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Retraction

Retracted: The Split Common Fixed Point Problem for Quasi-Total Asymptotically Nonexpansive Uniformly Lipschitzian Mappings

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Research Article

The Split Common Fixed Point Problem for Quasi-Total Asymptotically Nonexpansive Uniformly Lipschitzian Mappings

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We introduce an algorithm for solving the split common fixed point problem for quasi-total asymptotically nonexpansive uniformly Lipschitzian mapping in Hilbert spaces. The results presented in this paper improve and extend some recent corresponding results.

1. Introduction and Preliminaries

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem (SFP) is formulated as finding a point x^* with the property

$$x^* \in C, \quad Ax^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator.

The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [1, 3–7]. The SFP in infinite-dimensional Hilbert spaces can be found in [8–16].

The split common fixed point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP), see [4]. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two mappings satisfying $F(S) = \{x \in H_1 : Sx = x\} \neq \phi$ and $F(T) = \{x \in H_2 : Tx = x\} \neq \phi$, respectively. The split common fixed point problem for mappings S and T is to find a point $q \in H_1$ with the property

$$q \in F(S), \quad Aq \in F(T), \tag{1.2}$$

where A is a bounded linear operator from H_1 to H_2. We use Γ to denote the set of solutions of SCFP (1.2).

We first recall some definitions, notations, and conclusions which will be used in proving our main results.

Let E be a Banach space. A mapping $T : E \to E$ is said to be demiclosed at origin, if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$ and $\|(I - T)x_n\| \to 0$, then $x^* = Tx^*$.

A Banach space E is said to satisfy Opial's condition, if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x^*$ implies that

$$\liminf_{n\to\infty}\|x_n - x^*\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in E \text{ with } y \neq x^*. \tag{1.3}$$

It is well known that every Hilbert space satisfies Opial's condition.

Definition 1.1. Let H be a real Hilbert space, T a mapping from H into itself, and the fixed point set $F(T)$ of T nonempty.

(1) T is said to be *quasi-nonexpansive* if for all $(x, q) \in H \times F(T)$

$$\|Tx - q\| \leq \|x - q\|. \tag{1.4}$$

(2) T is said to be *quasi-asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for all $(x, q) \in H \times F(T)$

$$\|T^n x - q\| \leq k_n\|x - q\|, \quad \forall n \geq 1, \tag{1.5}$$

(3) T is said to be $(\{\mu_n\}, \{\xi_n\}, \phi)$-*quasi-total asymptotically nonexpansive* if for all $(x, q) \in H \times F(T)$

$$\|T^n x - q\|^2 \leq \|x - q\|^2 + \mu_n\phi(\|x - q\|) + \xi_n, \quad \forall n \geq 1, \tag{1.6}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\{\mu_n\}$ and $\{\xi_n\}$ are two nonnegative real sequences satisfying $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. The class of mappings was introduced by Alber et al. [17] in 2006.

(4) T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for all $(x, y) \in H \times H$

$$\|T^n x - T^n y\| \leq L\|x - y\|. \tag{1.7}$$

(5) T is said to be *semicompact* if for any bounded sequence $\{x_n\} \subset H$ with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a point $x^* \in H$.

Remark 1.2. From Definition 1.1 we can see that a quasi-nonexpansive mapping or an asymptotically quasi-nonexpansive mapping is a $(\{\mu_n\}, \{\xi_n\}, \phi)$-quasi-total asymptotically nonexpansive mapping. But the converse does not hold.

In [9], Moudafi proposed the following iterative algorithm for solving split common fixed problem of quasi-nonexpansive mappings: for arbitrarily chosen $x_1 \in H_1$,

$$\begin{aligned} &\left\{ u_n = x_n + \gamma\beta A^*(T - I)Ax_n, \right. \\ &x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U u_n, \quad n \in N, \end{aligned} \tag{1.8}$$

and proved that $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$, where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two quasi-nonexpansive mappings and $A : H_1 \to H_2$ is a bounded linear operator.

