Research Article

Nonoscillatory Solutions of Second-Order Superlinear Dynamic Equations with Integrable Coefficients

Quanwen Lin¹ and Baoguo Jia²

¹ Department of Mathematics, Maoming University, Maoming 525000, China

² School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou 510275, China

Correspondence should be addressed to Quanwen Lin, linquanwen@126.com

Received 9 September 2011; Revised 15 November 2011; Accepted 23 November 2011

Academic Editor: D. Anderson

Copyright © 2012 Q. Lin and B. Jia. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The asymptotic behavior of nonoscillatory solutions of the superlinear dynamic equation on time scales $(r(t)x^\Delta(t))^\Delta + p(t)|x(\sigma(t))|^\gamma \operatorname{sgn} x(\sigma(t)) = 0$, $\gamma > 1$, is discussed under the condition that $P(t) = \lim_{\tau \rightarrow \infty} \int_t^\tau p(s) \Delta s$ exists and $P(t) \geq 0$ for large t .

1. Introduction

Consider the second-order superlinear dynamic equation

$$\left(r(t)x^\Delta(t)\right)^\Delta + p(t)|x(\sigma(t))|^\gamma \operatorname{sgn} x(\sigma(t)) = 0, \quad \gamma > 1, \quad (1.1)$$

where

$$P(t) = \lim_{\tau \rightarrow \infty} \int_t^\tau p(s) \Delta s \quad (1.2)$$

exists and is finite. $P(t) \geq 0$ for large t .

When $\mathbb{T} = \mathbb{R}$, $r(t) = 1$, (1.1) is the second-order superlinear differential equation

$$x''(t) + p(t)|x(t)|^\gamma \operatorname{sgn} x(t) = 0, \quad \gamma > 1. \quad (1.3)$$

When $\mathbb{T} = \mathbb{N}$, $r(n) = 1$, (1.1) is the second-order superlinear difference equation

$$\Delta^2 x(n) + p(n)|x(n+1)|^\gamma \operatorname{sgn} x(n+1) = 0, \quad \gamma > 1. \quad (1.4)$$

The following condition is introduced in [1].

Condition (H). We say that $\mathbb{T}$ satisfies Condition (H) provided one of the following holds.

- (1) There exists a strictly increasing sequence $\{t_n\}_{n=0}^\infty \subset \mathbb{T}$ with $\lim_{n \rightarrow \infty} t_n = \infty$ and for each $n \geq 0$ either $\sigma(t_n) = t_{n+1}$ or the real interval $[t_n, t_{n+1}] \subset \mathbb{T}$;
- (2) $\mathbb{T} \cap \mathbb{R} = [T', \infty)$ for some $T' \in \mathbb{T}$.

We note that time scales which satisfy Condition (H) include most of the important time scales, such as $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0}$, where $q > 1$ and $\mathbb{N}_0$ is the nonnegative integers and harmonic numbers $\{\sum_{k=1}^n 1/k : n \in \mathbb{N}\}$ [2, Example 1.45].

In [3], Naito proved the following result.

Theorem 1.1. If $P(t) = \int_t^\infty p(s)ds \geq 0$ for large t , then a nonoscillatory solution $x(t)$ of (1.3) satisfies exactly one of the following three asymptotic properties:

$$\begin{aligned} \lim_{t \rightarrow \infty} x(t) &= c \neq 0, \\ \lim_{t \rightarrow \infty} \frac{x(t)}{t} &= 0, \quad \lim_{t \rightarrow \infty} x(t) = \pm\infty, \\ \lim_{t \rightarrow \infty} \frac{x(t)}{t} &= c \neq 0. \end{aligned} \quad (1.5)$$

In this paper, we extend Theorem 1.1 to superlinear dynamic equation (1.1) on time scale. As an application, we get the asymptotic behavior of each nonoscillation solution of the difference equation

$$\Delta^2 x(n) + \left(\frac{a}{n^{1+c}} + \frac{(-1)^n b}{n^c} \right) |x(n+1)|^\gamma \operatorname{sgn} x(n+1) = 0, \quad \gamma > 1, \quad (1.6)$$

where $b > 0$, $c > 1$, and $a/c > b/2$.

