Research Article

Quasimonotone and Almost Increasing Sequences and Their New Applications

Hüseyin Bor

P.O. Box 121, 06502 Bahçelievler, Ankara, Turkey

Correspondence should be addressed to Hüseyin Bor, hbor33@gmail.com

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Recently, we have proved a main theorem dealing with the absolute Nörlund summability factors of infinite series by using δ -quasimonotone sequences. In this paper, we prove that result under weaker conditions. A new result has also been obtained.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say $b_n = ne^{(-1)^n}$. A sequence (d_n) is said to be δ -quasimonotone if $d_n > 0$ ultimately and $\Delta d_n = d_n - d_{n+1} \geq -\delta_n$, where $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α , we denote the n th Cesàro means of order α , with $\alpha > -1$, of the sequences (s_n) and (na_n) , respectively, that is,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} s_v, \quad (1.1)$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v, \quad (1.2)$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for } n > 0. \quad (1.3)$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \quad (1.4)$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability.

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \quad (n \geq 0). \quad (1.5)$$

The sequence-to-sequence transformation

$$V_n = \frac{1}{P_n} \sum_{v=0}^n p_{n-v} s_v \quad (1.6)$$

defines the sequence (V_n) of the Nörlund mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) . The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |V_n - V_{n-1}|^k < \infty. \quad (1.7)$$

In the special case when

$$p_n = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)\Gamma(n + 1)}, \quad \alpha \geq 0, \quad (1.8)$$

the Nörlund mean reduces to the (C, α) mean and $|N, p_n|_k$ summability becomes $|C, \alpha|_k$ summability. For $p_n = 1$, we get the $(C, 1)$ mean and then $|N, p_n|_k$ summability becomes $|C, 1|_k$ summability. Also, if we take $k = 1$, then we get $|N, p_n|$ summability. For any sequence (λ_n) , we write $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$. Quite recently, in [5], we have proved the following theorem dealing with the absolute Nörlund summability factors of infinite series.

Theorem A. Let $p_0 > 0$, $p_n \geq 0$, and (p_n) be a nonincreasing sequence. Let (X_n) be an almost increasing sequence such that $|\Delta X_n| = O(X_n/n)$ and (λ_n) is a sequence such that

$$|\lambda_n| X_n = O(1) \quad \text{as } n \longrightarrow \infty. \quad (1.9)$$

Suppose also that there exists a sequence of numbers (A_n) such that it is δ -quasimonotone with $\sum n \delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, and $|\Delta \lambda_n| \leq |A_n|$ for all n . If the sequence (w_n^α) defined by (see [6])

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases} \quad (1.10)$$

satisfies the condition

$$\sum_{n=1}^m \frac{(w_n^\alpha)^k}{n} = O(X_m) \quad \text{as } m \rightarrow \infty, \quad (1.11)$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.

2. The Main Results

The aim of this paper is to prove Theorem A under weaker conditions. We will prove the following theorems.

Theorem 2.1. *If the sequences (X_n) , (A_n) , and (λ_n) are as in Theorem A and if conditions (1.9) and*

$$\sum_{n=1}^m \frac{(w_n^\alpha)^k}{n X_n^{k-1}} = O(X_m) \quad \text{as } m \rightarrow \infty \quad (2.1)$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|C, \alpha|_k$, $0 < \alpha \leq 1$ and $k \geq 1$.

Theorem 2.2. *Let (p_n) be as in Theorem A. If the sequences (X_n) , (A_n) , and (λ_n) are as in Theorem A and if conditions (1.9) and (2.1) are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.*

Remark 2.3. The following sequences satisfy the conditions of the theorems:

$$\delta_n = \frac{1}{n^3}, \quad A_n = \frac{1}{n^2}, \quad \lambda_n = \frac{1}{n}, \quad X_n = n^\epsilon, \quad 0 < \epsilon < 1. \quad (2.2)$$

Remark 2.4. It should be noted that condition (2.1) is the same as condition (1.11) when $k = 1$. When $k > 1$, condition (2.1) is weaker than condition (1.11), but the converse is not true. In fact, if (1.11) is satisfied, then we get that

$$\sum_{n=1}^m \frac{(w_n^\alpha)^k}{n X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^m \frac{(w_n^\alpha)^k}{n} = O(X_m). \quad (2.3)$$

To show that the converse is false when $k > 1$, the following example is sufficient. We can take $X_n = n^\epsilon$, $0 < \epsilon < 1$, and then construct a sequence (a_n) such that

$$\frac{(w_n^\alpha)^k}{n X_n^{k-1}} = X_n - X_{n-1}, \quad (2.4)$$

whence

$$\sum_{n=1}^m \frac{(w_n^\alpha)^k}{n X_n^{k-1}} = X_m = m^\epsilon, \quad (2.5)$$

and so

$$\begin{aligned}
 \sum_{n=1}^m \frac{(w_n^\alpha)^k}{n} &= \sum_{n=1}^m (X_n - X_{n-1}) X_n^{k-1} = \sum_{n=1}^m (n^\epsilon - (n-1)^\epsilon) n^{\epsilon(k-1)} \\
 &\geq \epsilon \sum_{n=1}^m (n-1)^{\epsilon-1} n^{\epsilon(k-1)} \\
 &= \epsilon \sum_{n=1}^m (n-1)^{\epsilon k-1} \sim \frac{m^{\epsilon k}}{k} \quad \text{as } m \rightarrow \infty.
 \end{aligned} \tag{2.6}$$

This is because $v^{\epsilon-1} \geq n^{\epsilon-1}$ for $n-1 \leq v \leq n$.