Inspired by the work of Moudafi [9, 10], very recently, Qin et al. [12] introduced the following iterative algorithm to study multiples-sets feasibility problem of a finite family of asymptotically quasi-nonexpansive mappings. For arbitrarily chosen $x_1 \in H_1$, $\{x_n\}$ is defined as follows:

$$\begin{aligned} y_n &= x_n + \gamma\beta A^*(T_n^n - I)Ax_n, \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n S_n^n y_n, \quad n \geq 1, \end{aligned} \tag{1.9}$$

where $S_n^n = S_{n(\bmod N)}^n$, $T_n^n = T_{n(\bmod N)}^n$, $n \geq 1$, $A : H_1 \to H_2$ is a bounded linear operator. They also proved that $\{x_n\}$ converges strongly or weakly to a multiple-sets split common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in Hilbert spaces under some suitable conditions.

Motivated and inspired by the work of Moudafi [9, 10] and Qin et al. [12], in this paper, we study the SCFP (1.2) of $(\{\mu_n\}, \{\xi_n\}, \phi)$-quasi-total asymptotically nonexpansive mappings in Hilbert spaces and obtain some of the strong and weak convergence of the presented algorithm to some $q \in \Gamma$. The results obtained in this paper improve and extend the result of Qin et al. [12], Xu [14], and Yang [16] and others.

By using the well-known equality $\langle x, y\rangle = (1/2)\|x\|^2 + (1/2)\|y\|^2 - (1/2)\|x - y\|^2$ in Hilbert spaces, we can easily show the following proposition. The proof is omitted.

Proposition 1.3. *Let $T : H \to H$ be a $(\{\mu_n\}, \{\xi_n\}, \phi)$-quasi-total asymptotically nonexpansive mapping. Then for each $q \in F(T)$ and $x \in H$, the following inequalities hold:*

$$\langle x - T^n x, x - q\rangle \geq \frac{1}{2}\|x - T^n x\|^2 - \frac{\mu_n}{2}\phi(\|x - q\|) - \frac{\xi_n}{2}, \tag{1.10}$$

$$\langle x - T^n x, q - T^n x\rangle \leq \frac{1}{2}\|x - T^n x\|^2 + \frac{\mu_n}{2}\phi(\|x - q\|) + \frac{\xi_n}{2}. \tag{1.11}$$

Lemma 1.4. *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n. \tag{1.12}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the $\lim_{n\to\infty} a_n$ exists.

2. Main Results

For solving split common fixed point problem (1.2), we assume that the following conditions are satisfied:

(i) H_1 and H_2 are two real Hilbert spaces, and $A : H_1 \to H_2$ is a bounded linear operator;

(ii) $S : H_1 \to H_1$ is uniformly L_1-Lipschitzian and $(\{\mu_n^{(1)}\}, \{\xi_n^{(1)}\}, \phi_1)$-quasi-total asymptotically nonexpansive mapping and $T : H_2 \to H_2$ is uniformly L_2-Lipschitzian and $(\{\mu_n^{(2)}\}, \{\xi_n^{(2)}\}, \phi_2)$-quasi-total asymptotically nonexpansive mapping satisfying the following conditions:

(a) $C := F(S) \neq \emptyset$, $Q := F(T) \neq \emptyset$;

(b) $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$, $\xi_n = \max\{\xi_n^{(1)}, \xi_n^{(2)}\}$, $n \geq 1$, and $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(c) $\phi = \max\{\phi_1, \phi_2\}$ and there exists two positive constant M and M^* such that $\phi(\lambda) \leq M^*\lambda^2$ for all $\lambda \geq M$.

Theorem 2.1. *Let $H_1, H_2, A, S, T, C, Q, L_1, L_2, \{\mu_n\}, \{\xi_n\}$, and ϕ be the same as above. Let $\{x_n\}$ be the sequence generated by: arbitrarily chosen $x_1 \in H_1$*

$$\begin{aligned} u_n &= x_n + \gamma A^*(T^n - I)Ax_n, \\ x_{n+1} &= (1-\alpha_n)u_n + \alpha_n S^n u_n, \quad n \geq 1, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$ is a sequence in $[\alpha, 1-\alpha]$ for some $\alpha \in (0,1)$ and $\gamma > 0$ is a constant satisfying the following condition:

(d) $\gamma \in (0, 1/\|A\|^2)$.

If T and S both are demi-closed at origin and $\Gamma \neq \emptyset$, then

(I) *the sequence $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$;*

(II) *in addition, if S is also semicompact, then $\{x_n\}$ and $\{u_n\}$ both converge strongly to a $x^* \in \Gamma$.*

Proof. (I) The proof will be divided into 4 steps.

