Approximation by the $q$-Szász-Mirakjan Operators

N. I. Mahmudov

Department of Mathematics, Eastern Mediterranean University, Gazimagusa, North Cyprus, Mersin 10, Turkey

Correspondence should be addressed to N. I. Mahmudov, nazim.mahmudov@emu.edu.tr

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This paper deals with approximating properties of the $q$-generalization of the Szász-Mirakjan operators in the case $q > 1$. Quantitative estimates of the convergence in the polynomial-weighted spaces and the Voronovskaja’s theorem are given. In particular, it is proved that the rate of approximation by the $q$-Szász-Mirakjan operators ($q > 1$) is of order $q^{-n}$ versus $1/n$ for the classical Szász-Mirakjan operators.

1. Introduction

The approximation of functions by using linear positive operators introduced via $q$-Calculus is currently under intensive research. The pioneer work has been made by Lupas [1] and Phillips [2] who proposed generalizations of Bernstein polynomials based on the $q$-integers. The $q$-Bernstein polynomials quickly gained the popularity, see [3–11]. Other important classes of discrete operators have been investigated by using $q$-Calculus in the case $0 < q < 1$, for example, $q$-Meyer-König operators [12–14], $q$-Bleimann, Butzer and Hahn operators [15–17], $q$-Szász-Mirakjan operators [18–21], and $q$-Baskakov operators [22, 23].

In the present paper, we introduce a $q$-generalization of the Szász operators in the case $q > 1$. Notice that different $q$-generalizations of Szász-Mirakjan operators were introduced and studied by Aral and Gupta [18, 19], by Radu [20], and by Mahmudov [21] in the case $0 < q < 1$. Since we define $q$-Szász-Mirakjan operators for $q > 1$, the rate of approximation by the $q$-Szász-Mirakjan operators ($q > 1$) is of order $q^{-n}$, which is essentially better than $1/n$ (rate of approximation for the classical Szász-Mirakjan operators). Thus our $q$-Szász-Mirakjan operators have better approximation properties than the classical Szász-Mirakjan operators and the other $q$-Szász-Mirakjan operators.
The paper is organized as follows. In Section 2, we give standard notations that will be used throughout the paper, introduce \( q \)-Szász-Mirakjan operators, and evaluate the moments of \( M_{n,q} \). In Section 3 we study convergence properties of the \( q \)-Szász-Mirakjan operators in the polynomial-weighted spaces. In Section 4, we give the quantitative Voronovskaja-type asymptotic formula.

2. Construction of \( M_{n,q} \) and Estimation of Moments

Throughout the paper we employ the standard notations of \( q \)-calculus, see [24, 25].

\( q \)-integer and \( q \)-factorial are defined by

\[
[n]_q := \begin{cases} 
\frac{1 - q^n}{1 - q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\}, \\
n, & \text{if } q = 1,
\end{cases} \quad \text{for } n \in \mathbb{N}, \quad [0] = 0, \\
\prod_{k=1}^n [n]_q := [1]_q [2]_q \cdots [n]_q, \quad \text{for } n \in \mathbb{N}, \quad [0]! = 1.
\] (2.1)

For integers \( 0 \leq k \leq n \) \( q \)-binomial is defined by

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.
\] (2.2)

The \( q \)-derivative of a function \( f(x) \), denoted by \( D_q f \), is defined by

\[
(D_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad (D_q f)(0) := \lim_{x \to 0} (D_q f)(x).
\] (2.3)

The formula for the \( q \)-derivative of a product and quotient are

\[
D_q(u(x)v(x)) = D_q(u(x))v(x) + u(qx)D_q(v(x)).
\] (2.4)

Also, it is known that

\[
D_q x^n = [n]_q x^{n-1}, \quad D_q E(ax) = a E(q ax).
\] (2.5)

If \( |q| > 1 \), or \( 0 < |q| < 1 \) and \( |z| < 1/(1-q) \), the \( q \)-exponential function \( e_q(z) \) was defined by Jackson

\[
e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}.
\] (2.6)
If $|q| > 1$, $e_q(z)$ is an entire function and

$$e_q(z) = \prod_{j=0}^{\infty} \left( 1 + (q - 1) \frac{z}{q^{j+1}} \right), \quad |q| > 1. \quad (2.7)$$

There is another $q$-exponential function which is entire when $0 < |q| < 1$ and which converges when $|z| < 1/|1 - q|$ if $|q| > 1$. To obtain it we must invert the base in (2.6), that is, $q \to 1/q$:

$$E_q(z) := e_{1/q}(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{[k]_q!}. \quad (2.8)$$

