtain (2.31) later, then (2.13) will be improved in the form (2.32).

Lemma 2.3. For \( \xi \gg 1 \),

\[
V_2 = -2^{-1/2} \pi^{-7/2} (1 + o(1)) \xi^{-5/2} \left\{ \frac{1}{\sqrt{2}} \cos \left( 2 \xi - \frac{1}{4} \pi \right) - \cos \xi \cos \left( \xi - \frac{1}{4} \pi \right) \right\} \\
+ o(\xi^{-5/2}).
\]


Proof. For $\xi \gg 1$ and $0 \leq s \leq 1$, by mean value theorem,

$$|B| \leq C \xi^{-1} (1 - s) \leq C \xi^{-1} \left(1 - s^2\right).$$  \hspace{1cm} (2.19)

By this and Lebesgue’s convergence theorem, we have

$$V_2 = \frac{-2}{\lambda \xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1 - s^2}} \times \left(\frac{1}{\sqrt{1 - s^2 + B (\sqrt{1 - s^2} + \sqrt{1 - s^2})}} - \frac{1}{\sqrt{1 - s^2 (2\sqrt{1 - s^2})}}\right) ds$$

$$= -(1 + o(1)) \frac{2}{\lambda \xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1 - s^2}} \times \frac{2(1 - s^2) - (1 - s^2 + B + \sqrt{1 - s^2} \sqrt{1 - s^2 + B})}{\sqrt{1 - s^2 + B (\sqrt{1 - s^2} + \sqrt{1 - s^2})} \sqrt{1 - s^2 \sqrt{1 - s^2}}} ds$$

$$= -(1 + o(1)) \frac{2}{\lambda \xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1 - s^2}} \cdot \frac{1 - s^2 - B - \sqrt{1 - s^2} \sqrt{1 - s^2 + B}}{4(1 - s^2)^2} ds$$  \hspace{1cm} (2.20)

$$= -(1 + o(1)) \frac{1}{\lambda \xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1 - s^2}} \cdot \frac{(1 - s^2 - B)^2 - (1 - s^2)(1 - s^2 + B)}{4(1 - s^2)^2 (1 - s^2 - B) + \sqrt{1 - s^2} \sqrt{1 - s^2 + B}} ds$$

$$= \frac{3}{4} (1 + o(1)) \frac{1}{\lambda \xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{\sqrt{1 - s^2}} \cdot \frac{(1 - s^2) B}{(1 - s^2)^3} ds$$

$$= \frac{3}{2} (1 + o(1)) \frac{1}{\lambda \xi^2} \int_0^1 \frac{\cos \xi - \cos \xi s}{(1 - s^2)^{3/2}} ds$$

$$= \frac{3}{2} (1 + o(1)) \frac{1}{\lambda \xi^2} \int_0^{\pi/2} \frac{(\cos \xi - \cos (\xi \sin \theta))^2}{\cos^3 \theta} d\theta$$

$$= \frac{3}{2} (1 + o(1)) \frac{1}{\lambda \xi^2} V_3,$$

where

$$V_3 := \int_0^{\pi/2} \frac{(\cos \xi - \cos (\xi \sin \theta))^2}{\cos^3 \theta} d\theta. \hspace{1cm} (2.21)$$

We know

$$\int \frac{1}{\cos^4 \theta} d\theta = \frac{1}{3} \sin \theta \left( \frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right). \hspace{1cm} (2.22)$$
Taking (2.22) into account and integration by parts in $V_5$, we obtain that

\[
V_5 = \lim_{\theta \to \pi/2} \left[ \frac{1}{3} \sin \theta \left( \frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right) (\cos \xi - \cos(\xi \sin \theta))^2 \right]_0^\theta \\
- \frac{2}{3} \int_0^{\pi/2} \sin \theta \left( \frac{1}{\cos^2 \theta} + 2 \right) (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) d\theta
\]

(2.23)

\[
:= \frac{1}{3} V_4 - \frac{2}{3} \xi (V_5 + V_6),
\]

where

\[
V_4 := \lim_{\theta \to \pi/2} \sin \theta \left( \frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right) (\cos \xi - \cos(\xi \sin \theta))^2
\]

(2.24)

\[
V_5 := \int_0^{\pi/2} \frac{\sin \theta}{\cos^2 \theta} (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) d\theta,
\]

(2.25)

\[
V_6 := 2 \int_0^{\pi/2} \sin \theta (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) d\theta.
\]