Step 1. We prove that $\lim_{n\to\infty}\|x_n - q\|$ and $\lim_{n\to\infty}\|u_n - q\|$ exist, and $\lim_{n\to\infty}\|x_n - q\| = \lim_{n\to\infty}\|u_n - q\|$for each $q \in \Gamma$. Since ϕ is a continuous and increasing function, as $\lambda \geq 0$, we can obtain that

$$\phi(\lambda) \leq \phi(M) + M^*\lambda^2. \tag{2.2}$$

Taking $q \in \Gamma$, that is $q \in F(S) = C$ and $Aq \in F(T) = Q$, and using (1.10), (2.1), and (2.2), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|u_n - q - \alpha_n(u_n - S^n u_n)\|^2 \\
&= \|u_n - q\|^2 - 2\alpha_n\langle u_n - q, u_n - S^n u_n\rangle + \alpha_n^2\|u_n - S^n u_n\|^2 \\
&\le \|u_n - q\|^2 - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2 \\
&\quad + \alpha_n\mu_n\left(\phi(M) + M^*(\|u_n - q\|)^2\right) + \alpha_n\xi_n \\
&= (1 + \alpha_n\mu_n M^*)\|u_n - q\|^2 - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2 \\
&\quad + \alpha_n\mu_n\phi(M) + \alpha_n\xi_n,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\|u_n - q\|^2 &= \|x_n + \gamma A^*(T^n - I)Ax_n - q\|^2 \\
&= \|x_n - q\|^2 + \gamma^2\|A^*(T^n - I)Ax_n\|^2 + 2\gamma\langle x_n - q, A^*(T^n - I)Ax_n\rangle \\
&= \|x_n - q\|^2 + \gamma^2\langle (T^n - I)Ax_n, AA^*(T^n - I)Ax_n\rangle \\
&\quad + 2\gamma\ \langle x_n - q, A^*(T^n - I)Ax_n\rangle,
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
\gamma^2\langle (T^n - I)Ax_n, AA^*(T^n - I)Ax_n\rangle &\le \|A\|^2\gamma^2\langle (T^n - I)Ax_n, (T^n - I)Ax_n\rangle \\
&\le \|A\|^2\gamma^2\|(T^n - I)Ax_n\|^2,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
2\gamma\langle x_n - q, A^*(T^n - I)Ax_n\rangle &= 2\gamma\langle A(x_n - q), (T^n - I)Ax_n\rangle \\
&= 2\gamma\langle A(x_n - q) + (T^n - I)Ax_n - (T^n - I)Ax_n, (T^n - I)Ax_n\rangle \\
&= 2\gamma\langle A(x_n - q) + (T^n - I)Ax_n - (T^n - I)Ax_n, (T^n - I)Ax_n\rangle \\
&= 2\gamma\left(\langle T^n(Ax_n) - Aq, (T^n - I)Ax_n\rangle - \|(T^n - I)Ax_n\|^2\right).
\end{aligned} \tag{2.6}$$

Using (1.11), we have

$$\begin{aligned}
&2\gamma\left(\langle T^n(Ax_n) - Aq, (T^n - I)Ax_n\rangle - \|(T^n - I)Ax_n\|^2\right) \\
&\quad \le \gamma\|(T^n - I)Ax_n\|^2 + \gamma\mu_n\phi(\|Ax_n - Aq\|) + \gamma\xi_n - 2\gamma\|(T^n - I)Ax_n\|^2 \\
&\quad \le -\gamma\|(T^n - I)Ax_n\|^2 + \gamma\mu_n M^*\|A\|^2\|x_n - q\| + \gamma\mu_n\phi(M) + \gamma\xi_n.
\end{aligned} \tag{2.7}$$

