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Research Article

The Uniform Boundedness Theorem in Asymmetric Normed Spaces

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We obtain a uniform boundedness type theorem in the frame of asymmetric normed spaces. The classical result for normed spaces follows as a particular case.

1. Introduction

Throughout this paper the letters $\mathbb{R}$ and $\mathbb{R}^+$ will denote the set of real numbers and the set of nonnegative real numbers, respectively.

The book of Aliprantis and Border [1] provides a good basic reference for functional analysis in our context.

Let X be a real vector space. A function $p : X \to \mathbb{R}^+$ is said to be an asymmetric norm on X [2, 3] if for all $x, y \in X$, and $r \in \mathbb{R}^+$,

(i) $p(x) = p(-x) = 0$ if and only if $x = 0$;

(ii) $p(rx) = rp(x)$;

(iii) $p(x + y) \leq p(x) + p(y)$.

The pair (X, p) is called an asymmetric normed space.

Asymmetric norms are also called quasinorms in [4–6], and nonsymmetric norms in [7].

If p is an asymmetric norm on X, then the function p^{-1} defined on X by $p^{-1}(x) = p(-x)$ is also an asymmetric norm on X, called the conjugate of p. The function p^s defined on X by

$p^s(x) = \max\{p(x), p^{-1}(x)\}$ is a norm on X. We say that (X, p) is a bi-Banach space if (X, p^s) is a Banach space [3]. The following is a simple instance of a biBanach space.

Example 1.1. Denote by u the function defined on $\mathbb{R}$ by $u(x) = x \vee 0$ for all $x \in \mathbb{R}$. Then u is an asymmetric norm on $\mathbb{R}$ such that u^s is the Euclidean norm on $\mathbb{R}$, that is, $(\mathbb{R}, u^s)$ is the Euclidean normed space $(\mathbb{R}, |\cdot|)$. Hence $(\mathbb{R}, u)$ is a biBanach space.

By a quasimetric on a nonempty set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

The function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is a quasi-metric on X called the conjugate of d.

Each quasi-metric d on X induces a T_0 topology $T(d)$ on X which has as a base the family of the balls $\{B_d(x, r) : x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

Each asymmetric norm p on a real vector space X induces a quasi-metric d_p defined by $d_p(x, y) = p(y - x)$, for all $x, y \in X$. We refer to the topology $T(d_p)$ as the topology induced by p. The terms p-neighborhood, p-open, p-closed, and so forth will refer to the corresponding topological concepts with respect to that topology. The ball $B_{d_p}(x, r)$ will be simply denoted by $B_p(x, r)$.

It was shown in [5] that for any normed lattice $(X, \|\cdot\|)$, the function p defined on X by $p(x) = \|x^+\|$, with $x^+ = x \vee 0$, is an asymmetric norm on X and the norm p^s is equivalent to the norm $\|\cdot\|$. Moreover, p determines both the topology and order of X. We will refer to p as the asymmetric norm associated to$(X, \|\cdot\|)$.

It seems interesting to point out that the recent development of the theory of asymmetric normed spaces has been motivated, in great part, by their applications. In fact, asymmetric norms (in particular, those that are associated to normed lattices) and other related nonsymmetric structures from topological algebra and functional analysis have been applied to construct suitable mathematical models in theoretical computer science [2, 8–10] as well as to some questions in approximation theory [6, 7, 11, 12].

The asymmetric normed spaces share some properties with usual normed spaces but there are also significant differences between them. In the last decade, several papers on general topology and functional analysis have been published in order to extend well-known results of the theory of normed spaces to the framework of asymmetric normed spaces (see, e.g., [3, 4, 11, 13–17]).

In this sense, the recent book of Cobzas [18] collects in a unified way the most interesting results obtained up to date in the context of nonsymmetric topology and fuctional analysis. Furthermore, in this monograph, the author also presents new results that significantly enrich the theory of asymmetric normed spaces. One of these new results which appears in the book is the uniform boundedness theorem that extends the classical one for normed spaces. In our terminology, this result is formulated as follows.

