Research Article

On the Stability Problem in Fuzzy Banach Space

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We investigate the generalized Ulam-Hyers stability of the Cauchy functional equation and pose two open problems in fuzzy Banach space.

1. Introduction and Preliminaries


Theorem 1.1 (Th. M. Rassias). Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality:

\[
\| f(x + y) - f(x) - f(y) \| \leq \epsilon (\|x\|^p + \|y\|^p) \tag{1.1}
\]

for all \( x, y \in E \), where \( \epsilon \) and \( p \) are constants with \( \epsilon > 0 \) and \( 0 \leq p < 1 \). Then, the limit \( L(x) = \lim_{n \to \infty} (1/2^n) f (2^n x) \) exists for all \( x \in E \) and \( L : E \to E' \) is the unique additive mapping which satisfies

\[
\| f(x) - L(x) \| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p \tag{1.2}
\]

for all \( x \in E \). Also, if for each \( x \in E \) the function \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then \( L \) is linear.
In 1990, Th. M. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for \( p \geq 1 \). In 1991, Gajda [6] gave an affirmative solution to this question for \( p > 1 \). It was shown by Gajda [6], as well as by Th. M. Rassias and Šemrl [7], that one cannot prove a Th. M. Rassias type theorem when \( p = 1 \). Găvruţa [8] proved that the function \( f(x) = x \ln|x|, \) if \( x \neq 0 \) and \( f(0) = 0 \) satisfies (1.1) with \( \varepsilon = p = 1 \) but

\[
\sup_{x \neq 0} \frac{|f(x) - A(x)|}{|x|} \geq \sup_{n \in \mathbb{N}} \frac{|n \ln n - A(n)|}{n} = \sup_{n \in \mathbb{N}} |\ln n - A(1)| = \infty
\]

for any additive function \( A : \mathbb{R} \to \mathbb{R} \). J. M. Rassias [9] replaced the factor \( ||x||^p + ||y||^p \) by \( ||x||^p \cdot ||y||^p \) for \( p_1, p_2 \in \mathbb{R} \) with \( p_1 + p_2 \neq 1 \) (see also [10, 11]) and has obtained the following theorem.

**Theorem 1.2.** Let \( X \) be a real normed linear space and \( Y \) a real complete normed linear space. Assume that \( f : X \to Y \) is an approximately additive mapping for which there exist constants \( \theta \geq 0 \) and \( p = p_1 + p_2 \neq 1 \) such that \( f \) satisfies the inequality:

\[
||f(x + y) - f(x) - f(y)|| \leq \theta ||x||^p ||y||^p
\]

for all \( x, y \in X \). Then, there exists a unique additive mapping \( L : X \to Y \) satisfying

\[
||f(x) - L(x)|| \leq \frac{\theta}{2^p - 2} ||x||^p
\]

for all \( x \in X \). If, in addition, \( f : X \to Y \) is a mapping such that the transformation \( t \to f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then \( L \) is an \( \mathbb{R} \)-linear mapping.

In the case \( p = 1 \), we do not have stability [12]. In 1994, a further generalization of Th. M. Rassias’ Theorem was obtained by Găvruţa [13], in which he replaced the bound \( \varepsilon(||x||^p + ||y||^p) \) by a general control function \( \varphi(x, y) \). Isac and Th. M. Rassias [14] replaced the factor \( ||x||^p + ||y||^p \) by \( ||x||^p \cdot ||y||^p \) in Theorem 1.1 and solved stability problem when \( p_2 \leq p_1 < 1 \) or \( 1 < p_2 \leq p_1 \), also they asked the question whether such a theorem can be proved for \( p_2 < 1 < p_1 \). Găvruţa [8] gave a negative answer to this question. Isac and Th. M. Rassias [15] applied the Ulam-Hyers-Rassias stability theory to prove fixed point theorems and study some new applications in nonlinear analysis. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Ulam-Hyers stability to a number of functional equations and mappings (see [16–40]). We also refer the readers to the books of Czerwik [41] and Hyers et al. [42].

