Research Article

On Subclasses of Analytic Functions with respect to Symmetrical Points

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In our present investigation, motivated from Noor work, we define the class $\mathcal{R}_k^s(b)$ of functions of bounded radius rotation of complex order b with respect to symmetrical points and learn some of its basic properties. We also apply this concept to define the class $\mathcal{L}_k^s(\alpha, b, \delta)$. We study some interesting results, including arc length, coefficient difference, and univalence sufficient condition for this class.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic function satisfying the condition $f(0) = 0$, $f'(0) - 1 = 0$ in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and in more simple form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \quad (1.1)$$

By $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{S}^*$, we means the well-known subclasses of $\mathcal{A}$ which consists of univalent, convex, and starlike functions, respectively. In [1], Sakaguchi introduced the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetrical points and is defined as follows: a function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_s^*$, if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0 \quad (z \in \mathcal{U}). \quad (1.2)$$

Motivated from Sakaguchi work, Das and Singh [2] extend the concepts of $\mathcal{S}_s^*$ to other class in $\mathcal{U}$, namely, convex functions with respect to symmetrical points. Let $\mathcal{C}_s$ denote the class of convex functions with respect to symmetrical points and satisfying the following condition:

$$\operatorname{Re} \left\{ \frac{(zf'(z))'}{f'(z) - f'(-z)} \right\} > 0 \quad (z \in \mathcal{U}). \quad (1.3)$$

Let $\mathcal{P}_k(\delta)$, $0 \leq \delta < 1$, be the class of functions $p(z)$ analytic in $\mathcal{U}$ with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \delta}{1 - \delta} \right| d\theta \leq k\pi, \quad z = re^{i\theta}, \quad k \geq 2. \quad (1.4)$$

This class was introduced in [3]. For $\delta = 0$, we obtain the class $\mathcal{P}_k$ defined by Pinchuk [4], and for $k = 2$, the class $\mathcal{P}_k$ reduces to the class $\mathcal{P}$ of functions with positive real part.

Now, with the help of the aforementioned concepts, we define the class $\mathcal{R}_k^s(b)$ of functions of bounded radius rotation of complex order b with respect to symmetrical points as follows.

Definition 1.1. Let $f(z) \in \mathcal{A}$ in $\mathcal{U}$. Then $f(z) \in \mathcal{R}_k^s(b)$, if and only if

$$1 + \frac{1}{b} \left\{ \frac{2zf'(z)}{f(z) - f(-z)} - 1 \right\} \in \mathcal{P}_k \quad (z \in \mathcal{U}), \quad (1.5)$$

where $k \geq 2$ and $b \in \mathbb{C} - \{0\}$.

Using the class $\mathcal{R}_k^s(b)$, we define the class $\mathcal{H}_k^s(\alpha, b, \delta)$ as follows.

Definition 1.2. Let $f(z) \in \mathcal{A}$ in $\mathcal{U}$. Then $f(z) \in \mathcal{H}_k^s(\alpha, b, \delta)$, if and only if there exists $g(z) \in \mathcal{R}_k^s(b)$ such that

$$\frac{zf'(z)}{f(z)} \left(\frac{2f(z)}{g(z) - g(-z)} \right)^\alpha \in \mathcal{P}(\delta), \quad (1.6)$$

where $\alpha > 0$, $0 \leq \delta < 1$, and $b \in \mathbb{C} - \{0\}$.

It is noticed that, by giving specific values to α , b , δ , and k in $\mathcal{R}_k^s(b)$ and $\mathcal{H}_k^s(\alpha, b, \delta)$, we obtain many well-known as well as new subclasses of analytic and univalent functions; for details see [5–11].

Throughout this paper, we will assume, unless otherwise stated, that $k \geq 2$, $\alpha > 0$, $0 \leq \delta < 1$, and $b \in \mathbb{C} - \{0\}$.

Lemma 1.3. Let $p(z)$ be analytic in $\mathcal{U}$ where $p(0) = 1$ belongs to $\mathcal{P}(\delta)$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1 + (4(1 - \delta)^2 - 1)r^2}{1 - r^2} \quad (1.7)$$

(see [8, 12]).