(2.26)

Then by l’Hôpital’s rule,

\[
V_4 = \lim_{\theta \to \pi/2} \sin \theta \left( \frac{1}{\cos^3 \theta} + \frac{2}{\cos \theta} \right) (\cos \xi - \cos(\xi \sin \theta))^2
\]

\[
= \lim_{\theta \to \pi/2} \frac{(1 + 2 \cos^2 \theta)(\cos \theta - \cos(\xi \sin \theta))^2}{\cos^3 \theta}
\]

\[
= \lim_{\theta \to \pi/2} \frac{(\cos \xi - \cos(\xi \sin \theta))^2}{\cos^3 \theta}
\]

\[
= \lim_{\theta \to \pi/2} -\frac{2 \xi (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta)}{3 \cos \theta \sin \theta}
\]

\[
= \lim_{\theta \to \pi/2} -\frac{2 \xi^2 \sin(\xi \sin \theta) \cos \theta}{3 \cos(2\theta)} = 0.
\]

(2.27)

We next calculate $V_5$. We know from [12, pages 442 and 972] that for $z \gg 1$,

\[
\int_0^{\pi/2} \cos(\xi \sin \theta) d\theta = \frac{\pi}{2} J_0(z)
\]

(2.28)

\[
= \sqrt{\frac{\pi}{2}} (1 + o(1)) z^{-1/2} \cos \left( z - \frac{1}{4} \pi \right),
\]
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where \( J_0(z) \) is Bessel function. Integration by parts in (2.25), applying the l’Hopital’s rule, putting \( \theta = \pi/2 - \eta \) and taking (2.28) into account, we obtain

\[
V_5 = \lim_{\theta \to \pi/2} \left[ \frac{1}{\cos \theta} (\cos \xi - \cos(\xi \sin \theta)) \sin(\xi \sin \theta) \right]_0^\theta \\
- \xi \int_0^{\pi/2} \left( \sin^2(\xi \sin \theta) + \cos \xi \cos(\xi \sin \theta) - \cos^2(\xi \sin \theta) \right) d\theta \\
= \xi \int_0^{\pi/2} \cos(2\xi \sin \theta) d\theta - \xi \cos \xi \int_0^{\pi/2} \cos(\xi \sin \theta) d\theta \\
= \xi \int_0^{\pi/2} \cos(2\xi \eta) d\eta - \xi \cos \xi \int_0^{\pi/2} \cos(\xi \eta) d\eta \\
= \sqrt{\frac{\pi}{2}} \xi^{1/2} (1 + o(1)) \left( \frac{1}{\sqrt{\xi}} \cos \left( \frac{2\xi - \frac{1}{4} \pi}{\xi} \right) - \cos \xi \cos \left( \frac{1}{4} \pi \right) \right).
\]

Clearly,

\[
V_6 = O(1). \tag{2.30}
\]

By (1.4), (2.20), (2.23), (2.27), (2.29), and (2.30), we obtain (2.18). Thus the proof is complete.

\( \square \)

\textit{Proof of Theorem 1.1.} By (2.8), Lemmas 2.1 and 2.3,

\[
\lambda = \pi^2 + 4\pi V + 4V^2 = \pi^2 + 4\pi V_1 + O(\xi^{-5/2}) = \pi^2 + O(\xi^{-3/2}). \tag{2.31}
\]

By this and Lemma 2.1,

\[
V_1 = \sqrt{\frac{\pi}{2}} \xi^{3/2} \left( \pi^2 + O(\xi^{-3/2}) \right)^{-1} \\
\times \left( \cos(\xi - \frac{3}{4} \pi) - \frac{3}{8} (1 + o(1)) \xi^{-1} \sin(\xi - \frac{3}{4} \pi) + O(\xi^{-2}) \right) \\
= 2^{-1/2} \pi^{-3/2} \xi^{-3/2} \left( \cos(\xi - \frac{3}{4} \pi) - \frac{3}{8} \xi^{-1} \sin(\xi - \frac{3}{4} \pi) \right) + o(\xi^{-5/2}). \tag{2.32}
\]
By this, (2.31) and Lemmas 2.1 and 2.3,

\[ \lambda = \pi^2 + 4\pi(V_1 + V_2) + O(V^2) \]

\[ = \pi^2 + 4\pi \left[ 2^{-1/2}\pi^{-3/2}\xi^{-3/2} \left( \cos \left( \xi - \frac{3}{4}\pi \right) - \frac{3}{8}\xi^{-1} \sin \left( \xi - \frac{3}{4}\pi \right) \right) \right. \]

\[ - 2^{-1/2}\pi^{-7/2}\xi^{-3/2} \left( \frac{1}{\sqrt{2}} \cos \left( 2\xi - \frac{1}{4}\pi \right) - \cos \xi \cos \left( \xi - \frac{1}{4}\pi \right) \right) \]

\[ + o(\xi^{-5/2}). \]