Th. M. Rassias [43] has obtained the following theorem and posed a problem.

**Theorem 1.3.** Let \( E_1 \) and \( E_2 \) be two Banach spaces, and let \( f : E_1 \to E_2 \) be a mapping such that \( f(tx) \) is continuous in \( t \) for each fixed \( x \). Assume that there exist \( \theta \geq 0 \) and \( p \in (0, 1) \) such that

\[
||f(x + y) - f(x) - f(y)|| \leq \theta (||x||^p + ||y||^p)
\]
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for all \(x, y \in X\). Let \(k\) be a positive integer \(k > 2\). Then, there exists a unique linear mapping \(T : E_1 \rightarrow E_2\) such that

\[
\|f(x) - T(x)\| \leq \frac{k\theta}{k - k^p}\|x\|^p s(k, p) \tag{1.7}
\]

for all \(x \in X\), where

\[
s(k, p) = 1 + \frac{1}{k} \sum_{m=2}^{k-1} m^p. \tag{1.8}
\]

**Th. M. Rassias Problem**

What is the best possible value of \(k\) in Theorem 1.3?

Gâvruţa et al. have given a generalization of [13] and have answered to Th. M. Rassias problem [44].

In [45], J. M. Rassias et al. have investigated the generalized Ulam-Hyers “product-sum” stability of functional equations and have obtained the following theorem.

**Theorem 1.4 (see [45]).** Let \(f : E \rightarrow F\) be a mapping which satisfies the inequality

\[
\left\|f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2\left(m^2 - 2\right)f(x) + 2f(y)\right\| \leq e \left(\|x\|_E^p \|y\|_E^p + \|x\|_E^{2p} + \|y\|_E^{2p}\right) \tag{1.9}
\]

for all \(x, y \in E\) with \(x \perp y\), where \(e\) and \(p\) are constants with \(e, p > 0\) and either \(m > 1, p < 1\) or \(m < 1, p > 1\) with \(m \neq 0, m \neq \pm 1, m \neq \sqrt{2}\), and \(-1 \neq |m|^{-1} < 1\). Then, the limit \(\lim_{n \rightarrow \infty} m^{-2n}f(m^n x)\) exists for all \(x \in E\) and \(Q : E \rightarrow F\) is the unique orthogonally Euler-Lagrange quadratic mapping such that

\[
\left\|f(x) - Q(x)\right\| \leq \frac{e}{2[m^2 - m^{2p}]} \|x\|_E^{2p} \tag{1.10}
\]

for all \(x \in E\).

Note that the mixed “product-sum” function was introduced by J. M. Rassias in 2008-2009 [46–48].

We recall some basic facts concerning fuzzy normed space.

Let \(X\) be a real linear space. A function \(N : X \times \mathbb{R} \rightarrow [0, 1]\) (so-called fuzzy subset) is said to be a fuzzy norm on \(X\) if for all \(x, y \in X\) and all \(c, t \in \mathbb{R}\),

(N1) \(N(x, c) = 0\) for \(c \leq 0\);
(N2) \(x = 0\) if and only if \(N(x, c) = 1\) for all \(c > 0\);
(N3) \(N(cx, t) = N(x, t/|c|)\) if \(c \neq 0\);
(N4) \(N(x + y, t) \geq \min\{N(x, t), N(y, t)\}\);
(N5) \( N(x, \cdot) \) is a nondecreasing function of \( \mathbb{R} \) and
\[
\lim_{t \to \infty} N(x, t) = 1.
\] (1.11)

The pair \((X, N)\) is called a fuzzy normed linear space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [49–51].

Let \((X, N)\) be a fuzzy normed space and let \(\{x_n\}\) be a sequence in \(X\). Then, \(\{x_n\}\) is said to be convergent if there exists \(x \in X\) such that \(\lim_{n \to \infty} N(x_n - x, t) = 1\) for all \(t > 0\). In that case, \(x\) is called the limit of the sequence \(\{x_n\}\) and we denote it by \(\lim_{n \to \infty} x_n = x\).