Lemma 1.4. Let $s_1(z)$ be univalent function in $\mathcal{U}$. Then there exists ξ with $|\xi| = r$ such that for all z , $|z| = r$,

$$|z - \xi| |s_1(z)| \leq \frac{2r^2}{1 - r^2} \quad (1.8)$$

(see [13]).

2. Some Properties of the Classes $\mathcal{R}_k^s(b)$ and $\mathcal{L}_k^s(\alpha, b, \delta)$

Theorem 2.1. Let $f(z) \in \mathcal{R}_k^s(b)$. Then the odd function

$$\phi(z) = \frac{1}{2} [f(z) - f(-z)] \quad (2.1)$$

belongs to $\mathcal{R}_k(b)$ in $\mathcal{U}$.

Proof. Let $f(z) \in \mathcal{R}_k^s(b)$ and consider

$$\phi(z) = \frac{1}{2} [f(z) - f(-z)]. \quad (2.2)$$

From logarithmic differentiation of the previous relation, we have

$$\frac{\phi'(z)}{\phi(z)} = \frac{f'(z) - f'(-z)}{f(z) - f(-z)}, \quad (2.3)$$

or, equivalently,

$$\frac{z\phi'(z)}{\phi(z)} = \frac{1}{2} [p_1(z) + p_2(z)] \quad (2.4)$$

with

$$p_1(z) = \frac{2zf'(z)}{f(z) - f(-z)}, \quad p_2(z) = \frac{2(-z)f'(-z)}{f(-z) - f(z)} \quad (2.5)$$

belongs to $\mathcal{P}_k(b)$. Since $\mathcal{P}_k(b)$ is a convex set, we have

$$\frac{z\phi'(z)}{\phi(z)} \in \mathcal{P}_k(b) \quad (z \in \mathcal{U}), \quad (2.6)$$

and hence $\phi(z) \in \mathcal{R}_k(b)$. □

Theorem 2.2. Let $f(z) \in \mathcal{R}_k^s(b)$. Then

$$f'(z) = \frac{1}{2} [b(p(z) - 1) + 1] \exp \left\{ \frac{b}{2} \int_0^z \frac{1}{\xi} \left(p(\xi) - p(-\xi) - \frac{2}{b} \right) d\xi \right\}. \quad (2.7)$$

Proof. Let $f(z) \in \mathcal{R}_k^s(b)$. Then by definition we have

$$1 + \frac{1}{b} \left[\frac{2zf'(z)}{f(z) - f(-z)} - 1 \right] = p(z), \quad p(z) \in \mathcal{P}_k. \quad (2.8)$$

Simple computation yields us

$$\frac{f(z) - f(-z)}{z} = \exp \left\{ \frac{b}{2} \int_0^z \frac{1}{\xi} \left[p(\xi) - p(-\xi) - \frac{2}{b} \right] d\xi \right\}. \quad (2.9)$$

Using (2.8) in (2.9), we can easily obtain (2.7).

If we put $b = 1$ and $k = 2$ in Theorem 2.1, we obtain the integral representation for $\mathcal{S}_s^*$ given by Stankiewicz in [14]. $\square$

Theorem 2.3. Let $f(z) \in \mathcal{R}_k^s(b)$. Then

$$|a_2| \leq \frac{k|b|}{2}. \quad (2.10)$$

The function $f_0(z) \in \mathcal{R}_k^s(b)$ defined by

$$f_0'(z) = \frac{(1+z^2)^{(k-2)/4}}{(1-z^2)^{(k+2)/4}} \left[b \left\{ \left(\frac{k+2}{4} \right) \left(\frac{1-z}{1+z} \right) - \left(\frac{k-2}{4} \right) \left(\frac{1+z}{1-z} \right) \right\} + (1-b) \right] \quad (2.11)$$

shows that this bound is sharp.