We immediately obtain from (2.7) that

$$E_q(z) = \prod_{j=0}^{\infty} \left( 1 + (1 - q) z q^j \right), \quad 0 < |q| < 1. \quad (2.9)$$

The $q$-difference equations corresponding to $e_q(z)$ and $E_q(z)$ are

$$\begin{align*}
D_q e_q(az) &= a e_q(qz), \\
D_q E_q(az) &= a E_q(qaz), \\
D_{1/q} e_q(z) &= D_{1/q} E_{1/q}(z) = E_{1/q} \left( q^{-1} z \right) = e_q \left( q^{-1} z \right), \quad q \neq 0.
\end{align*} \quad (2.10)$$

Let $C_p$ be the set of all real valued functions $f$, continuous on $[0, \infty)$, such that $w_p f$ is uniformly continuous and bounded on $[0, \infty)$ endowed with the norm

$$\|f\|_p := \sup_{x \in [0, \infty)} w_p(x) |f(x)|. \quad (2.11)$$

Here

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \quad \text{if } p \in \mathbb{N}. \quad (2.12)$$

The corresponding Lipschitz classes are given for $0 < \alpha \leq 2$ by

$$\begin{align*}
\Delta_h^2 f(x) &= f(x + 2h) - 2 f(x + h) + f(x), \\
\omega_p^2(f; \delta) &:= \sup_{0 < h \leq \delta} \|\Delta_h^2 f\|_p, \quad \text{Lip}_p^{\alpha} := \{ f \in C_p : \omega_p^2(f; \delta) = 0(\delta^\alpha), \ \delta \to 0^+ \}. \quad (2.13)
\end{align*}$$

Now we introduce the $q$-parametric Szász-Mirakjan operator.
Definition 2.1. Let \( q > 1 \) and \( n \in \mathbb{N} \). For \( f : [0, \infty) \to \mathbb{R} \) one defines the Szász-Mirakjan operator based on the \( q \)-integers

\[
M_{n,q}(f; x) := \sum_{k=0}^{\infty} f\left( \frac{[k]_q}{[n]_q} x \right) \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left( [-n]_q q^{-k} x \right).
\]  

(2.14)

Similarly as a classical Szász-Mirakjan operator \( S_n \), the operator \( M_{n,q} \) is linear and positive. Furthermore, in the case of \( q \to 1^+ \) we obtain classical Szász-Mirakjan operators.

Moments \( M_{n,q}(t^m; x) \) are of particular importance in the theory of approximation by positive operators. From (2.14) one easily derives the following recurrence formula and explicit formulas for moments \( M_{n,q}(t^m; x) \), \( m = 0, 1, 2, 3, 4 \).

Lemma 2.2. Let \( q > 1 \). The following recurrence formula holds

\[
M_{n,q}(t^{m+1}; x) = \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[n]_q^{j+1}} M_{n,q}(t^j; q^{-1} x).
\]

(2.15)

Proof. The recurrence formula (2.15) easily follows from the definition of \( M_{n,q} \) and \( q[k]_q + 1 = [k + 1]_q \) as show below:

\[
M_{n,q}(t^{m+1}; x)
\]

\[
= \sum_{k=0}^{\infty} \frac{[k]_q^{m+1}}{[n]_q^{m+1}} \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left( [-n]_q q^{-k} x \right)
\]

\[
= \sum_{k=0}^{\infty} \frac{[k]_q^m}{[n]_q^m} \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^{k-1} x^k}{[k]_q!} e_q\left( [-n]_q q^{-k} x \right)
\]

\[
= \sum_{k=0}^{\infty} \left( q[k]_q + 1 \right)^m \frac{1}{[n]_q^m} \frac{1}{q^{k(k+1)/2}} \frac{[n]_q^{k+1} x^{k+1}}{[k]_q!} e_q\left( [-n]_q q^{-k} q^{-1} x \right)
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{[n]_q^m} \sum_{j=0}^{m} \binom{m}{j} q^j [k]_q^j \frac{1}{q^{k(k+1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left( [-n]_q q^{-k} q^{-1} x \right)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[n]_q^m} \sum_{k=0}^{\infty} \frac{[k]_q^j}{[n]_q^j} \frac{1}{q^{k(k-1)/2}} \frac{[n]_q^k x^k}{[k]_q!} e_q\left( [-n]_q q^{-k} q^{-1} x \right)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[n]_q^m} M_{n,q}(t^j; q^{-1} x).
\]
Lemma 2.3. The following identities hold for all \( q > 1, \ x \in [0, \infty), \ n \in \mathbb{N}, \) and \( k \geq 0: \)