By this, we obtain (1.10). Thus, the proof is complete. \(\square\)

3. Proof of Theorem 1.2

We write \(\lambda = \lambda(\xi)\) for simplicity. We prove (1.11) by showing the calculation to get \(a_2\). The argument to obtain \(a_n\) \((n \geq 3)\) is the same as that to obtain \(a_2\). The argument in this section is a variant used in [11, Section 2]. By (2.8) and (2.10), we have

\[ \frac{1}{2} = \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{1}{\sqrt{1 - s^2 + B}} ds. \]  

(3.1)

Since \(0 < \xi \ll 1\), by Taylor expansion, for \(0 \leq s \leq 1\), we obtain

\[ \cos \xi - \cos \xi s = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \xi^{2k} \left( 1 - s^{2k} \right). \]  

(3.2)

By this and (3.1),

\[ \sqrt{\lambda} = 2 \int_0^1 \frac{1}{\sqrt{1 - s^2}} \left( 1 + \frac{2}{\lambda \xi^2} \frac{1}{1 - s^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \xi^{2k} \left( 1 - s^{2k} \right) \right)^{-1/2} ds. \]  

(3.3)

By using this, direct calculation gives us Theorem 1.2. For completeness, we calculate (1.11) up to the third term.

**Step 1.** We have

\[ 1 + \frac{2}{\lambda \xi^2} \frac{1}{1 - s^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \xi^{2k} \left( 1 - s^{2k} \right) = 1 - \frac{1}{\lambda} + \frac{1}{12\lambda} \frac{1 - s^4}{1 - s^2} + o(\xi^2). \]  

(3.4)
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By (3.3), (3.4), and Taylor expansion,

\[ \sqrt{\lambda} = 2 \int_0^1 \frac{1}{\sqrt{1 - s^2}} \left( 1 - \frac{1}{\lambda} + \frac{1}{12\lambda} (1 + s^2) \xi^2 + o(\xi^2) \right)^{-1/2} ds \]

\[ = \frac{2\sqrt{\lambda}}{\sqrt{\lambda - 1}} \int_0^1 \frac{1}{\sqrt{1 - s^2}} \left( 1 - \frac{1}{24(\lambda - 1)} (1 + s^2) \xi^2 + o(\xi^2) \right) ds. \]  

(3.5)

By this, (1.5) and direct calculation, we obtain

\[ \sqrt{\lambda - 1} = \pi - \frac{1}{16\pi} \xi^2 + o(\xi^2). \]  

(3.6)

This implies

\[ \lambda = \pi^2 + 1 - \frac{1}{8} \xi^2 + o(\xi^2). \]  

(3.7)

**Step 2.** Now we calculate the third term of \( \lambda(\xi) \). First, we note that

\[ \int_0^1 \frac{1 + s^2 + s^4}{\sqrt{1 - s^2}} ds = \frac{15}{16\pi}, \quad \int_0^1 (1 + s^2)^2 ds = \frac{19}{16\pi}. \]  

(3.8)

By this, (1.5), (3.3), (3.7), Taylor expansion, and the same calculation as that to obtain (3.5),

\[ \sqrt{\lambda - 1} = 2 \int_0^1 \frac{1}{\sqrt{1 - s^2}} \left\{ 1 - \frac{1}{2} \left( \frac{1}{12(\lambda - 1)} (1 + s^2) \xi^2 \right) - \frac{1}{360(\lambda - 1)} (1 + s^2 + s^4) \xi^2 \right\} ds \]

\[ = \pi - \frac{1}{16\pi} \xi^2 \left( 1 + \frac{1}{8\pi^2} \xi^2 + o(\xi^2) \right) + \frac{1}{360\pi} \frac{15}{16} \xi^4 + \frac{1}{192\pi^3} \frac{19}{16} \xi^4 + o(\xi^4) \]

\[ = \pi - \frac{1}{16\pi} \xi^2 + \frac{1}{384\pi} \left( 1 - \frac{5}{8\pi^2} \right) \xi^4 + o(\xi^4). \]  