Let (X, p) be a right K-complete asymmetric normed space, and let (Y, q) be an asymmetric normed space. If $\mathcal{F}$ is a family of continuous linear operators from (X, p) to (Y, q) such that for each $x \in X$, there exist $b_x > 0$ and $c_x > 0$ with $q(f(x)) \leq b_x$ and $q^{-1}(f(x)) \leq c_x$ for all $f \in \mathcal{F}$, then there exist $b > 0$ and $c > 0$ such that

$$\sup\left\{q(f(x)) : p^{-1}(x) \leq 1\right\} \leq b, \qquad \sup\left\{q^{-1}(f(x)) : p(x) \leq 1\right\} \leq c, \tag{1.1}$$

for all $f \in \mathcal{F}$.

The condition of right K-completeness for (X,p) leaves outside the scope of this theorem an important class of asymmetric normed spaces, the asymmetric normed spaces associated to normed lattices because these spaces are right K-complete only for the trivial case [13]. In this paper, we give a uniform boundedness type theorem in the setting of asymmetric normed spaces which extends the classical result for normed spaces and it is less restrictive than Cobzas' theorem. For this purpose we introduce the notion of quasi-metric space of the half second category. These spaces play in our context a similar role that the metric spaces of the second category have played in the realm of normed spaces.

2. The Results

The result for normed linear spaces usually called Uniform Boundedness or Banach-Steinhauss theorem is formulated as follows.

Theorem 2.1 (see [1]). *Let X be a Banach space and let Y be a normed space. If $\mathcal{F}$ is a family of continuous linear operators from X to Y such that for each $x \in X$ there exists $b_x > 0$ with $\|f(x)\| \leq b_x$ for all $f \in \mathcal{F}$, then there exists $b > 0$ such that $\|f\| \leq b$ for all $f \in \mathcal{F}$.*

A natural way of extending the preceding result to the context of asymmetric normed spaces consists of replacing Banach space by biBanach asymmetric normed spaces. Thus one may conjecture that the next result would be desirable.

Let X be a biBanach asymmetric normed space and let Y be an asymmetric normed space. If $\mathcal{F}$ is a family of continuous linear operators from (X,p) to (Y,q) such that for each $x \in X$ there exists $b_x > 0$ with $q(f(x)) \leq b_x$ for all $f \in \mathcal{F}$, then there exists $b > 0$ such that

$$\sup\{q(f(x)) : p(x) \leq 1\} \leq b, \tag{2.1}$$

for all $f \in \mathcal{F}$.

Nevertheless, the following example shows that such a result does not hold in our context.

Example 2.2. Let $X_k = \{(x_n) \in l_1 : 2kx_{2k-1} + x_{2k} = 0\}$. Since X_k is a closed linear subspace of l_1 for every $k \in \mathbb{N}$, the subspace

$$X = \bigcap_{k=1}^{\infty} X_k, \tag{2.2}$$

is a closed linear subspace of l_1. We consider the asymmetric normed space (l_1,p), where

$$p(x) = \|x^+\|_1 = \sum_{n=1}^{\infty} x_n^+. \tag{2.3}$$

Since the norm p^s is equivalent to the norm $\|\cdot\|_1$ on l_1, we have that X is a closed linear subspace of the Banach space (l_1,p^s) and then (X,p^s) is a Banach space. Therefore (X,p) is a biBanach space.

Let $f_n : X \to \mathbb{R}$ be given by $f_n(x) = (2n+1)x_{2n-1}$ for every $n \in \mathbb{N}$. Since

$$f_n(x) = (2n+1)x_{2n-1} \leq (2n+1)x_{2n-1}^{+} \leq (2n+1)p(x), \tag{2.4}$$

we have that f_n is a continuous linear map from (X,p) to $(\mathbb{R},u)$, for every $n \in \mathbb{N}$.