\[
x D_q s_{nk}(q; x) = \left[ n \right]_q \left( \frac{[k]_q}{[n]_q} - x \right) s_{nk}(q; x),
\]

\[
M_{n,q}(t^{m+1}; x) = \frac{x}{[n]_q} D_q M_{n,q}(t^m; x) + x M_{n,q}(t^m; x),
\]

where \( s_{nk}(q; x) := (1/q^{k(k-1)/2})([n]_q^k x^k/[k]_q^k) e_q(−[n]_q q^{-k} x). \)

Proof. The first identity follows from the following simple calculations

\[
x D_q s_{nk}(q; x)
\]

\[
= \left[ k \right]_q q^{k(k-1)/2} \frac{[n]_q^k x^k}{[k]_q^k} e_q(−[n]_q q^{-k} x) - x q^{-k} \left[ n \right]_q q^{k(k-1)/2} \frac{[n]_q^k x^k}{[k]_q^k} e_q(−[n]_q q^{-k} x)
\]

\[
= \left[ k \right]_q s_{nk}(q; x) - x \left[ n \right]_q s_{nk}(q; x) = \left[ n \right]_q \left( \frac{[k]_q}{[n]_q} - x \right) s_{nk}(q; x).
\]

The second one follows from the first:

\[
x D_q M_{n,q}(t^m; x) = \left[ n \right]_q \sum_{k=0}^{\infty} \left( \frac{[k]_q}{[n]_q} \right)^m \left( \frac{[k]_q}{[n]_q} - x \right) s_{nk}(q; x)
\]

\[
= \left[ n \right]_q \sum_{k=0}^{\infty} \left( \frac{[k]_q}{[n]_q} \right)^{m+1} s_{nk}(q; x) - \left[ n \right]_q x \sum_{k=0}^{\infty} \left( \frac{[k]_q}{[n]_q} \right)^m s_{nk}(q; x)
\]

\[
= \left[ n \right]_q M_{n,q}(t^{m+1}; x) - \left[ n \right]_q x M_{n,q}(t^m; x). \]

Lemma 2.4. Let \( q > 1. \) One has

\[
M_{n,q}(1; x) = 1, \quad M_{n,q}(t; x) = x, \quad M_{n,q}\left( t^2; x \right) = x^2 + \frac{1}{[n]_q} x,
\]

\[
M_{n,q}(t^3; x) = x^3 + \frac{2 + q}{[n]_q} x^2 + \frac{1}{[n]_q^2} x,
\]

\[
M_{n,q}(t^4; x) = x^4 + \left( 3 + 2q + q^2 \right) \frac{x^3}{[n]_q} + \left( 3 + 3q + q^2 \right) \frac{x^2}{[n]_q^2} + \frac{1}{[n]_q^3} x.
\]

Proof. For a fixed \( x \in \mathbb{R}_+, \) by the \( q \)-Taylor theorem [24], we obtain

\[
\varphi_n(t) = \sum_{k=0}^{\infty} \frac{\left( t - x \right)_k^q}{[k]_q^k} D_{1/q}^k \varphi_n(x).
\]
Choosing $t = 0$ and taking into account

$$(-x)_{1/q}^k = (-1)^k x^k q^{-k(1-k)/2}, \quad D_{1/q}^k e_q\left(-[n]_q x\right) = (-1)^k q^{-k(1-k)/2} [n]_q^k e_q\left(-[n]_q q^{-k} x\right) \quad (2.22)$$

we get for $\varphi_n(x) = e_q\left(-[n]_q x\right)$ that

$$1 = \varphi_n(0) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{[k]_q^k} D_{1/q}^k \varphi_n(x)$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{[k]_q^k} \left(-1\right)^k q^{-k(1-k)/2} [n]_q^k e_q\left(-[n]_q q^{-k} x\right) \quad (2.23)$$
$$= \sum_{k=0}^{\infty} \left([n]_q^k x^k\right) k q^{-k(1-k)/2} e_q\left(-[n]_q q^{-k} x\right).$$

In other words $M_{n,q}(1; x) = 1$.