This shows that, when $k > 1$, (1.11) implies (2.1) but not conversely.

We need the following lemmas for the proof of our theorem.

Lemma 2.5 (see [7]). *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\left| \sum_{p=0}^v A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^m A_{m-p}^{\alpha-1} a_p \right|. \tag{2.7}$$

Lemma 2.6 (see [8]). *If $-1 < \alpha \leq \beta$, $k > 1$ and the series $\sum a_n$ is summable $|C, \alpha|_k$, then it is also summable $|C, \beta|_k$.*

Lemma 2.7 (see [9]). *Let (X_n) be an almost increasing sequence such that $n|\Delta X_n| = O(X_n)$.*

If (A_n) is a δ -quasimonotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, then

$$\begin{aligned}
 nA_n X_n &= O(1) \quad \text{as } n \rightarrow \infty, \\
 \sum_{n=1}^{\infty} nX_n |\Delta A_n| &< \infty.
 \end{aligned} \tag{2.8}$$

Lemma 2.8 (see [10]). *Let $p_0 > 0$, $p_n \geq 0$, and (p_n) be a nonincreasing sequence. If the series $\sum a_n$ is summable $|C, 1|_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.*

3. Proof of Theorem 2.1

Let (T_n^α) be the n th (C, α) , with $0 < \alpha \leq 1$, mean of the sequence $(na_n \lambda_n)$. Then, by (1.2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{3.1}$$

First applying Abel's transformation and then using Lemma 2.5, we have that

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v, \quad (3.2)$$

$$\begin{aligned} |T_n^\alpha| &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} |\Delta \lambda_v| \left| \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p \right| + \frac{|\lambda_n|}{A_n^\alpha} \left| \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \right| \\ &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta \lambda_v| + |\lambda_n| w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha. \end{aligned} \quad (3.3)$$

To complete the proof of Theorem 2.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1} |T_{n,r}^\alpha|^k < \infty \quad \text{for } r = 1, 2. \quad (3.4)$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k' , where $(1/k) + (1/k') = 1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} n^{-1} |T_{n,1}^\alpha|^k &\leq \sum_{n=2}^{m+1} n^{-1} (A_n^\alpha)^{-k} \left\{ \sum_{v=1}^m A_v^\alpha w_v^\alpha |\Delta \lambda_v| \right\}^k \\ &\leq \sum_{n=2}^{m+1} n^{-1-\alpha k} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} (w_v^\alpha)^k |A_v| \right\}^k \times \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1} \\ &= O(1) \sum_{v=1}^m v^{\alpha k} (w_v^\alpha)^k |A_v| |A_v|^{k-1} \sum_{n=v+1}^{m+1} \frac{1}{n^{2+(\alpha-1)k}} \\ &= O(1) \sum_{v=1}^m v^{\alpha k} (w_v^\alpha)^k |A_v| \frac{1}{(v X_v)^{k-1}} \int_v^\infty \frac{dx}{x^{2+(\alpha-1)k}} \\ &= O(1) \sum_{v=1}^m v |A_v| \frac{(w_v^\alpha)^k}{v X_v^{k-1}} = O(1) \sum_{v=1}^{m-1} \Delta(v |A_v|) \sum_{r=1}^v \frac{(w_r^\alpha)^k}{r X_r^{k-1}} \\ &\quad + O(1) m |A_m| \sum_{v=1}^m \frac{(w_v^\alpha)^k}{v X_v^{k-1}} = O(1) \sum_{v=1}^{m-1} |\Delta(v |A_v|)| X_v \\ &\quad + O(1) m |A_m| X_m = O(1) \sum_{v=1}^{m-1} v |\Delta A_v| X_v + O(1) \sum_{v=1}^m |A_v| X_v \\ &\quad + O(1) m |A_m| X_m = O(1) \quad \text{as } m \longrightarrow \infty, \end{aligned} \quad (3.5)$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.7. Again, we have that

$$\begin{aligned}
 \sum_{n=1}^m n^{-1} |T_{n,2}^\alpha|^k &= O(1) \sum_{n=1}^m |\lambda_n| |\lambda_n|^{k-1} \frac{(w_n^\alpha)^k}{n} \\
 &= O(1) \sum_{n=1}^m |\lambda_n| \frac{(w_n^\alpha)^k}{n X_n^{k-1}} = O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^n \frac{(w_v^\alpha)^k}{v X_v^{k-1}} \\
 &\quad + O(1) |\lambda_m| \sum_{n=1}^m \frac{(w_n^\alpha)^k}{n X_n^{k-1}} = O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n \\
 &\quad + O(1) |\lambda_m| X_m = O(1) \sum_{n=1}^{m-1} |A_n| X_n + O(1) |\lambda_m| X_m \\
 &= O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{3.6}$$

by virtue of the hypotheses of Theorem 2.1. This completes the proof of Theorem 2.1. If we take $\alpha = 1$, then we get a new result dealing with $|C, 1|_k$ summability factors.

Proof of Theorem 2.2. In order to prove Theorem 2.2, we need to consider only the special case in which (N, p_n) is (C, α) . Therefore, Theorem 2.2 will then follow by means of Theorem 2.1, Lemma 2.6 (for $\beta = 1$), and Lemma 2.8. If we take $\alpha = 1$, then we get a new result for the absolute Nörlund summability factors of infinite series. $\square$

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