If $x \in X$, then $\|x\|_1 = \sum_{n=1}^{\infty}(2n+1)|x_{2n-1}|$, so that

$$(2n+1)x_{2n-1} \leq (2n+1)|x_{2n-1}| \leq \|x\|_1. \tag{2.5}$$

Therefore, $f_n(x) = (2n+1)x_{2n-1} \leq \|x\|_1$, for every $n \in \mathbb{N}$.

Now, we will prove that $\sup\{f_n(x) : p(x) \leq 1\} = 2n+1$, for every $n \in \mathbb{N}$. Indeed, if $p(x) \leq 1$, then $f_n(x) \leq 2n+1$. If we consider $x = (x_i^n)_{i=1}^{\infty}$ such that $x_{2n-1}^n = 1$, $x_{2n}^n = -2n$ and $x_i^n = 0$ if $i \in \mathbb{N} - \{2n-1, 2n\}$, then $x \in X, p(x) = 1$ and $f_n(x) = 2n+1$. Hence,

$$\sup\{f_n(x) : p(x) \leq 1\} = 2n+1. \tag{2.6}$$

Consequently,

$$\sup\{\sup\{f_n(x) : p(x) \leq 1\} : n \in \mathbb{N}\} = \infty. \tag{2.7}$$

Before introducing the notion of a quasi-metric space of the half second category we recall some pertinent concepts in order to help the reader.

A topological space X is said to be of the first category if it is the union of a countable collection of nowhere dense subsets. X is said to be of the second category if it is not of the first category (see, e.g., [19, page 348]) From the characterization of nowhere dense subsets given in Proposition 11.13 [19], it follows that a topological space X is of the second category if and only if condition $X = \bigcup_{n=1}^{\infty} E_n$ implies $\operatorname{int}(\operatorname{cl} E_m) \neq \emptyset$ for some $m \in \mathbb{N}$.

If (X,d) is a quasi-metric space and A is a subset of X, we denote by $\operatorname{cl}_{d^{-1}} A$ the closure of A in the topological space $(X, T(d^{-1}))$ and by $\operatorname{int}_d A$ the interior of A in the topological space $(X, T(d))$. If (X,p) is an asymmetric normed space and A is a subset of X, we will write $\operatorname{cl}_{p^{-1}} A$ and $\operatorname{int}_p A$ instead of $\operatorname{cl}_{(d_p)^{-1}} A$ and $\operatorname{int}_{d_p} A$, respectively.

Definition 2.3. We say that a quasi-metric space (X,d) is of the half second category if $X = \bigcup_{n=1}^{\infty} E_n$, implies $\operatorname{int}_d(\operatorname{cl}_{d^{-1}} E_m) \neq \emptyset$ for some $m \in \mathbb{N}$.

Note that if d is a metric on X, the notion of space of the half second category coincides with the classical notion of space of the second category.

The quasi-metric space $(\mathbb{R}, d_u)$ given in Example 1.1 is of the half second category. Indeed, if $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$, take $m \in \mathbb{N}$ with $E_m \neq \emptyset$. If $E_m = \mathbb{R}$, then $\operatorname{cl}_{(d_u)^{-1}}(\operatorname{int}_{d_u} E_m) = \mathbb{R}$. Otherwise, since nonempty proper $(d_u)^{-1}$-closed subsets of $\mathbb{R}$ are of the form $]-\infty, a]$, $a \in \mathbb{R}$, there exists $a_m \in \mathbb{R}$ such that $\operatorname{cl}_{(d_u)^{-1}} E_m =]-\infty, a_m]$. Therefore, $\operatorname{int}_{d_u}(\operatorname{cl}_{(d_u)^{-1}} E_m) =]-\infty, a_m[$.

Note that $(\mathbb{R}, d_u)$ is not of the second category. Indeed, in Proposition 1 of [20] it is proved that if $(X, \|\cdot\|)$ is a normed lattice and $p(x) = \|x^+\|$, then (X, d_p) is not of the second category.