Calculation of $M_{n,q}(t^i; x)$, $i = 1, 2, 3, 4$, based on the recurrence formula (2.17) (or (2.15)). We only calculate $M_{n,q}(t^i; x)$ and $M_{n,q}(t^4; x)$:

$$M_{n,q}(t^i; x) = \frac{x}{[n]_q} D_q M_{n,q}(t^i; x) + x M_{n,q}(t^i; x)$$
$$= \frac{x}{[n]_q} \left(2 q x + \frac{[1]}{[n]_q} x\right) + x \left(x^2 + \frac{[1]}{[n]_q} x\right)$$
$$= \frac{x}{[n]_q} \left(2 + q\right) x^2 + x^3,$$

$$M_{n,q}(t^4; x) = \frac{x}{[n]_q} D_q M_{n,q}(t^4; x) + x M_{n,q}(t^4; x) \quad (2.24)$$
$$= \frac{x}{[n]_q} \left(\frac{1}{[n]_q^2} + \frac{2 + q}{[n]_q} [2]_q x + [3]_q x^2\right) + x \left(\frac{1}{[n]_q^2} x^2 + \frac{2 + q}{[n]_q} x^3 + x^3\right)$$
$$= \frac{x}{[n]_q} \left(3 + 3 q + q^2\right) \frac{x^2}{[n]_q^2} + \left(3 + 2 q + q^2\right) \frac{x^3}{[n]_q} + x^4.$$
Lemma 2.5. Assume that \( q > 1 \). For every \( x \in [0, \infty) \) there hold

\[
M_{n,q}(t - x)^2; x) = \frac{x}{[n]_q},
\]

\[
M_{n,q}(t - x)^3; x) = \frac{1}{[n]_q^2} x + (q - 1) \frac{x^2}{[n]_q},
\]

\[
M_{n,q}(t - x)^4; x) = \frac{1}{[n]_q^3} x + (q^2 + 3q - 1) \frac{x^2}{[n]_q} + (q - 1)^2 \frac{x^3}{[n]_q}. \tag{2.27}
\]

Proof. First of all we give an explicit formula for \( M_{n,q}((t - x)^4; x) \).

\[
M_{n,q}(t - x)^3; x) = M_{n,q}(t^3; x) - 3x M_{n,q}(t^2; x) + 3x^2 M_{n,q}(t; x) - x^3
\]

\[
= x^3 + \frac{2 + q}{[n]_q} x^2 + \frac{1}{[n]_q^2} x - 3x \left( x^2 + \frac{x}{[n]_q} \right) + 3x^3 - x^3
\]

\[
= \frac{1}{[n]_q^2} x + (q - 1) \frac{x^2}{[n]_q},
\]

\[
M_{n,q}(t - x)^4; x) = M_{n,q}(t^4; x) - 4x M_{n,q}(t^3; x) + 6x^2 M_{n,q}(t^2; x) - 4x^3 M_{n,q}(t; x) + x^4
\]

\[
= \frac{1}{[n]_q^3} x + \left( 3 + 3q + q^2 \right) \frac{x^2}{[n]_q} + \left( 3 + 2q + q^2 \right) \frac{x^3}{[n]_q} + x^4
\]

\[
- 4x \left( \frac{1}{[n]_q} x + \frac{2 + q}{[n]_q} x^2 + x^3 \right) + 6x^2 \left( x^2 + \frac{x}{[n]_q} \right) - 4x^4 + x^4
\]

\[
= \frac{1}{[n]_q^3} x + \left( -1 + 3q + q^2 \right) \frac{x^2}{[n]_q^2} + (q - 1)^2 \frac{x^3}{[n]_q}. \tag{2.28}
\]

Now we prove explicit formula for the moments \( M_{n,q}(t^m; x) \), which is a \( q \)-analogue of a result of Becker, see [26, Lemma 3].

Lemma 2.6. For \( q > 1, \ m \in \mathbb{N} \) there holds

\[
M_{n,q}(t^m; x) = \sum_{j=1}^{m} \binom{m}{j} \frac{x^j}{[n]_q^{m-j}}. \tag{2.29}
\]
Moreover for every \( f \), let \( M_{n,q} \in L^p(w) \), then by Stirling numbers of the second type.

\[
S_q(m+1,j) = [j]S_q(m,j) + S_q(m,j-1), \quad m \geq 0, \quad j \geq 1,
\]

\[
S_q(0,0) = 1, \quad S_q(m,0) = 0, \quad m > 0, \quad S_q(m,j) = 0, \quad m < j.
\] (2.30)

In particular \( M_{n,q}(t^n;x) \) is a polynomial of degree \( m \) without a constant term.

**Proof.** Because of \( M_{n,q}(t;x) = x, \) \( M_{n,q}(t^2;x) = x^2 + x/\binom{1}{q} \), the representation (2.29) holds true for \( m = 1, 2 \) with \( S_q(2,1) = 1, \) \( S_q(1,1) = 1 \).

