Research Article

Strong Convergence of an Implicit S-Iterative Process for Lipschitzian Hemicontractive Mappings

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1. Introduction

Let \( H \) be a Hilbert space and let \( T : H \rightarrow H \) be a mapping.

The mapping \( T \) is called \textit{Lipschitzian} if there exists \( L > 0 \) such that

\[
\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H.
\] (1.1)

If \( L = 1 \), then \( T \) is called \textit{nonexpansive} and if \( 0 \leq L < 1 \), then \( T \) is called \textit{contractive}.

The mapping \( T \) is said to be \textit{pseudocontractive} ([1, 2]) if

\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H,
\] (1.2)

and the mapping \( T \) is said to be \textit{strongly pseudocontractive} if there exists \( k \in (0, 1) \) such that

\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H.
\] (1.3)
Let \( F(T) := \{ x \in H : Tx = x \} \) and the mapping \( T \) is called \textit{hemicontractive} if \( F(T) \neq \emptyset \) and
\[
\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx\|^2, \quad \forall x \in H, x^* \in F(T).
\] (1.4)

It is easy to see the class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractive mappings. For the importance of fixed points of pseudocontractions the reader may consult [1].

In 1974, Ishikawa [3] proved the following result.

\textbf{Theorem 1.1.} Let \( K \) be a compact convex subset of a Hilbert space \( H \) and let \( T : K \to K \) be a Lipschitzian pseudocontractive mapping.

For arbitrary \( x_1 \in K \), let \( \{ x_n \} \) be a sequence defined iteratively by
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 1,
\end{align*}
\] (1.5)

where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are sequences satisfying the conditions:

(i) \( 0 \leq \alpha_n \leq \beta_n \leq 1 \),

(ii) \( \lim_{n \to \infty} \beta_n = 0 \),

(iii) \( \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty \).

Then the sequence \( \{ x_n \} \) converges strongly to a fixed point of \( T \).

Another iteration scheme which has been studied extensively in connection with fixed points of pseudocontractive mappings.


Let \( K \) be a nonempty convex subset of a normed space \( X \) and let \( T : K \to K \) be a mapping. Then, for arbitrary \( x_1 \in K \), the \( S \)-iterative process is defined by
\[
\begin{align*}
x_{n+1} &= Ty_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 1,
\end{align*}
\] (1.6)

where \( \{ \beta_n \} \) is a real sequence in \( [0, 1] \).

In this paper, we establish the strong convergence for the implicit \( S \)-iterative process associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

\section{Main Results}

We need the following lemma.

\textbf{Lemma 2.1 (see [6]).} For all \( x, y \in H \) and \( \lambda \in [0, 1] \), the following well-known identity holds
\[
\| (1 - \lambda)x + \lambda y \|^2 = (1 - \lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.
\] (2.1)
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Now we prove our main results.

**Theorem 2.2.** Let $K$ be a compact convex subset of a real Hilbert space $H$ and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying

$$
\| x - Ty \| \leq \| Tx - Ty \|, \quad \forall x, y \in K.
$$

(C)

Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

1. $\sum_{n=1}^{\infty} \beta_n = \infty,$
2. $\sum_{n=1}^{\infty} \beta^2_n < \infty.$

For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$
x_n = Ty_n, \\
y_n = (1 - \beta_n)x_{n-1} + \beta_nTx_n, \quad n \geq 1.
$$

(2.2)

Then the sequence $\{x_n\}$ converges strongly to the fixed point $x^*$ of $T$.