A sequence \(\{x_n\}\) in a fuzzy normed space \((X, N)\) is called Cauchy if, for each \(\epsilon > 0\) and \(\delta > 0\), one can find some \(n_0\) such that
\[
N(x_m - x_n, \delta) > 1 - \epsilon
\] (1.12)
for all \(n, m \geq n_0\).

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If, in a fuzzy-normed space, each Cauchy sequence is convergent, then the fuzzy-norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Stability of Cauchy, Jensen, quadratic, and cubic function equation in fuzzy normed spaces have first been investigated in [50–53].

In this paper, we give a generalization of the results from [13] and pose two open problems in fuzzy Banach space. For convenience, we use the following abbreviation for a given mapping \(f\):
\[
Df(x, y) =: f(x + y) - f(x) - f(y).
\] (1.13)

2. Stability of the Cauchy Functional Equation

Hereafter, unless otherwise stated, we will assume that \(X\) is real vector space, \((Y, N)\) is a complete fuzzy norm space and \(k\) is a fixed integer greater than 1.

**Theorem 2.1.** Let \((Z, N')\) be a fuzzy normed space and \(\varphi : X \times X \to Z\) be a mapping such that, \(\varphi(\alpha x, \alpha y) = \varphi(x, y)\) for some \(\alpha\) with \(0 < \alpha < 1\). Suppose that \(f : X \to Y\) be mapping such that
\[
N(Df(x, y), t) \geq N'(\varphi(x, y), t)
\] (2.1)
for all \(x, y \in X\) and all positive real number \(t\). Then, there is a unique additive mapping \(T_k : X \to Y\) such that \(T_k(x) = \lim_{n \to \infty} f(k^n x) / k^n\) and
\[
N(T_k(x) - f(x), t) \geq M_k(x, (k - \alpha)t),
\] (2.2)
where \(M_k(x, t) := \min\{N'(\varphi(x, i x), t) : 1 \leq i < k\}\).

**Proof.** By induction on \(k\), we show that
\[
N(f(kx) - kf(x), t) \geq M_k(x, t) := \min\{N'(\varphi(x, i x), t) : 1 \leq i < k\}
\] (2.3)
for all \( x \in X \) and all positive real number \( t \). Letting \( y = x \) in (2.1), we get

\[
N \left( f(2x) - 2f(x), t \right) \geq N'(\varphi(x, x), t). \tag{2.4}
\]

So we get (2.3) for \( k = 2 \).

Assume that (2.3) holds for \( k \) with \( k > 2 \). Letting \( y = kx \) in (2.1), we get

\[
N \left( f((k+1)x) - f(kx), t \right) \geq N'(\varphi(x, kx), t). \tag{2.5}
\]

for all \( x \in X \). By using (2.3) and (2.5), we get (2.3) for \( k + 1 \) and this completes the induction argument. Replacing \( x \) by \( k^nx \) in (2.3), we get

\[
N \left( f(k^{n+1}x) - kf(k^n x), t \right) \geq M_k(k^n x, t). \tag{2.6}
\]

Thus

\[
N \left( \frac{1}{k^{n+1}} f(k^n x) - \frac{1}{k^n} f(k^{n-1} x), t \right) \geq \frac{1}{k^n} f(k^{n-1} x) \tag{2.7}
\]

for all \( x \in X \) and all positive real number \( t \). Hence,

\[
N \left( \frac{1}{k^{n+1}} f(k^n x) - \frac{1}{k^n} f(k^{n-1} x), \sum_{i=m}^{n} \frac{\alpha^i}{k^{i+1}} t \right) \geq \frac{1}{k^n} f(k^{n-1} x) \tag{2.8}
\]