Now assume (2.29) to be valued for \( m \) then by Lemma 2.3 we have

\[
M_{n,q}(t^{m+1};x) = \frac{x}{[n]_q}D_q M_{n,q}(t^m;x) + x M_{n,q}(t^m;x)
\]

\[
= \frac{x}{[n]_q} \sum_{j=1}^{m} [j]_q S_q(m,j) \frac{x^{j-1}}{[n]_{q-1}^{m-j}} + x \sum_{j=1}^{m} S_q(m,j) \frac{x^j}{[n]_{q-1}^{m-j}}
\]

\[
= \sum_{j=1}^{m} [j]_q S_q(m,j) \frac{x^{j-1}}{[n]_{q-1}^{m-j}} + \sum_{j=1}^{m} S_q(m,j) \frac{x^j}{[n]_{q-1}^{m-j}}
\] (2.31)

\[
= \frac{x}{[n]_m} S_q(m,1) + x^{m+1} S_q(m,m)
\]

\[
+ \sum_{j=2}^{m} \left( [j]_q S_q(m,j) + S_q(m,j-1) \right) \frac{x^j}{[n]_{q-1}^{m-j+1}}.
\]

**Remark 2.7.** Notice that \( S_q(m,j) \) are Stirling numbers of the second kind introduced by Goodman et al. in [8]. For \( q = 1 \) the formulae (2.30) become recurrence formulas satisfied by Stirling numbers of the second type.

### 3. \( M_{n,q} \) in Polynomial-Weighted Spaces

**Lemma 3.1.** Let \( p \in \mathbb{N} \cup \{0\} \) and \( q \in (1, \infty) \) be fixed. Then there exists a positive constant \( K_1(q,p) \) such that

\[
\| M_{n,q}(1/w_p;x) \|_p \leq K_1(q,p), \quad n \in \mathbb{N}.
\] (3.1)

Moreover for every \( f \in \mathcal{C}_p \) one has

\[
\| M_{n,q}(f) \|_p \leq K_1(q,p) \| f \|_p, \quad n \in \mathbb{N}.
\] (3.2)

Thus \( M_{n,q} \) is a linear positive operator from \( \mathcal{C}_p \) into \( \mathcal{C}_p \) for any \( p \in \mathbb{N} \cup \{0\} \).
Proof. The inequality (3.1) is obvious for \( p = 0 \). Let \( p \geq 1 \). Then by (2.29) we have

\[
\omega_p(x) M_{n,q} \left( \frac{1}{\omega_p(x) x} \right) = \omega_p(x) + \omega_p(x) \sum_{j=1}^{p} S_q(p, j) \frac{x^j}{[n]_q^p} \leq K_1(q, p),
\]

(3.3)

\( K_1(q, p) \) is a positive constant depending on \( p \) and \( q \). From this follows (3.1). On the other hand

\[
\| M_{n,q}(f) \|_p \leq \| f \|_p \left\| M_{n,q} \left( \frac{1}{\omega_p} \right) \right\|_p,
\]

(3.4)

for every \( f \in C_p \). By applying (3.1), we obtain (3.2). \( \square \)

Lemma 3.2. Let \( p \in \mathbb{N} \cup \{0\} \) and \( q \in (1, \infty) \) be fixed. Then there exists a positive constant \( K_2(q, p) \) such that

\[
\left\| M_{n,q} \left( \frac{(t - x)^2}{\omega_p(t)} \right) \right\|_p \leq K_2(q, p) \frac{1}{[n]_q}, \quad n \in \mathbb{N}.
\]

(3.5)

Proof. The formula (2.25) imply (3.5) for \( p = 0 \). We have

\[
M_{n,q} \left( \frac{(t - x)^2}{\omega_p(t)} \right) = M_{n,q}((t - x)^2; x) + M_{n,q}((t - x)^2 t; x),
\]

(3.6)

for \( p, n \in \mathbb{N} \). If \( p = 1 \) then we get

\[
M_{n,q}((t - x)^2(1 + t); x) = M_{n,q}((t - x)^2; x) + M_{n,q}((t - x)^2 t; x)
\]

\[
= M_{n,q}((t - x)^2; x) + (1 + x) M_{n,q}((t - x)^2; x),
\]

(3.7)

which by Lemma 2.5 yields (3.5) for \( p = 1 \).