**Proof.** From Schauder’s fixed point theorem, $F(T)$ is nonempty since $K$ is a convex compact set and $T$ is continuous, let $x^* \in F(T).$ Using the fact that $T$ is hemicontractive we obtain

$$
\| Tx_n - x^* \|^2 \leq \| x_n - x^* \|^2 + \| x_n - Tx_n \|^2, 
$$

(2.3)

$$
\| Ty_n - x^* \|^2 \leq \| y_n - x^* \|^2 + \| y_n - Ty_n \|^2.
$$

(2.4)

Now by (v), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$
\beta_n \leq \min \left\{ \frac{1}{3'}, \frac{1}{L^2} \right\},
$$

(2.5)

which implies that

$$
\frac{2\beta_n}{1 - \beta_n} \leq 1.
$$

(2.6)
With the help of (2.2), (2.3), and Lemma 2.1, we obtain the following estimates:

\[
\|y_n - x^*\|^2 = \|(1 - \beta_n) x_{n-1} + \beta_n T x_n - x^*\|^2
\]
\[
= \|(1 - \beta_n) (x_{n-1} - x^*) + \beta_n (T x_n - x^*)\|^2
\]
\[
= (1 - \beta_n) \|x_{n-1} - x^*\|^2 + \beta_n \|T x_n - x^*\|^2
\]
\[
- \beta_n (1 - \beta_n) \|x_{n-1} - T x_n\|^2
\]
\[
\leq (1 - \beta_n) \|x_{n-1} - x^*\|^2 + \beta_n \left( \|x_{n-1} - x^*\|^2 + \|x_n - T x_n\|^2 \right)
\]
\[
- \beta_n (1 - \beta_n) \|x_{n-1} - T x_n\|^2.
\] (2.7)

\[
\|y_n - T y_n\|^2 = \|(1 - \beta_n) x_{n-1} + \beta_n T x_n - T y_n\|^2
\]
\[
= \|(1 - \beta_n) (x_{n-1} - T y_n) + \beta_n (T x_n - T y_n)\|^2
\]
\[
= (1 - \beta_n) \|x_{n-1} - T y_n\|^2 + \beta_n \|T x_n - T y_n\|^2
\]
\[
- \beta_n (1 - \beta_n) \|x_{n-1} - T x_n\|^2.
\]

Substituting (2.7) in (2.4) we obtain

\[
\|T y_n - x^*\|^2 \leq (1 - \beta_n) \|x_{n-1} - x^*\|^2 + \beta_n \left( \|x_{n-1} - x^*\|^2 + \|x_n - T x_n\|^2 \right)
\]
\[
+ (1 - \beta_n) \|x_{n-1} - T y_n\|^2 + \beta_n \|T x_n - T y_n\|^2
\]
\[
- 2\beta_n (1 - \beta_n) \|x_{n-1} - T x_n\|^2.
\] (2.8)

Also with the help of condition \((C)\) and (2.8), we have

\[
\|x_{n+1} - x^*\|^2 = \|T y_n - x^*\|^2
\]
\[
\leq (1 - \beta_n) \|x_{n-1} - x^*\|^2 + \beta_n \left( \|x_{n-1} - x^*\|^2 + \|x_n - T x_n\|^2 \right)
\]
\[
+ (1 - \beta_n) \|x_{n-1} - T y_n\|^2 + \beta_n \|T x_n - T y_n\|^2
\]
\[
- 2\beta_n (1 - \beta_n) \|x_{n-1} - T x_n\|^2
\]
\[
\leq (1 - \beta_n) \|x_{n-1} - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|x_{n-1} - T y_n\|^2
\]
\[
+ 2\beta_n \|T x_n - T y_n\|^2 - 2\beta_n (1 - \beta_n) \|x_{n-1} - T x_n\|^2.
\] (2.9)
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which implies that

\[
\|x_{n+1} - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 + \|x_{n-1} - Ty_n\|^2 \\
+ \frac{2\beta_n}{1 - \beta_n} \|Tx_{n-1} - Ty_n\|^2 \leq \|x_{n-1} - x^*\|^2 + \|x_{n-1} - Ty_n\|^2 \leq \left(\frac{1}{1 - \beta_n}\right)^2 \|x_{n-1} - Ty_n\|^2, \\
\]

(2.10)

where

\[
\|x_{n-1} - Ty_n\|^2 \leq \|Tx_{n-1} - Ty_n\|^2 \\
\leq L^2 \|x_{n-1} - y_n\|^2 \\
= L^2 \beta_n^2 \|x_{n-1} - Tx_n\|^2, \\
\]