Lemma 2.4 (uniform boundedness principle). *If (X,d) is a quasi-metric space of the half second category and $\mathcal{F}$ is a family of real valued lower semicontinuous functions on (X,d^{-1}) such that for each $x \in X$ there exists $b_x > 0$ such that $f(x) \le b_x$ for all $f \in \mathcal{F}$, then there exist a nonempty open set U in (X,d) and $b > 0$ such that $f(x) \le b$ for all $f \in \mathcal{F}$ and $x \in U$.*

Proof. For each $n \in \mathbb{N}$ let

$$E_n = \bigcap_{f\in\mathcal{F}} f^{-1}]-\infty, n]. \tag{2.8}$$

Then each E_n is closed in (X,d^{-1}). Moreover $X = \bigcup_{n=1}^{\infty} E_n$. Indeed, by our hypothesis, given $x \in X$ there exists $n_x \in \mathbb{N}$ such that $f(x) \le n_x$ for all $f \in \mathcal{F}$, so $x \in E_{n_x}$.

Hence, there exists $m \in \mathbb{N}$ such that $U \neq \emptyset$, where $U = \mathrm{int}_d(\mathrm{cl}_{d^{-1}} E_m) = \mathrm{int}_d E_m$. Then, for each $x \in U$ and $f \in \mathcal{F}$ we obtain $f(x) \le m$. This completes the proof. □

Definition 2.5. We say that an asymmetric normed space (X,p) is of the half second category if the quasi-metric space (X,d_p) is of the half second category.

Theorem 2.6 (uniform boundedness theorem). *Let (X,p) and (Y,q) be two asymmetric normed spaces such that (X,p) is of the half second category. If $\mathcal{F}$ is a family of continuous linear operators from (X,p) to (Y,q) such that for each $x \in X$ there exists $b_x > 0$ with $q(f(x)) \le b_x$ for all $f \in \mathcal{F}$, then there exists $b > 0$ such that*

$$\sup\{q(f(x)) : p(x) \le 1\} \le b, \tag{2.9}$$

for all $f \in \mathcal{F}$.

Proof. For each $g \in \mathcal{F}$ define a function h_g by $h_g(x) = q(g(x))$ for all $x \in X$. We first show that h_g is lower semicontinuous on $(X,(d_p)^{-1})$. Indeed, let $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $(d_p)^{-1}(x,x_n) \to 0$. Then $p(x-x_n) \to 0$. By the linear continuity of g we deduce that $q(g(x)-g(x_n)) \to 0$. From the fact that $q(g(x)) - q(g(x_n)) \le q(g(x)-g(x_n))$ for all $n \in \mathbb{N}$, it follows that for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $h_g(x) - h_g(x_n) < \varepsilon$. Since x is arbitrary, we conclude that h_g is lower semicontinuous on $(X,(d_p)^{-1})$.

Put $\mathcal{H} = \{h_g : g \in \mathcal{F}\}$. Since for each $x \in X, h_g(x) = q(g(x)) \le b_x$ for all $g \in \mathcal{F}$, we can apply Lemma 2.4 and thus there exists a nonempty p-open subset U of X and a $\delta > 0$ such that $h_g(x) \le \delta$ for all $g \in \mathcal{F}$ and $x \in U$.

Fix $z \in U$. Then, there exists $r > 0$ such that $B_p(z,r) \subset U$. Take an $\varepsilon \in]0,r[$. Then, $\overline{B}_p(z,\varepsilon) \subset U$, where $\overline{B}_p(z,\varepsilon) = \{y \in X : p(y-z) \le \varepsilon\}$. Put $b = (\delta + b_{-z})/\varepsilon$, and let $x \in X$ such that $p(x) \le 1$. We will prove that $q(f(x)) \le b$ for all $f \in \mathcal{F}$. Indeed, first note that $p(\varepsilon x + z - z) = \varepsilon p(x) \le \varepsilon$, so $\varepsilon x + z \in U$.