Let \( p \geq 2 \). By applying (2.29), we get

\[
\omega_p(x) M_{n,q}((t - x)^2 t^{p}; x)
\]

\[
= \omega_p(x) \left( M_{n,q}((t - p + 1)^2; x) - 2x M_{n,q}(p + 1; x) + x^2 M_{n,q}(t^{p}; x) \right)
\]

\[
= \omega_p(x) \left( x^{p+2} + \sum_{j=1}^{p+1} S_q(p + 2, j) \frac{x^j}{[n]_q^{p+2-j}} - 2x^{p+2} + 2 \sum_{j=1}^{p} S_q(p + 1, j) \frac{x^{j+1}}{[n]_q^{p+1-j}} + x^{p+2} + \sum_{j=1}^{p+1} S_q(p, j) \frac{x^{j+1}}{[n]_q^{p+1-j}} \right)
\]
\[ w_p(x) = w_p(x) \left( \sum_{j=2}^{p} (S_q(p+2,j) - 2S_q(p+1,j) + S_q(p,j)) \frac{x^{j+1}}{[n]_q^{p+1-j}} \right) \]
\[ + S_q(p+2,1) \frac{x}{[n]_q^{p+1}} + (S_q(p+2,2) - 2S_q(p+2,1)) \frac{x^2}{[n]_q^{p+2}} \]
\[ = w_p(x) \frac{x}{[n]_q} P_p(q;x), \]

(3.8)

where \( P_p(q;x) \) is a polynomial of degree \( p \). Therefore one has
\[
wp(x) M_{n,q}(t-x)^2 p^n; x) \leq K_2(q,p) \frac{x}{[n]_q}. \tag{3.9}
\]

Our first main result in this section is a local approximation property of \( M_{n,q} \) stated below.

**Theorem 3.3.** There exists an absolute constant \( C > 0 \) such that
\[
wp(x) M_{n,q}(g;x) - g(x) \leq K_3(q,p) \|g''\| \frac{x}{[n]_q}, \tag{3.10}
\]

where \( g \in C^2_p, q > 1 \) and \( x \in [0, \infty) \).

**Proof.** Using the Taylor formula
\[
g(t) = g(x) + g'(x)(t-x) + \int_x^t \int_x^s g''(u)du ds, \quad g \in C^2_p, \tag{3.11}
\]
we obtain that
\[
w_p(x) |M_{n,q}(g;x) - g(x)| = w_p(x) \left| M_{n,q}\left( \int_x^t \int_x^s g''(u)du ds; x \right) \right|
\]
\[
\leq w_p(x) M_{n,q} \left( \left| \int_x^t \int_x^s g''(u)du ds \right|; x \right)
\]
\[
\leq w_p(x) M_{n,q} \left( \|g''\|_p \left| \int_x^t \int_x^s (1+u''')du ds \right|; x \right)
\]
\[
\leq w_p(x) \frac{1}{2} \|g''\|_p M_{n,q} \left( (t-x)^2 \frac{1}{w_p(x)} + 1/w_p(t) \right); x \right)
\]
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\[
\leq \frac{1}{2} \|g''\|_p \left( M_{n,q} \left( (t - x)^2; x \right) + w_p(x) M_{n,q} \left( (t - x)^2 w_p(t); x \right) \right) \\
\leq K_3(q,x) \|g''\|_p \frac{x}{[n]_q}.
\]  

(3.12)

Now we consider the modified Steklov means

\[
f_h(x) := \frac{4}{h^2} \int_0^{h/2} \left[ 2f(x + s + t) - f(x + 2(s + t)) \right] ds \, dt.
\]  

(3.13)

\(f_h(x)\) has the following properties:

\[
f(x) - f_h(x) = \frac{4}{h^2} \int_0^{h/2} \Delta_{s+t}^2 f(x) \, ds \, dt, \quad f''_h(x) = h^{-2} \left( 8 \Delta_{h/2}^2 f(x) - \Delta_h^2 f(x) \right)
\]  

(3.14)

and therefore

\[
\|f - f_h\|_p \leq \omega^2_p(f;h), \quad \|f''_h\|_p \leq \frac{1}{9h^2} \omega^2_p(f;h).
\]  

(3.15)

We have the following direct approximation theorem.