(3.9)

By this, we obtain (1.11) up to the third term. Thus, the proof is complete.
4. Proof of Theorem 1.3

In this section, we assume that $\alpha \gg 1$. We write $\lambda = \lambda(\alpha)$ for simplicity. We consider the solution pair $(\lambda(\alpha), u_\alpha) \in \mathbb{R}_+ \times M_\alpha$. We obtain from the same argument as that in [10, Theorem 1.2] that

$$\frac{u_\alpha(t)}{\alpha} \rightarrow \sqrt{2}\sin \pi t$$

(4.1)

uniformly on $[0, 1]$ as $\alpha \to \infty$. By this, we have

$$\|u_\alpha\|_\infty = \sqrt{2} \alpha (1 + o(1)).$$

(4.2)

Furthermore, by [13, Lemma 2.4], we see that $\beta(\alpha)$ is continuous for $\alpha > 0$. By multiplying $u_\alpha$ by (1.1) and integration by parts, we obtain

$$\lambda(\alpha)\alpha^2 = \|u'_\alpha\|^2_2 + \int_0^1 u_\alpha(t) \sin u_\alpha(t) dt.$$  

(4.3)

By this and (1.16), for $\alpha \gg 1$,

$$\lambda(\alpha)\alpha^2 = 2\beta(\alpha) + \int_0^1 u_\alpha(t) \sin u_\alpha(t) dt - 2 \int_0^1 (1 - \cos u_\alpha(t)) dt.$$  

(4.4)

This along with (4.1) implies that $\lambda(\alpha)$ is continuous for $\alpha \gg 1$.

**Lemma 4.1.** For $\alpha \gg 1$,

$$\|u_\alpha\|^2_\infty = \left(1 - \frac{2}{\sqrt{\lambda}} \left(\frac{\pi}{4} + U\right)\right)^{-1} \alpha^2,$$

(4.5)

where

$$U = -\int_0^1 \frac{\sqrt{1 - s^2}B}{\sqrt{1 - s^2 + B} \left(\sqrt{1 - s^2} + B + \sqrt{1 - s^2}\right)} ds.$$  

(4.6)
Proof. By (2.7), (2.10), and putting $\theta = u_\alpha$ and $s = \theta/\|u_\alpha\|_{\infty}$,

$$
\|u_\alpha\|_{\infty} - \alpha^2 = 2 \int_0^{1/2} \frac{\left(\|u_\alpha\|_{\infty}^2 - u_\alpha(t)^2\right) u_\alpha'(t)}{\sqrt{\lambda\left(\|u_\alpha\|_{\infty}^2 - u_\alpha(t)^2\right)} + 2(\cos \|u_\alpha\|_{\infty} - \cos u_\alpha(t))} dt
\]
$$

$$
= 2 \int_0^{\|u_\alpha\|_{\infty}} \frac{\|u_\alpha\|_{\infty}^2 - \theta^2}{\sqrt{\lambda\left(\|u_\alpha\|_{\infty}^2 - \theta^2\right)} + 2(\cos \|u_\alpha\|_{\infty} - \cos \theta)} d\theta
\]

$$
= 2 \frac{\|u_\alpha\|_{\infty}^2}{\sqrt{\lambda}} \int_0^1 \frac{1 - s^2}{\sqrt{1 - s^2} + B} ds
\]

$$
= 2 \frac{\|u_\alpha\|_{\infty}^2}{\sqrt{\lambda}} \left[ \int_0^1 \sqrt{1 - s^2} ds + \int_0^1 \left( \frac{1 - s^2}{\sqrt{1 - s^2} + B} - \sqrt{1 - s^2} \right) ds \right]
\]

$$
= 2 \frac{\|u_\alpha\|_{\infty}^2}{\sqrt{\lambda}} \left( \frac{\pi}{4} + U \right).
\]

Now, the result follows easily from (4.7). Thus, the proof is complete. □

Lemma 4.2. For $\alpha \gg 1$,

$$
\|u_\alpha\|_{\infty} = \sqrt{2} \alpha - 2^{-3/4} \pi^{-5/2} \alpha^{-1/2} \cos \left( \sqrt{2} \alpha - \frac{3}{4} \pi \right) + o\left(\alpha^{-1/2}\right).
\]