Now take $f \in \mathcal{F}$. Then

$$\begin{aligned} q(f(x)) &= \frac{1}{\varepsilon} q(f(\varepsilon x)) = \frac{1}{\varepsilon} q(f(\varepsilon x + z - z)) = \frac{1}{\varepsilon} q(f(\varepsilon x + z) + f(-z)) \\ &\le \frac{1}{\varepsilon}\left[q(f(\varepsilon x + z)) + q(f(-z))\right] = \frac{1}{\varepsilon}\left[h_f(\varepsilon x + z) + h_f(-z)\right] \\ &\le \frac{1}{\varepsilon}(\delta + b_{-z}) = b. \end{aligned} \tag{2.10}$$

□

Since the Banach spaces are of the second category, the classical result for normed space is a corollary of this theorem.

Corollary 2.7. *Let X be a Banach space and let Y be a normed space. If $\mathcal{F}$ is a family of continuous linear operators from X to Y such that for each $x \in X$ there exists $b_x > 0$ with $\|f(x)\| \leq b_x$ for all $f \in \mathcal{F}$, then there exists $b > 0$ such that $\|f\| \leq b$ for all $f \in \mathcal{F}$.*

In the remainder of this section we give examples of asymmetric normed spaces of the half second category.

Definition 2.8. Let (X, p) be an asymmetric normed space. (X, p) is right bounded if there is $r > 0$ such that $rB_p(0,1) \subset B_{p^s}(0,1) + \theta(0)$, being $\theta(0) = \{y \in X : p(y) = 0\}$.

This definition is equivalent to Definition 16 of [16].

Remark 2.9. The class of right bounded asymmetric normed spaces contains the asymmetric normed spaces given by normed lattices. Indeed, in Lemma 1 of [14] it is proved that if $(X, \|\cdot\|)$ is a normed lattice and $p(x) = \|x^+\|$, then the asymmetric normed space (X, p) is right-bounded with constant $r = 1$.

Lemma 2.10. *Let (X, p) be an asymmetric normed space and let $\theta(0) = \{y \in X : p(y) = 0\}$.*

(1) *If G is a p-open subset of X, then $G + \theta(0) \subset G$.*

(2) *If G is p^{-1}-open subset of X, then $G - \theta(0) \subset G$.*

(3) *If F is a p^{-1}-closed subset of X, then $F + \theta(0) \subset F$*

(4) *If (X, p) right bounded, then for all $a \in X$ and for all $\delta > 0$, there is $k > 0$ such that $B_p(a, k) \subset B_{p^s}(a, \delta) + \theta(0)$.*

Proof. (1) Let $y \in G + \theta(0)$. There exists $x \in G$ such that $y - x \in \theta(0)$. Since G is p-open, there is $\varepsilon > 0$ such that $B_p(x, \varepsilon) \subset G$. Since $p(y - x) = 0$, we have that $(y - x) \in B_p(0, \varepsilon)$, then $y \in x + B_p(0, \varepsilon) = B_p(x, \varepsilon) \subset G$.

(2) Let $y \in G - \theta(0)$. There exists $x \in G$ such that $y - x \in -\theta(0)$, that is, $x - y \in \theta(0)$. Since G is p^{-1}-open, there is $\varepsilon > 0$ such that $B_{p^{-1}}(x, \varepsilon) \subset G$. Since $p^{-1}(y - x) = p(x - y) = 0$, we have that $(y - x) \in B_{p^{-1}}(0, \varepsilon)$, then $y \in x + B_{p^{-1}}(0, \varepsilon) = B_{p^{-1}}(x, \varepsilon) \subset G$.

(3) Suppose that $y \in F + \theta(0)$. Then, $y = x + z$ with $x \in F$ and $z \in \theta(0)$. Suppose that $y \notin F$. Since $X \setminus F$ is p^{-1}-open, by (2) we have that $y - z = x \in X \setminus F$ and this yields a contradiction.