Let \( \epsilon > 0 \) and \( \delta > 0 \) be given. Since \( \lim_{n \to \infty} M_k(x, t) = 1 \), there is some \( t_0 > 0 \) such that \( M_k(x, t_0) > 1 - \epsilon \). Since \( \sum_{n=0}^{\infty} (a^n/k^n) t_0 < \infty \), there is some \( n_0 \in N \) such that \( \sum_{i=m}^{n} (a^i/k^i) t_0 < k\delta \) for all \( n > m \geq n_0 \). It follows that

\[
N \left( \frac{1}{k^{n+1}} f(k^n x) - \frac{1}{k^n} f(k^{n-1} x), \delta \right) \geq \frac{1}{k^n} f(k^{n-1} x) \tag{2.9}
\]

for all \( x \in X \) and all nonnegative integers \( n \) and \( m \) with \( n > m \geq n_0 \). Therefore, the sequence \( \{1/k^n f(k^n x)\} \) is a Cauchy sequence in \((Y, N)\) for all \( x \in X \). Since \((Y, N)\) is complete, the
sequence \( \{1/k^n f(k^n x)\} \) converges in \( Y \) for all \( x \in X \). So one can define the mapping \( T_k : X \to Y \) by

\[
T_k(x) := \lim_{n \to \infty} \frac{1}{k^n} f(k^n x) \tag{2.10}
\]

for all \( x \in X \). Now, we show that \( T_k \) is an additive mapping. It follows from (2.1) and (2.10) that

\[
N(DT_k(x, y), t) = \lim_{n \to \infty} N\left( \frac{Df(k^n x, k^n y)}{k^n}, t \right) \\
\geq \lim_{n \to \infty} N\left( \frac{\varphi(k^n x, k^n y)}{k^n}, t \right) \\
= \lim_{n \to \infty} N\left( \varphi(x, y), \frac{k^n}{\alpha^n} t \right) \\
= 1
\]

for all \( x, y \in X \) and all positive real number \( t \). Therefore, the mapping \( T_k \) is additive. Moreover, if we put \( m = 0 \) in (2.8), we observe that

\[
N\left( \frac{1}{k^{n+1}} f(k^{n+1} x) - f(x), \sum_{i=0}^{n} \frac{\alpha^i}{k^{i+1}} t \right) \geq M_k(x, t) \tag{2.12}
\]

Therefore,

\[
N\left( \frac{1}{k^{n+1}} f(k^{n+1} x) - f(x), t \right) \geq M_k\left( x, \frac{t}{\sum_{i=0}^{n} (\alpha^i/k^{i+1})} \right) \tag{2.13}
\]

It follows from (2.13), for large enough \( n \), that

\[
N(T_k(x) - f(x), t) \geq \min \left\{ N\left( \frac{f(k^{n+1} x)}{k^{n+1}} - f(x), t \right), N\left( T_k(x) - \frac{f(k^{n+1} x)}{k^{n+1}}, t \right) \right\} \\
\geq M_k\left( x, \frac{t}{\sum_{i=0}^{n} (\alpha^i/k^{i+1})} \right) \\
\geq M_k(x, (k - \alpha)t) \tag{2.14}
\]

Now, we show that \( T_k \) is unique. Let \( T' \) be another additive mapping from \( X \) into \( Y \), which satisfies the required inequality. Then, for each \( x \in X \) and \( t > 0 \), we have

\[
N(T_k(x) - T'(x), t) \geq \min \left\{ N(T_k(x) - f(x), t), N(f(x) - T'(x), t) \right\} \\
\geq M_k(x, (k - \alpha)t) \tag{2.15}
\]
So,

\[ N\left( T_k(x) - T'(x), t \right) = N\left( \frac{T_k(k^n x)}{k^n} - \frac{T'(k^n x)}{k^n}, t \right) \]
\[ = N\left( T_k(k^n x) - T'(k^n x), k^n t \right) \]
\[ \geq M_k \left( x, (k - \alpha) \frac{k^n t}{\alpha n} \right). \]

Hence, the right-hand side of the above inequality tends to 1 as \( n \to \infty \). It follows that \( T_k(x) = T'(x) \) for all \( x \in X \).

\[ \square \]