Substituting (2.5) and (2.7) into (2.4) and simplifying, we obtain

$$\begin{aligned}\|u_n - q\|^2 &\le \|x_n - q\|^2 + \gamma^2\|A\|^2\|(T^n - I)Ax_n\|^2 - \gamma\|(T^n - I)Ax_n\|^2 \\ &\quad + \gamma\mu_n M^*\|A\|^2\|x_n - q\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n \\ &\le \left(1 + \gamma\mu_n M^*\|A\|^2\right)\|x_n - q\|^2 - \gamma\left(1 - \gamma\|A\|^2\right)\|(T^n - I)Ax_n\|^2 \\ &\quad + \gamma\mu_n\phi(M) + \gamma\xi_n.\end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.3) and simplifying, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le (1 + \alpha_n\mu_n M^*)\left\{\left(1 + \gamma\mu_n M^*\|A\|^2\right)\|x_n - q\|^2\right. \\ &\quad \left. -\gamma\left(1 - \gamma\|A\|^2\right)\|(T^n - I)Ax_n\|^2 + \gamma\mu_n\phi(M) + \gamma\xi_n\right\} \\ &\quad - \alpha_n(1 - \alpha_n)\|u_n - S^n u_n\|^2 + \alpha_n\mu_n\phi(M) + \alpha_n\xi_n \\ &= (1 + b_n)\|x_n - q\|^2 - (1 + \alpha_n\mu_n M^*)\gamma\left(1 - \gamma\|A\|^2\right)\|(T^n - I)Ax_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n)\|u_n - S^n u_n\|^2 + c_n,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}b_n &= \alpha_n\mu_n M^* + \gamma\mu_n M^*\|A\|^2 + \gamma\|A\|^2\alpha_n\mu_n^2(M^*)^2, \\ c_n &= (1 + \alpha_n\mu_n M^*)(\gamma\mu_n\phi(M) + \alpha_n\xi_n) + \alpha_n\mu_n\phi(M) + \alpha_n\xi_n.\end{aligned} \tag{2.10}$$

Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\xi_n < \infty$, so $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. By Condition (4), we have

$$\|x_{n+1} - q\|^2 \le (1 + b_n)\|x_n - q\|^2 + c_n. \tag{2.11}$$

Therefore, it follows from Lemma 1.4 that $\lim_{n\to\infty}\|x_n - q\|$ exists.We now prove that $\lim_{n\to\infty}\|u_n - q\|$ exists for each $q \in \Gamma$. Since $\lim_{n\to\infty}\|x_n - q\|$ exists, from (2.9), we have

$$\begin{aligned}&(1 + \alpha_n\mu_n M^*)\gamma\left(1 - \gamma\|A\|^2\right)\|(T^n - I)Ax_n\|^2 + \alpha_n(1 - \alpha_n)\|u_n - S^n u_n\|^2 \\ &\quad \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 - b_n\|x_n - q\|^2 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.12}$$

This together with Condition (4) implies that

$$\lim_{n\to\infty}\|u - S^n u_n\| = 0, \tag{2.13}$$

$$\lim_{n\to\infty}\|(T^n - I)Ax_n\| = 0. \tag{2.14}$$

Thus, since $\lim_{n\to\infty}\|x_n - q\|$ exists, it follows from (2.4) and (2.14) that $\lim_{n\to\infty}\|u_n - q\|$ exists and $\lim_{n\to\infty}\|x_n - q\| = \lim_{n\to\infty}\|u_n - q\|$.

Step 2. Now we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$.

As a matter of fact, it follows from (2.1) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)u_n + \alpha_n S^n u_n - x_n\| \\
&= \left\|(1-\alpha_n)\left(x_n + \gamma A^*(T^n - I)Ax_n\right) + \alpha_n S^n u_n - x_n\right\| \\
&= \left\|(1-\alpha_n)\gamma A^*(T^n - I)Ax_n + \alpha_n(S^n u_n - x_n)\right\| \\
&= \left\|(1-\alpha_n)\gamma A^*(T^n - I)Ax_n + \alpha_n(S^n u_n - u_n) + \alpha_n(u_n - x_n)\right\| \\
&= \left\|(1-\alpha_n)\gamma A^*(T^n - I)Ax_n + \alpha_n(S^n u_n - u_n) + \alpha_n\gamma A^*(T^n - I)Ax_n\right\| \\
&= \left\|\gamma A^*(T^n - I)Ax_n + \alpha_n(S^n u_n - u_n)\right\|.
\end{aligned} \tag{2.15}$$

In view of (2.13) and (2.14) we have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.16}$$

Similarly, it follows from (2.1), (2.14), and (2.16) that

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \left\|x_{n+1} + \gamma A^*\left(T^{n+1} - I\right)Ax_{n+1} - \left(x_n + \gamma A^*(T^n - I)Ax_n\right)\right\| \\
&\le \|x_{n+1} - x_n\| + \gamma\left\|A^*\left(T^{n+1} - I\right)Ax_{n+1}\right\| \\
&\quad + \gamma\|A^*(T^n - I)Ax_n\| \longrightarrow 0, \quad (n \longrightarrow \infty).
\end{aligned} \tag{2.17}$$

Step 3. Next, we prove that $\|u_n - Su_n\| \to 0$ and $\|Ax_n - TAx_n\| \to 0$ as $n \to \infty$.