(2.11)

\[
\|Tx_{n-1} - Ty_n\|^2 \leq L^2 \|x_n - y_n\|^2 \\
\leq L^2 (\|x_n - x_{n-1}\| + \|x_{n-1} - y_n\|)^2 \\
\leq L^2 (\|x_n - x_{n-1}\| + \beta_n \|x_{n-1} - Tx_n\|)^2 \\
\leq L^2 (\|x_n - x_{n-1}\| + \beta_n M)^2, \\
\]

(2.12)

\[
\|x_n - x_{n-1}\| = \|x_{n-1} - Ty_n\| \\
\leq \|Tx_{n-1} - Ty_n\| \\
\leq L \|x_{n-1} - y_n\| \\
= L \beta_n \|x_{n-1} - Tx_n\| \\
\leq L \beta_n M \\
\]

and consequently from (2.12), we obtain

\[
\|Tx_n - Ty_n\|^2 \leq L^2 (1 + L)^2 M^2 \beta_n^2, \\
\]

(2.13)
Hence by (2.5), (2.10), (2.11), and (2.13), we have
\[
\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 + L^2 \beta_n^2 \|x_{n-1} - Tx_n\|^2 \\
+ L^2 (1 + L)^2 M^2 \beta_n^2 - \beta_n \|x_{n-1} - Tx_n\|^2 \\
= \|x_{n-1} - x^*\|^2 + L^2 (1 + L)^2 M^2 \beta_n^2 \\
- \beta_n \left( 2 - L^2 \beta_n \right) \|x_{n-1} - Tx_n\|^2 \\
\leq \|x_{n-1} - x^*\|^2 + L^2 (1 + L)^2 M^2 \beta_n^2 - \beta_n \|x_{n-1} - Tx_n\|^2,
\]
which implies that
\[
\beta_n \|x_{n-1} - Tx_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + L^2 (1 + L)^2 M^2 \beta_n^2,
\]
so that
\[
\frac{1}{2} \sum_{j=N}^{n} \beta_j \|x_{j-1} - Tx_j\|^2 \leq \|x_N - x^*\|^2 - \|x_n - x^*\|^2 + L^2 (1 + L)^2 M^2 \sum_{j=N}^{n} \beta_j^2.
\]
Hence by conditions (iv) and (v), we get
\[
\sum_{j=0}^{\infty} \|x_{j-1} - Tx_j\|^2 < \infty.
\]
It implies that
\[
\lim_{n \to \infty} \|x_{n-1} - Tx_n\| = 0.
\]
Consider
\[
\|x_n - Tx_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - Tx_n\|,
\]
which implies that
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\]

The rest of the argument follows exactly as in the proof of Theorem of [3]. This completes the proof. □

**Theorem 2.3.** Let $K$ be a compact convex subset of a real Hilbert space $H$ and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying the condition (C). Let $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying the conditions (iv) and (v).
Assume that $P_K : H \to K$ be the projection operator of $H$ onto $K$. Let $\{x_n\}$ be a sequence defined iteratively by

$$x_n = P_K(Ty_n),$$
$$y_n = P_K(((1 - \beta_n)x_{n-1} + \beta_nTx_n), \quad n \geq 1.$$ (2.21)

Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof.** The operator $P_K$ is nonexpansive (see, e.g., [2]). $K$ is a Chebyshev subset of $H$ so that, $P_K$ is a single-valued mapping. Hence, we have the following estimate:

$$\|x_n - x^*\|^2 = \|P_K(Ty_n) - P_Kx^*\|^2 \leq \|Ty_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 + L^2(1 + L)^2M^2\beta_n^2 - \beta_n\|x_{n-1} - Tx_n\|^2.$$ (2.22)

The set $K = K \cup T(K)$ is compact and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 2.2. This completes the proof. $\square$

**Remark 2.4.** In main results, the condition (C) is not new and it is due to Liu et al. [7].

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**References**


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