Proof. Since $f(z) \in \mathcal{R}_k^s(b)$, there exists an odd function $\phi(z) \in \mathcal{R}_k(b)$ with

$$\phi(z) = \frac{1}{2} [f(z) - f(-z)], \quad (2.12)$$

such that

$$zf'(z) = \phi(z)p(z), \quad (2.13)$$

with $p(z) \in \mathcal{P}_k(b)$. Let

$$\phi(z) = z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \quad (2.14)$$

Then (2.13) implies that

$$z + \sum_{n=2}^{\infty} n a_n z^n = \left[z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1} \right] \left[1 + \sum_{n=1}^{\infty} c_n z^n \right]. \quad (2.15)$$

Equating the coefficients of z^2 , we have $2a_2 = c_1$, and so

$$|a_2| \leq \frac{k|b|}{2}, \quad (2.16)$$

where we have used the coefficient bounds $|c_1| \leq k|b|$ for the class $\mathcal{P}_k(b)$. $\square$

Corollary 2.4. *The range of every univalent function $f(z) \in \mathcal{R}_k^s(b)$ contains the disc*

$$|w| < \frac{2}{4 + k|b|}. \quad (2.17)$$

Proof. The Koebe one-quarter theorem states that each omitted value w of the univalent function $f(z)$ of the form (1.1) satisfies

$$|w| > \frac{1}{2 + |a_2|}. \quad (2.18)$$

Using (2.18) and Theorem 2.3, we obtain the required result. $\square$

By using the same method as in [1], we obtain the following result.

Theorem 2.5. *Let $f(z) \in \mathcal{R}_k^s(b)$. Then, for $z = re^{i\theta}$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -(k-1)|b|\pi. \quad (2.19)$$

Theorem 2.6. *Let $f(z) \in \mathcal{J}_k^s(\alpha, b, 0)$. Then, for $z = re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} J(\alpha, f(z)) d\theta > -(\alpha|b|(k-1) + 1)\pi, \quad (2.20)$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and

$$J(\alpha, f(z)) = \left(1 + \frac{zf''(z)}{f'(z)} \right) + (\alpha - 1) \frac{zf'(z)}{f(z)}. \quad (2.21)$$

Proof. We can define, for $z = re^{i\theta}$, $r < 1$, θ real, the following:

$$S(r, \theta) = \arg \left[z f'(z) f^{\alpha-1}(z) \right], \quad (2.22)$$

$$V(r, \theta) = \arg \left[\frac{g(z) - g(-z)}{2} \right]^\alpha. \quad (2.23)$$

The functions $S(z)$ and $V(z)$ are periodic and continuous with period 2π . Since $f(z) \in \mathcal{H}_k^s(\alpha, b, 0)$, therefore from (2.22), it follows that we can choose the branches of argument of $S(z)$ and $V(z)$ as

$$|S(r, \theta) - V(r, \theta)| \leq \frac{\pi}{2}. \quad (2.24)$$

Now we have from (2.22)

$$V(r, \theta_2) - V(r, \theta_1) = \alpha \int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{z \phi'(z)}{\phi(z)} d\theta, \quad (2.25)$$

where $\phi(z)$ is an odd function of the following form:

$$\phi(z) = \frac{1}{2} [g(z) - g(-z)]. \quad (2.26)$$

Since $g(z) \in \mathcal{R}_k^s(b)$, therefore by using Theorem 2.5, we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{z \phi'(z)}{\phi(z)} d\theta > -(k-1)|b|\pi. \quad (2.27)$$

From (2.22), (2.23), (2.24), and (2.27), we have

$$\begin{aligned} |S(r, \theta_2) - S(r, \theta_1)| &= |S(r, \theta_2) - V(r, \theta_2) - (S(r, \theta_1) - V(r, \theta_1)) + (V(r, \theta_2) - V(r, \theta_1))| \\ &< \frac{\pi}{2} + \frac{\pi}{2} + \alpha(k-1)|b|\pi = (\alpha|b|(k-1) + 1)\pi. \end{aligned} \quad (2.28)$$