**Theorem 2.2.** Let \((Z, N')\) be a fuzzy normed space and, \( \Phi : X \times X \to Z \) be a mapping such that \( \Phi(k^{-1}x, k^{-1}y) = \alpha^{-1}\Phi(x, y) \) for some \( \alpha \) with \( \alpha > k \). Suppose that \( f : X \to Y \) be mapping such that

\[ N(Df(x, y), t) \geq N'(\Phi(x, y), t) \] (2.17)

for all \( x, y \in X \) and all positive real number \( t \). Then, there is a unique additive mapping \( T_k : X \to Y \) such that \( T_k(x) = \lim_{n \to \infty} k^n f\left( x / k^n \right) \) and

\[ N(T_k(x) - f(x), t) \geq M_k(x, (\alpha - k)t), \]

where \( M_k(x, t) := \min\{ N'(\Phi(x, ix), t) : 1 \leq i < k \} \).

**Proof.** Similarly to the proof of Theorem 2.1, we have

\[ N\left( f(kx) - kf(x), t \right) \geq M_k(x, t) \] (2.19)

for all \( x \in X \) and all positive real number \( t \). Replacing \( x \) by \( x / k^{n+1} \) in (2.19), we get

\[ N\left( f\left( \frac{x}{k^n} \right) - kf\left( \frac{x}{k^{n+1}} \right), t \right) \geq M_k\left( \frac{x}{k^{n+1}}, t \right). \] (2.20)

Thus,

\[ N\left( k^n f\left( \frac{x}{k^n} \right) - k^{n+1} f\left( \frac{x}{k^{n+1}} \right), k^n t \right) \geq M_k\left( x, \alpha^{n+1} t \right) \] (2.21)
for all $x \in X$ and all positive real number $t$. Hence,
\[
N\left( f(k^{n+1}x) - k^n f\left( \frac{x}{k^n} \right) \right) \geq N\left( \sum_{i=m}^{n} k^i f\left( \frac{x}{k^{i+1}} \right) - k^m f\left( \frac{x}{k^m} \right) \sum_{i=m}^{n} \frac{k^i}{\alpha^{i+1}}t \right) \\
\geq \min \bigcup_{i=m}^{n} \left\{ N\left( k^{i+1} f\left( \frac{x}{k^{i+1}} \right) - k^i f\left( \frac{x}{k^i} \right) , \frac{k^i}{\alpha^{i+1}}t \right) \right\} \\
\geq M_k(x, t).
\]  
(2.22)

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{k \to \infty} M_k(x, t) = 1$, there is some $t_0 > 0$ such that $M_k(x, t_0) > 1 - \epsilon$. Since $\sum_{n=0}^{\infty} (k^n / \alpha^n) t_0 < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{n=0}^{n_0} (k^n / \alpha^n) t_0 < \alpha \delta$ for all $n > m \geq n_0$. It follows from (2.22) that
\[
N\left( f(k^{n+1}x) - k^n f\left( \frac{x}{k^n} \right) , \delta \right) \geq N\left( f(k^{n+1}x) - k^n f\left( \frac{x}{k^n} \right) , \sum_{i=m}^{n} \frac{k^i}{\alpha^{i+1}}t_0 \right) \\
\geq M_k(x, t_0) > 1 - \epsilon
\]  
(2.23)

for all $x \in X$ and all nonnegative integers $n$ and $m$ with $n > m \geq n_0$. Therefore, the sequence \( \{k^n f(x/k^n)\} \) is a Cauchy sequence in $\mathcal{Y}$ for all $x \in X$. Since $\mathcal{Y}$ is complete, the sequence \( \{k^n f(x/k^n)\} \) converges in $\mathcal{Y}$ for all $x \in X$. So one can define the mapping $T_k : X \to \mathcal{Y}$ by
\[
T_k(x) := \lim_{n \to \infty} k^n f\left( \frac{x}{k^n} \right)
\]  
(2.24)

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.1.