**Theorem 3.4.** For every \(p \in \mathbb{N} \cup \{0\}, f \in C_p\) and \(x \in [0, \infty), \ q > 1\), one has

\[
\omega_p(x) \mid M_{n,q}(f;x) - f(x) \mid \leq M_p \omega^2_p \left( f; \sqrt{\frac{x}{[n]_q}} \right) = M_p \omega^2_p \left( f; \sqrt{\frac{(q-1)x}{(q^n-1)}} \right).
\]  

(3.16)

Particularly, if \(\text{Lip}_p^2(\alpha)\) for some \(\alpha \in (0, 2]\), then

\[
\omega_p(x) \mid M_{n,q}(f;x) - f(x) \mid \leq M_p \left( \frac{x}{[n]_q} \right)^{\alpha/2}.
\]  

(3.17)

**Proof.** For \(f \in C_p\) and \(h > 0\)

\[
\mid M_{n,q}(f;x) - f(x) \mid \leq \mid M_{n,q}((f - f_h);x) - (f - f_h)(x) \mid + \mid M_{n,q}(f_h;x) - f_h(x) \mid
\]  

(3.18)

and therefore

\[
\omega_p(x) \mid M_{n,q}(f;x) - f(x) \mid \leq \|f - f_h\|_p \left( \frac{1}{w_p(t)}; x \right) \left( \omega_p(x) M_{n,q} \left( \frac{1}{w_p(t)}; x \right) + 1 \right)
\]  

(3.19)

\[+ K_3(q,p) \|f''_h\|_p \frac{x}{[n]_q}.
\]
Since \( w_p(x) M_{n,q}(1/w_p(t); x) \leq K_1(q,p) \), we get that
\[
\omega_p(x) |M_{n,q}(f;x) - f(x)| \leq M(q,p) \omega_p^2(f;h) \left[ 1 + \frac{x}{[n]_q h^q} \right]. \tag{3.20}
\]

Thus, choosing \( h = \sqrt{x/[n]_q} \) the proof is completed. \( \square \)

**Corollary 3.5.** If \( p \in \mathbb{N} \cup \{0\}, f \in C_p, q > 1 \) and \( x \in [0,\infty) \), then
\[
\lim_{n \to \infty} M_{n,q}(f;x) = f(x). \tag{3.21}
\]

This convergence is uniform on every \([a,b], 0 \leq a < b\).

**Remark 3.6.** Theorem 3.4 shows the rate of approximation by the \( q \)-Szász-Mirakjan operators \((q > 1)\) is of order \( q^{-n} \) versus \( 1/n \) for the classical Szász-Mirakjan operators.

### 4. Convergence of \( q \)-Szász-Mirakjan Operators

An interesting problem is to determine the class of all continuous functions \( f \) such that \( M_{n,q}(f) \) converges to \( f \) uniformly on the whole interval \([0,\infty)\) as \( n \to \infty \). This problem was investigated by Totik [27, Theorem 1] and de la Cal and Cárceño [28, Theorem 1]. The following result is a \( q \)-analogue of Theorem 1 [28].

**Theorem 4.1.** Assume that \( f : [0,\infty) \to \mathbb{R} \) is bounded or uniformly continuous. Let
\[
f^*(z) = f\left(z^2\right), \quad z \in [0,\infty). \tag{4.1}
\]

One has, for all \( t > 0 \) and \( x \geq 0 \),
\[
|M_{n,q}(f;x) - f(x)| \leq 2\omega\left(f^*;\sqrt{\frac{1}{[n]_q}}\right). \tag{4.2}
\]

Therefore, \( M_{n,q}(f;x) \) converges to \( f \) uniformly on \([0,\infty)\) as \( n \to \infty \), whenever \( f^* \) is uniformly continuous.

**Proof.** By the definition of \( f^* \) we have
\[
M_{n,q}(f;x) = M_{n,q}(f^*(\sqrt{\cdot});x). \tag{4.3}
\]
Thus we can write

\[
|M_{n,q}(f; x) - f(x)| = |M_{n,q}(f^*(\sqrt{\cdot}); x) - f^*(\sqrt{x})|
\]

\[
= \left| \sum_{k=0}^{\infty} \left( f^* \left( \sqrt{\frac{k}{n^q}} - \sqrt{\frac{n}{n^q}} \right) s_{n,k}(q; x) \right) \right|
\]

\[
\leq \sum_{k=0}^{\infty} \left( f^* \left( \sqrt{\frac{k}{n^q}} - \sqrt{\frac{n}{n^q}} \right) s_{n,k}(q; x) \right)
\]

\[
\leq \sum_{k=0}^{\infty} \omega \left( f^* \left| \sqrt{\frac{k}{n^q}} - \sqrt{\frac{n}{n^q}} \right| s_{n,k}(q; x) \right)
\]

\[
\leq \sum_{k=0}^{\infty} \omega \left( f^* \left| \sqrt{\frac{k}{n^q}} - \sqrt{\frac{n}{n^q}} \right| M_{n,q} \left( \left| \sqrt{\cdot} - \sqrt{x} \right| ; x \right) \right) s_{n,k}(q; x).
\]