Moreover, from (2.22)

$$\frac{d}{d\theta} S(r, \theta) = \operatorname{Re} \left[\left(1 + \frac{z f''(z)}{f'(z)} \right) + (\alpha-1) \frac{z f'(z)}{f(z)} \right]. \quad (2.29)$$

Therefore

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} J(\alpha, f(z)) d\theta > -(\alpha|b|(k-1) + 1)\pi. \quad (2.30)$$

□

Theorem 2.7. Let $f(z) \in \mathcal{H}_k^s(\alpha, b, \delta)$. Then for $\alpha(k/2 + 1) \operatorname{Re} b > 1$,

$$L_r f(z) \leq \begin{cases} C(\alpha, b, \delta, k) M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{\alpha(k/2+1) \operatorname{Re} b}, & 0 < \alpha \leq 1, \\ C(\alpha, b, \delta, k) m^{\alpha-1}(r) \left(\frac{1}{1-r} \right)^{\alpha(k/2+1) \operatorname{Re} b}, & \alpha > 1, \end{cases} \quad (2.31)$$

where $m(r) = \min_{|z|=r} |f(z)|$, $M(r) = \max_{|z|=r} |f(z)|$, and $C(\alpha, b, \delta, k)$ is a constant depending upon α, b, δ , and k only.

Proof. We know that

$$L_r f(z) = \int_0^{2\pi} |z f'(z)| d\theta, \quad z = r e^{i\theta}, \quad 0 < r < 1. \quad (2.32)$$

Since $f(z) \in \mathcal{H}_k^s(\alpha, b, \delta)$, therefore

$$\frac{z f'(z)}{f(z)} \left(\frac{2f(z)}{g(z) - g(-z)} \right)^\alpha = p(z), \quad p(z) \in \mathcal{P}(\delta). \quad (2.33)$$

By Theorem 2.1, we have, for $g(z) \in \mathcal{R}_k^s(b)$, the odd function $\phi(z) = (1/2)[g(z) - g(-z)] \in \mathcal{R}_k(b)$. This implies that

$$z f'(z) = (f(z))^{1-\alpha} (\phi(z))^\alpha p(z). \quad (2.34)$$

Therefore, we have

$$\begin{aligned} L_r f(z) &\leq \int_0^{2\pi} |f(z)|^{1-\alpha} |\phi(z)|^\alpha |p(z)| d\theta, \\ &\leq M^{1-\alpha}(r) \int_0^{2\pi} |\phi(z)|^\alpha |p(z)| d\theta. \end{aligned} \quad (2.35)$$

Since $\phi(z) \in \mathcal{R}_k(b)$, therefore we have for odd functions $s_1(z)$, $s_1(z) \in \mathcal{S}^*$,

$$\begin{aligned} &\leq M^{1-\alpha}(r) \int_0^{2\pi} \left| \frac{(s_1(z))^{(k/4+1/2)b}}{(s_2(z))^{(k/4-1/2)b}} \right|^\alpha |p(z)| d\theta, \\ &\leq c M^{1-\alpha}(r) \int_0^{2\pi} \frac{|(s_1(z))|^{\alpha(k/4+1/2) \operatorname{Re} b}}{|(s_2(z))|^{\alpha(k/4-1/2) \operatorname{Re} b}} |p(z)| d\theta, \quad c = e^{(\pi/2) \operatorname{Im} b}. \\ &\leq c M^{1-\alpha}(r) 2^{\alpha(k/2-1) \operatorname{Re} b} r^{-\alpha(k/4-1/2) \operatorname{Re} b} \int_0^{2\pi} |(s_1(z))|^{\alpha(k/4+1/2) \operatorname{Re} b} |p(z)| d\theta. \end{aligned} \quad (2.36)$$

Now using Cauchy Schwarz inequality, we have

$$\begin{aligned} L_r f(z) &\leq cM^{1-\alpha}(r)2^{\alpha(k/2-1)\operatorname{Re} b}r^{-\alpha(k/4-1/2)\operatorname{Re} b}\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2d\theta\right)^{1/2} \\ &\times\left(\frac{1}{2\pi}\int_0^{2\pi}|(s_1(z))|^{2\alpha(k/4+1/2)\operatorname{Re} b}d\theta\right)^{1/2}. \end{aligned} \quad (2.37)$$