**Theorem 2.3.** Let $X$ be a normed space, let $(\mathcal{Z}, N')$ be a fuzzy normed space, and let $\psi : [0, \infty) \to [0, \infty)$ be a function such that
(1) $\psi(ts) = \psi(t)\psi(s)$,
(2) $\psi(t) < t$ for all $t > 1$.

Suppose that a mapping $f : X \to \mathcal{Y}$ satisfies the inequality:
\[
N(Df(x, y), t) \geq N'((\psi(||x||) + \psi(||y||))z_0, t)
\]  
(2.25)

for all $x, y \in X$ and all positive real number $t$, where $z_0$ is a fixed vector of $\mathcal{Z}$. Then, there exists a unique additive mapping $T_k : X \to \mathcal{Y}$ satisfying $T_k(x) := \lim_{n \to \infty} f(k^n x/k^n)$ and
\[
N(T_k(x) - f(x), t) \geq N'\left( \psi(||x||)z_0, \frac{k - \psi(k)}{\alpha_k(\psi)}t \right)
\]  
(2.26)

for all $x \in X$, where $\alpha_k(\psi) = \max\{1 + \psi(i) : 1 \leq i < k\}$. Moreover, $T_k = T_2$ for all $k \geq 2$. 
Proof. Let
\[ \varphi(x, y) = (\varphi(\|x\|) + \varphi(\|y\|))z_0 \] (2.27)
for all \( x, y \in X \). So,
\[ \varphi(kx, ky) = \varphi(k)\varphi(x, y). \] (2.28)
where \( \varphi(k) < k \). By using Theorem 2.1, we can get (2.26). Now, we show that \( T_k = T_2 \). It follows from (1) that \( \varphi(k^n) = (\varphi(k))^n \). Replacing \( x \) by \( 2^n x \) in (2.26), we get
\[ N(T_k(2^n x) - f(2^n x), t) \geq N' \left( \varphi(\|2^n x\|)z_0, \frac{k - \varphi(k)}{\sigma_k(\varphi)}t \right) \] (2.29)
for all \( x \in X \). So we have
\[ N \left( T_k(x) - \frac{f(2^n x)}{2^n}, t \right) \geq N' \left( \varphi(\|x\|)z_0, \frac{k - \varphi(k)}{\sigma_k(\varphi)}\varphi(2^n)2^n t \right) \] (2.30)
Using (2) and passing the limit \( n \to \infty \) in (2.30), we get \( T_k = T_2 \). □

**Theorem 2.4.** Let \( X \) be a normed space, let \( (Z, N') \) be a fuzzy normed space, and let \( \varphi : [0, \infty) \to [0, \infty) \) be a function such that
1. \( \varphi(ts) = \varphi(t)\varphi(s) \),
2. \( \varphi(t) > t \) for all \( t > 1 \).

Suppose that a mapping \( f : X \to Y \) satisfies the inequality:
\[ N(Df(x, y), t) \geq N'((\varphi(\|x\|) + \varphi(\|y\|))z_0, t) \] (2.31)
for all \( x, y \in X \) and all positive real number \( t \), where \( z_0 \) is a fixed vector of \( Z \). Then, there exists a unique additive mapping \( T_k : X \to Y \) satisfying \( T_k(x) := \lim_{n \to \infty} k^n f(x/k^n) \) and
\[ N(T_k(x) - f(x), t) \geq N' \left( \varphi(\|x\|)z_0, \frac{\varphi(k) - k}{\sigma_k(\varphi)}t \right) \] (2.32)
for all \( x \in X \), where
\[ \sigma_k(\varphi) = \max\{1 + \varphi(i) : 1 \leq i < k\}. \] (2.33)
Moreover, \( T_k = T_2 \) for all \( k \geq 2 \).