(4) Since X is right-bounded there is $r > 0$ such that $B_p(0, r) \subset B_{p^s}(0,1) + \theta(0)$, then $\delta B_p(0, r) \subset \delta(B_{p^s}(0,1) + \theta(0)) \subset B_{p^s}(0, \delta) + \theta(0)$. Thus, if $k = r\delta$ then

$$B_p(a, k) = a + B_p(0, k) \subset a + B_{p^s}(0, \delta) + \theta(0) = B_{p^s}(a, \delta) + \theta(0). \tag{2.11}$$

□

Theorem 2.11. *If (X, p) is a biBanach right-bounded asymmetric normed space, then (X, p) is of the half second category.*

Proof. Suppose $X = \bigcup_{n=1}^{\infty} F_n$. Then $X = \bigcup_{n=1}^{\infty} \mathrm{cl}_{p^{-1}} F_n$. Since $\mathrm{cl}_{p^{-1}} F_n$ is p^s-closed, for all $n \in \mathbb{N}$, and (X, p^s) is of the second category, because it is a Banach space, there is $n_0 \in \mathbb{N}$ such that $\mathrm{int}_{p^s} \mathrm{cl}_{p^{-1}} F_{n_0} \neq \emptyset$. Therefore, there exist $a \in \mathrm{cl}_{p^{-1}} F_{n_0}$ and $t > 0$ such that $B_{p^s}(a, t) \subset cl_{p^{-1}} F_{n_0}$.

Since X is right bounded, by (4) of Lemma 2.10, there exists $k > 0$ such that $B_p(a,k) \subset B_{p^s}(a,t) + \theta(0)$. Therefore, by (3) of Lemma 2.10, we have that

$$B_p(a,k) \subset B_{p^s}(a,t) + \theta(0) \subset \mathrm{cl}_{p^{-1}} F_{n_0} + \theta(0) \subset \mathrm{cl}_{p^{-1}} F_{n_0}. \tag{2.12}$$

□

The following result is immediate by Remark 2.9.

Corollary 2.12. *If $(X, \|\cdot\|)$ is a Banach lattice and $p(x) = \|x^+\|$, then (X,p) is of the half second category.*

Lemma 2.13. *Let (X,p) be a right-bounded asymmetric normed space such that $\theta(0)$ has a nonempty p^s-interior. If F is p^{-1}-closed, then the p-interior of F is nonempty.*

Proof. Since $\mathrm{int}_{p^s}\theta(0) \neq \emptyset$, there exist $a \in \theta(0)$ and $\delta > 0$ such that the ball $B_{p^s}(a,\delta) \subset \theta(0)$. Since (X,p) is right-bounded, by (4) of Lemma 2.10, there is $r > 0$ such that $B_p(a,r) \subset B_{p^s}(a,1) + \theta(0)$. Then

$$B_p(a,r) \subset \left(\frac{1}{\delta}\right) B_{p^s}(a,\delta) + \theta(0) \subset \theta(0) + \theta(0) = \theta(0). \tag{2.13}$$

If $x \in F$, by (3) of Lemma 2.10, $x + \theta(0) \subset F$, and so

$$x + B_p(a,r) \subset x + \theta(0) \subset F. \tag{2.14}$$

□

Hence, $\mathrm{int}_p F \neq \emptyset$.

The following proposition is an immediate consequence of Lemma 2.13.

Proposition 2.14. *Let (X,p) be a right-bounded asymmetric normed space. If $\theta(0)$ has a nonempty p^s-interior, then (X,p) is of the half second category.*

Note that in the class of right-bounded asymmetric normed spaces there are spaces with empty p^s-interior of $\theta(0)$. In fact, if $(X, \|\cdot\|)$ is an AL-space (Banach normed lattice where $\|x \vee y\| = \|x\| \vee \|y\|$), then its positive cone $P = \{x \in X : x \geq 0\}$ has empty interior (see [1, page 357]). If we consider the canonical asymmetric normed space (X,p), then $\theta(0) = -P$ and so $\theta(0)$ has empty p^s-interior since the norm p^s is equivalent to the original norm of X.

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