By Lemma 1.3 and distortion results for the class $\mathcal{S}^*$ with a subordination result, we obtain

$$\begin{aligned} L_r f(z) &\leq cM^{1-\alpha}(r)2^{\alpha(k/2-1)\operatorname{Re} b}r^{\alpha\operatorname{Re} b}\left(\frac{1}{(1-r)^{4\alpha(k/4+1/2)\operatorname{Re} b-1}}\right)^{1/2}\left(\frac{1+(4(1-\delta)^2-1)r^2}{1-r^2}\right)^{1/2} \\ &= C(\alpha, b, \delta, k)M^{1-\alpha}(r)\left(\frac{1}{1-r}\right)^{\alpha(k/2+1)\operatorname{Re} b}. \end{aligned} \quad (2.38)$$

Similarly for $\alpha > 1$, we have

$$L_r f(z) \leq C(\alpha, b, \delta, k)m^{\alpha-1}(r)\left(\frac{1}{1-r}\right)^{\alpha(k/2+1)\operatorname{Re} b}. \quad (2.39)$$

□

Theorem 2.8. Let $f(z) \in \mathcal{H}_k^s(\alpha, b, \delta)$. Then for $\alpha(k/2+1)\operatorname{Re} b > 1$

$$|a_n| \leq \begin{cases} C_1(\alpha, b, \delta, k)M^{1-\alpha}(n)(n)^{\alpha(k/2+1)\operatorname{Re} b-1}, & 0 < \alpha \leq 1, \\ C_1(\alpha, b, \delta, k)m^{\alpha-1}(n)(n)^{\alpha(k/2+1)\operatorname{Re} b-1}, & \alpha > 1, \end{cases} \quad (2.40)$$

where m and M are the same as in Theorem 2.7 and $C_1(\alpha, b, \delta, k)$ is a constant depending upon α, b, δ , and k only.

Proof. Since, with $z = re^{i\theta}$ Cauchy theorem gives

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} z f'(z) e^{-in\theta} d\theta, \quad (2.41)$$

therefore

$$n|a_n| \leq \frac{1}{2\pi r^n} L_r f(z). \quad (2.42)$$

Now using Theorem 2.7 for $0 < \alpha \leq 1$, we have

$$n|a_n| \leq \frac{1}{2\pi r^n} C(\alpha, b, \delta, k) M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{\alpha(k/2+1) \operatorname{Re} b}. \quad (2.43)$$

Putting $r = 1 - 1/n$, we have

$$|a_n| \leq C_1(\alpha, b, \delta, k) M^{1-\alpha}(r) (n)^{\alpha(k/2+1) \operatorname{Re} b - 1}. \quad (2.44)$$

Similarly we obtain the required result for $\alpha > 1$. $\square$

Theorem 2.9. Let $f(z) \in \mathcal{H}_k^s(\alpha, b, \delta)$. Then for $\alpha(k/2 + 1) \operatorname{Re} b > 3$,

$$\|a_{n+1} - a_n\| \leq \begin{cases} C_2(\alpha, b, \delta, k) M^{1-\alpha}(r) (n)^{\alpha(k/2+1) \operatorname{Re} b - 2}, & 0 < \alpha \leq 1, \\ C_2(\alpha, b, \delta, k) m^{\alpha-1}(r) (n)^{\alpha(k/2+1) \operatorname{Re} b - 2}, & \alpha > 1. \end{cases} \quad (2.45)$$

Proof. We know that for $\xi \in \mathcal{U}$ and $n \geq 1$,

$$|(n+1)\xi a_{n+1} - na_n| \leq \int_0^{2\pi} |z - \xi| |zf'(z)| d\theta, \quad z = re^{i\theta}, \quad 0 < r < 1, \quad 0 \leq \theta \leq 2\pi. \quad (2.46)$$