Finally, from the inequality

\[
\omega(f^*; \alpha \delta) \leq (1 + \alpha)\omega(f^*; \delta), \quad \alpha, \delta \geq 0,
\]

we obtain

\[
|M_{n,q}(f; x) - f(x)| \leq \omega \left( f^*; M_{n,q} \left( \left| \sqrt{\cdot} - \sqrt{x} \right| ; x \right) \right) \sum_{k=0}^{\infty} \left( 1 + \frac{\left| \sqrt{\frac{k}{n^q}} - \sqrt{\frac{n}{n^q}} \right|}{M_{n,q} \left( \left| \sqrt{\cdot} - \sqrt{x} \right| ; x \right)} \right) s_{n,k}(q; x)
\]

\[
= 2\omega(f^*; M_{n,q} \left( \left| \sqrt{\cdot} - \sqrt{x} \right| ; x \right)).
\]

In order to complete the proof we need to show that we have for all \( t > 0 \) and \( x > 0 \),

\[
M_{n,q} \left( \left| \sqrt{\cdot} - \sqrt{x} \right| ; x \right) \leq \sqrt{\frac{1}{[n]_q}}.
\]
Indeed we obtain from the Cauchy-Schwarz inequality:

\[ M_{n,q}(\sqrt{\cdot} - \sqrt{x}; x) = \sum_{k=0}^{\infty} \sqrt{\frac{[k]_q}{[n]_q}} - \sqrt{x} \left| s_{n,k}(q; x) \right| \]

\[ = \sum_{k=0}^{\infty} \frac{[k]_q / [n]_q - x}{\sqrt{[k]_q / [n]_q + \sqrt{x}}} s_{n,k}(q; x) \leq \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \frac{[k]_q}{[n]_q} - x \left| s_{n,k}(q; x) \right| \]

\[ \leq \frac{1}{\sqrt{x}} \sqrt{\sum_{k=0}^{\infty} \frac{[k]_q}{[n]_q} - x \left( s_{n,k}(q; x) \right)^2} = \frac{1}{\sqrt{x}} \sqrt{M_{n,q}(\cdot - x)^2; x} \]

\[ = \frac{1}{\sqrt{x}} \sqrt{\frac{1}{[n]_q} x} = \sqrt{\frac{1}{[n]_q}} \]

(4.8)

showing (4.2), and completing the proof. □

Next we prove Voronovskaja type result for \( q \)-Szász-Mirakjan operators.

**Theorem 4.2.** Assume that \( q \in (1, \infty) \). For any \( f \in C_p^2 \) the following equality holds

\[ \lim_{n \to \infty} [n]_q (M_{n,q}(f; x) - f(x)) = \frac{1}{2} f''(x) x, \]

(4.9)

for every \( x \in [0, \infty) \).

**Proof.** Let \( x \in [0, \infty) \) be fixed. By the Taylor formula we may write

\[ f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + r(t; x)(t - x)^2, \]

(4.10)

where \( r(t; x) \) is the Peano form of the remainder, \( r(\cdot; x) \in C_p \), and \( \lim_{t \to x} r(t; x) = 0 \). Applying \( M_{n,q} \) to (4.10) we obtain

\[ [n] (M_{n,q}(f; x) - f(x)) = f'(x)[n]_q M_{n,q}(t - x; x) \]

\[ + \frac{1}{2} f''(x)[n]_q M_{n,q}(t - x)^2; x) + [n]_q M_{n,q}(r(t; x)(t - x)^2; x). \]

(4.11)

By the Cauchy-Schwartz inequality, we have

\[ M_{n,q}(r(t; x)(t - x)^2; x) \leq \sqrt{M_{n,q}(r^2(t; x); x)} \sqrt{M_{n,q}(t - x)^4; x}. \]

(4.12)
Observe that \( r^2(x; x) = 0 \). Then it follows from Corollary 3.5 that

\[
\lim_{n \to \infty} M_n,q(r^2(t; x); x) = r^2(x; x) = 0.
\] (4.13)

Now from (4.12), (4.13), and Lemma 2.5 we get immediately

\[
\lim_{n \to \infty} [n]_q M_{n,q}(r(t; x)(t - x)^2; x) = 0.
\] (4.14)

The proof is completed. \( \square \)

References