Proof. By (2.10) and (4.6),

$$
|U| \leq \frac{C}{\lambda \|u_\alpha\|_{\infty}^2} \int_0^1 \frac{\cos \|u_\alpha\|_{\infty} - \cos(\|u_\alpha\|_{\infty}s)}{\sqrt{1 - s^2}} ds \leq C \left(\|u_\alpha\|_{\infty}^2\right).
\]

By this, (2.8), Lemma 2.1, and Taylor expansion,

$$
1 - \frac{2}{\sqrt{\lambda}} \left( \frac{\pi}{4} + U \right) = 1 - 2(\pi + 2V)^{-1} \left( \frac{\pi}{4} + U \right)
\]

$$
= \frac{1}{2} - \frac{2}{\pi} \left( U - \frac{V}{2} (1 + o(1)) \right) = \frac{1}{2} + \frac{1}{\pi} V (1 + o(1)).
\]

(4.10)
Proof of Theorem 1.3.

By this, (4.5), (2.12), (2.13), (2.18), Taylor expansion, and (4.2),

\[ \|u_\alpha\|_\infty = \left( \frac{1}{2} + \frac{1}{\pi} V(1 + o(1)) \right)^{-1/2} \alpha \]

\[ = \sqrt{2} \left( 1 - \frac{1}{\pi} V(1 + o(1)) \right) \alpha \]

\[ = \sqrt{2} \alpha - 2^{-3/4} \pi^{-5/2} \alpha^{-1/2} \cos\left( \sqrt{2} \alpha - \frac{3}{4} \pi \right) + o\left( \alpha^{-1/2} \right). \quad (4.11) \]

Thus, the proof is complete. \( \square \)

Proof of Theorem 1.3. By Lemma 4.2, we put

\[ \|u_\alpha\|_\infty = \sqrt{2} \alpha + A \alpha^{-1/2} + o\left( \alpha^{-1/2} \right), \]

\[ A = -2^{-3/4} \pi^{-5/2} \cos\left( \sqrt{2} \alpha - \frac{3}{4} \pi \right). \quad (4.12) \]

Then substitute (4.12) for (1.10) and use Taylor expansion to obtain

\[ \lambda = \pi^2 + 2\sqrt{\frac{2}{\pi}} \left( \sqrt{2} \alpha + A \alpha^{-1/2} + o\left( \alpha^{-1/2} \right) \right)^{-3/2} \cos\left( \sqrt{2} \alpha + A \alpha^{-1/2} + o\left( \alpha^{-1/2} \right) - \frac{3}{4} \pi \right) \]

\[ + 2\sqrt{\frac{2}{\pi}} \left( \sqrt{2} \alpha \right)^{-5/2} \left\{ -\frac{3}{8} \sin\left( \sqrt{2} \alpha - \frac{3}{4} \pi \right) - \frac{1}{\sqrt{2} \pi^2} \cos\left( 2\sqrt{2} \alpha - \frac{1}{4} \pi \right) \right. \]

\[ + \frac{1}{\pi^2} \cos\left( \sqrt{2} \alpha \right) \cos\left( \sqrt{2} \alpha - \frac{1}{4} \pi \right) \left\} + o\left( \alpha^{-5/2} \right) \]

\[ = \pi^2 + 2^{3/4} \pi^{-1/2} \alpha^{-3/2} \left( 1 + \frac{1}{\sqrt{2}} A \alpha^{-3/2} + o\left( \alpha^{-3/2} \right) \right)^{-3/2} \]

\[ \times \left\{ \cos\left( \sqrt{2} \alpha - \frac{3}{4} \pi \right) \cos\left( A \alpha^{-1/2} (1 + o(1)) \right) \right. \]

\[ - \sin\left( \sqrt{2} \alpha - \frac{3}{4} \pi \right) \sin\left( A \alpha^{-1/2} (1 + o(1)) \right) \}

\[ + 2\sqrt{\frac{2}{\pi}} \left( \sqrt{2} \alpha \right)^{-5/2} \left\{ -\frac{3}{8} \sin\left( \sqrt{2} \alpha - \frac{3}{4} \pi \right) - \frac{1}{\sqrt{2} \pi^2} \cos\left( 2\sqrt{2} \alpha - \frac{1}{4} \pi \right) \right. \]

\[ + \frac{1}{\pi^2} \cos\left( \sqrt{2} \alpha \right) \cos\left( \sqrt{2} \alpha - \frac{1}{4} \pi \right) \left\} + o\left( \alpha^{-5/2} \right) \]
Thus, we obtain (1.17) and the proof is complete.

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References


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