Since $f(z) \in \mathcal{H}_k^s(\alpha, b, \delta)$, therefore

$$\frac{zf'(z)}{f(z)} \left(\frac{2f(z)}{g(z) - g(-z)} \right)^\alpha = p(z), \quad p(z) \in \mathcal{P}(\delta). \quad (2.47)$$

By Theorem 2.1, we have, for $g(z) \in \mathcal{R}_k^s(b)$, the odd function $\phi(z) = (1/2)[g(z) - g(-z)] \in \mathcal{R}_k(b)$. This implies that

$$zf'(z) = (f(z))^{1-\alpha} (\phi(z))^\alpha p(z). \quad (2.48)$$

Thus, for $\xi \in \mathcal{U}$ and $n \geq 1$, we have

$$|(n+1)\xi a_{n+1} - na_n| \leq M^{1-\alpha}(r) \int_0^{2\pi} |z - \xi| |\phi(z)|^\alpha |p(z)| d\theta. \quad (2.49)$$

Since $\phi(z) \in \mathcal{R}_k(b)$, therefore we have for odd functions $s_1(z)$, $s_1(z) \in \mathcal{S}^*$,

$$|(n+1)\xi a_{n+1} - na_n| \leq \frac{2^{\alpha(k/2-1) \operatorname{Re} b} e^{\operatorname{Im} b(\pi/2)} M^{1-\alpha}(r)}{2\pi r^{n+1-\alpha(k/2-1) \operatorname{Re} b}} \int_0^{2\pi} |z - \xi| |(s_1(z))|^{\alpha(k/4+1/2) \operatorname{Re} b} |p(z)| d\theta. \quad (2.50)$$

By using Lemma 1.4, we have

$$\leq \frac{2^{\alpha(k/2-1)\operatorname{Re} b} e^{\operatorname{Im} b(\pi/2)} M^{1-\alpha}(r)}{2\pi r^{n-1-\alpha(k/2-1)\operatorname{Re} b}(1-r)} \int_0^{2\pi} |(s_1(z))|^{\alpha(k/4+1/2)\operatorname{Re} b-1} |p(z)| d\theta. \quad (2.51)$$

Now using Cauchy Schwarz inequality, we have

$$\begin{aligned} |(n+1)\xi a_{n+1} - na_n| &\leq \frac{2^{\alpha(k/2-1)\operatorname{Re} b} e^{\operatorname{Im} b(\pi/2)} M^{1-\alpha}(r)}{2\pi r^{n-1-\alpha(k/2-1)\operatorname{Re} b}(1-r)} \left(\frac{1}{2\pi} \int_0^{2\pi} |(s_1(z))|^{2\alpha(k/4+1/2)\operatorname{Re} b-2} d\theta \right)^{1/2} \\ &\quad \times \left(\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \right)^{1/2}. \end{aligned} \quad (2.52)$$

By Lemma 1.3 and distortion result for the class S^* with a subordination result, we obtain

$$\begin{aligned} |(n+1)\xi a_{n+1} - na_n| &\leq c M^{1-\alpha}(r) 2^{\alpha(k/4)-\operatorname{Re} b} r^{\alpha\operatorname{Re} b-n+1} \left(\frac{1}{(1-r)} \right)^{\alpha(k/2+1)\operatorname{Re} b-5/2+1} \\ &\quad \times \left(\frac{1 + (4(1-\delta)^2 - 1)r^2}{1-r^2} \right)^{1/2}. \end{aligned} \quad (2.53)$$

Now putting $|\xi| = r = n/(n+1)$, we obtain

$$||a_{n+1}| - |a_n|| \leq C_2(\alpha, b, \delta, k) M^{1-\alpha}(r) (n)^{\alpha(k/2+1)\operatorname{Re} b-2}. \quad (2.54)$$

Similarly for $\alpha > 1$, we have

$$||a_{n+1}| - |a_n|| \leq C_2(\alpha, b, \delta, k) m^{\alpha-1}(r) (n)^{\alpha(k/2+1)\operatorname{Re} b-2}. \quad (2.55)$$

□

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