Setting $\eta_n := \|u_n - S^n u_n\|$, since S is uniformly L-Lipschitzian continuous, it follows from (2.13), (2.16), and (2.17) that

$$\begin{aligned}
\|u_n - Su_n\| &\le \|u_n - S^n u_n\| + \|S^n u_n - Su_n\| \\
&\le \eta_n + L\left\|S^{n-1}u_n - u_n\right\| \\
&\le \eta_n + L\left(\left\|S^{n-1}u_n - S^{n-1}u_{n-1}\right\| + \left\|S^{n-1}u_{n-1} - u_n\right\|\right) \\
&\le \eta_n + L^2\|u_n - u_{n-1}\| + L\left(\left\|S^{n-1}u_{n-1} - u_{n-1}\right\| + \|u_{n-1} - u_n\|\right) \\
&\le \eta_n + L(1+L)\|u_n - u_{n-1}\| + L\eta_{n-1} \longrightarrow 0, \quad (n \longrightarrow \infty).
\end{aligned} \tag{2.18}$$

Similarly, we have

$$\|Ax_n - TAx_n\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{2.19}$$

Step 4. Finally, we prove that $x_n \rightharpoonup x^*$ and $u_n \rightharpoonup x^*$, where $x^* \in \Gamma$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup x^*$(some point in H_1). From (2.18) we have $\lim_{i\to\infty}\|u_{n_i} - Su_{n_i}\| = 0$. Since S is demi-closed at zero, we know that $x^* \in F(S)$. Moreover, it follows from (2.1) and (2.14) that

$$x_{n_i} = u_{n_i} - \gamma A^*(T^{n_i} - I)Ax_{n_i} \rightharpoonup x^*. \tag{2.20}$$

Since A is a linear bounded operator, it gets $Ax_{n_i} \rightharpoonup Ax^*$. In view of (2.19) we have $\lim_{i\to\infty}\|Ax_{n_i} - TAx_{n_i}\| = 0$.

Again since T is demi-closed at zero, we know that $Ax^* \in F(T)$. This implies that $x^* \in \Gamma$.

Assume that there exists another subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to a point $y^* \in H$ with $y^* \neq x^*$. Using the same argument above, we know that $y^* \in \Gamma$. Since each Hilbert space possesses Opial's property, we have

$$\begin{aligned}
\liminf_{i\to\infty}\|u_{n_i} - x^*\| &< \liminf_{i\to\infty}\left\|u_{n_i} - y^*\right\| = \lim_{n\to\infty}\left\|u_n - y^*\right\| \\
&= \liminf_{k\to\infty}\left\|u_{n_k} - y^*\right\| < \liminf_{k\to\infty}\|u_{n_k} - x^*\| \\
&= \lim_{n\to\infty}\|u_n - x^*\| = \liminf_{i\to\infty}\|u_{n_i} - x^*\|,
\end{aligned} \tag{2.21}$$

which is a contradiction. This implies that $\{u_n\}$ converges weakly to the point $x^* \in \Gamma$. Since $x_n = u_n - \gamma A^*(T^n - I)Ax_n$, we know that $\{x_n\}$ converges weakly to $x^* \in \Gamma$. The proof of conclusion(I) is completed.

□

Proof of Conclusion (II). Since S is semi-compact, it follows from Step 4 that there exists a subsequence of $\{u_{n_i}\}$ (without loss of generality, we still denote it by $\{u_{n_i}\}$) such that $\{u_{n_i}\} \to u^* \in H$ (some point in H). Since $\{u_{n_i}\} \rightharpoonup x^*$, this implies that $x^* = u^*$. And so $\{u_{n_i}\} \to x^* \in \Gamma$ as $i \to \infty$. Since $\lim_{n\to\infty}\|x_n - q\|$ and $\lim_{n\to\infty}\|u_n - q\|$ exist, and $\lim_{n\to\infty}\|x_n - q\| = \lim_{n\to\infty}\|u_n - q\|$ for each $q \in \Gamma$, we know that $\lim_{n\to\infty}\|u_n - x^*\| = \lim_{n\to\infty}\|x_n - x^*\| = 0$. This implies that $\{x_n\}$ and $\{u_n\}$ both converge strongly to a $x^* \in \Gamma$. The proof is completed.

□

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