Proof. Let
\[ \Phi(x, y) = (\varphi(\|x\|) + \varphi(\|y\|))z_0 \] (2.34)
for all $x, y \in X$. So, we have

$$
\Phi\left( k^{-1}x, k^{-1}y \right) = q(k^{-1})\Phi(x, y),
$$

(2.35)

where $q(k^{-1}) = q(k)^{-1} < k^{-1}$. It follows from (1) that $q(k^{-n}) = (q(k))^{-n}$. By using Theorem 2.2, we can get (2.32). Now, we show that $T_k = T_2$. Replacing $x$ by $x/2^n$ in (2.32), we get

$$
N\left( T_k \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right), t \right) \geq N\left( q\left( \left\| \frac{x}{2^n} \right\| \right)z_0, \frac{q(k) - k}{\sigma_k(q)}t \right).
$$

(2.36)

for all $x \in X$. So we have

$$
N\left( T_k(x) - 2^nf \left( \frac{x}{2^n} \right), t \right) \geq N\left( q(\|x\|)z_0, \frac{q(k) - k}{2^n\sigma_k(q)q(2^{-n})}t \right).
$$

(2.37)

Using (2) and passing the limit $n \to \infty$ in (2.37), we get $T_k = T_2$. □

**Theorem 2.5.** Let $X$ be a normed space, let $p$ be a nonnegative real number such that $p \neq 1$, and let $H : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a homogeneous function of degree $p$. Suppose that $(Z, N')$ be a fuzzy normed space and let $f : X \to Y$ be mapping such that

$$
N(Df(x, y), t) \geq N'(H(\|x\|, \|y\|)z_0, t)
$$

(2.38)

for all $x, y \in X$ and all positive real number $t$, where $z_0$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$
N(T_k(x) - f(x), t) \geq M_k(x, |k^p - k|t),
$$

(2.39)

where $M_k(x, t) := \min\{ N'(\|x\|^pH(1, t)z_0, t) : 1 \leq i < k \}$.

**Proof.** The proof follows from Theorems 2.1 and 2.2. □

For the particular cases $H(x, y) = \theta(x^p + y^p)$, $H(x, y) = x^r y^s$, $H(x, y) = x^r y^s + x^{r+s} + y^{r+s} (r + s = p)$, and $H(x, y) = \min\{x^p, y^p\}$, we have the following corollaries.

**Corollary 2.6.** Let $X$ be a normed space, let $p$ be a nonnegative real number such that $p \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and $f : X \to Y$ be mapping such that

$$
N(Df(x, y), t) \geq N'((\|x\|^p + \|y\|^p)\theta, t)
$$

(2.40)

for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$
N(T_k(x) - f(x), t) \geq N\left( \|x\|^p\theta, \frac{|k^p - k|}{1 + (k - 1)p^t} \right).
$$

(2.41)
Corollary 2.7. Let $X$ be a normed space, $r, s$ be non-negative real numbers such that $p := r + s \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and $f : X \to Y$ be mapping such that

$$N(Df(x, y), t) \geq N'(|x|^p \|y\|^s \theta, t)$$

(2.42)

for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^p} t\right).$$

(2.43)

Corollary 2.8. Let $X$ be a normed space, and let $r, s$ be nonnegative real numbers such that $p := r + s \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and let $f : X \to Y$ be mapping such that

$$N(Df(x, y), t) \geq N'\left(\theta \|x\|^p \|y\|^s + \theta \|x\|^{r+s} + \theta \|y\|^{r+s}, t\right)$$

(2.44)

for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^p + (k-1)^s + 1} t\right).$$

(2.45)

Corollary 2.9. Let $X$ be a normed space, let $p$ be a nonnegative real number such that $p \neq 1$. Suppose that $(Z, N')$ be a fuzzy normed space and let $f : X \to Y$ be mapping such that

$$N(Df(x, y), t) \geq N'(\min\{\|x\|^p, \|y\|^p\} \theta, t)$$

(2.46)

for all $x, y \in X$ and all positive real number $t$, where $\theta$ is a fixed vector of $Z$. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'(\|x\|^p \theta, |k^p - k| t).$$

(2.47)

Problem 1. Whether Theorem 2.5 and/or such Corollaries can be proved for $p = 1$?

Problem 2. What is the best possible value of $k$ in Corollaries 2.6 and 2.7?

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References


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