2. Main Theorems

Consider the second-order nonlinear dynamic equation

$$\left(r(t)x^\Delta(t) \right)^\Delta + p(t)|x(\sigma(t))|^\gamma \operatorname{sgn} x(\sigma(t)) = 0, \quad \gamma > 0, \quad (2.1)$$

where $r(t), p(t) \in C(\mathbb{T}, \mathbb{R})$, $r(t) > 0$, $t_0 \in \mathbb{T}$, and $\int_{t_0}^\infty [r(t)]^{-1} dt = \infty$. $\lim_{t \rightarrow \infty} \int_{t_0}^t p(s) \Delta s$ exists and is finite.

Lemma 2.1. Suppose that $\mathbb{T}$ satisfies Condition (H). If $x(t)$ is a positive solution of (1.1) on $[T, \infty)$, then the integral equation

$$\frac{r(t)x^\Delta(t)}{x^\gamma(t)} = \alpha + P(t) + \int_t^\infty \frac{r(s)\gamma \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \quad (2.2)$$

is satisfied for $t \geq T$, where α is a nonnegative constant.

Proof. Suppose that $x(t)$ is a positive solution of (1.1) on $[T, \infty)$.

In the first place, we will prove

$$\int_T^\infty \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s < \infty, \quad (2.3)$$

where $x_h(t) = x(t) + h\mu(t)x^\Delta(t) = (1-h)x(t) + hx(\sigma(t)) > 0$.

Multiplying both sides of (1.1) by $1/x^\gamma(\sigma(t))$, we get that

$$\left(\frac{r(t)x^\Delta(t)}{x^\gamma(t)} \right)^\Delta = -p(t) - \frac{r(t)\gamma \int_0^1 [x_h(t)]^{\gamma-1} dh [x^\Delta(t)]^2}{x^\gamma(t)x^\gamma(\sigma(t))}. \quad (2.4)$$

Integrating from T to t ,

$$\frac{r(t)x^\Delta(t)}{x^\gamma(t)} - \frac{r(T)x^\Delta(T)}{x^\gamma(T)} = - \int_T^t p(s) \Delta s - \int_T^t \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s. \quad (2.5)$$

If (2.3) fails to hold, that is,

$$\int_T^\infty \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s = \infty, \quad (2.6)$$

from (2.5), we have

$$\lim_{t \rightarrow \infty} \frac{r(t)x^\Delta(t)}{x^\gamma(t)} = -\infty. \quad (2.7)$$

Without loss of generality, we can assume that for $t \geq T$

$$\frac{r(T)x^\Delta(T)}{x^\gamma(T)} - \int_T^t p(s) \Delta s < -1. \quad (2.8)$$

Otherwise, let $L = \max_{t \geq T} |\int_T^t p(s) \Delta s|$. By (2.7), we can take a large $T_1 > T$ such that $r(T_1)x^\Delta(T_1)/x^\gamma(T_1) < -(2L+1)$. So we have

$$\begin{aligned} \frac{r(T_1)x^\Delta(T_1)}{x^\gamma(T_1)} - \int_{T_1}^t p(s) \Delta s &< -(2L+1) - \left[\int_T^t p(s) \Delta s - \int_T^{T_1} p(s) \Delta s \right] \\ &\leq -(2L+1) - [-2L] = -1. \end{aligned} \quad (2.9)$$

So we can replace T by $T_1 > T$ such that (2.8) still holds.

From (2.5) and (2.8), we get for $t \geq T$

$$\frac{r(t)x^\Delta(t)}{x^\gamma(t)} + \int_T^t \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s < -1. \quad (2.10)$$

In particular, we have

$$x^\Delta(t) < 0, \quad \text{for } t \geq T. \quad (2.11)$$

Therefore, $x(t)$ is strictly decreasing.

Assume that $t = t_{i-1} < t_i = \sigma(t)$. Then, $x(\sigma(t)) < x(t)$, and so

$$\begin{aligned} \gamma \int_0^1 [x_h(s)]^{\gamma-1} dh &= \gamma \int_0^1 [(1-h)x(s) + hx(\sigma(s))]^{\gamma-1} dh \\ &= \frac{[(1-h)x(s) + hx(\sigma(s))]^\gamma \Big|_0^1}{x(\sigma(s)) - x(s)} = \frac{x^\gamma(\sigma(s)) - x^\gamma(s)}{x(\sigma(s)) - x(s)}. \end{aligned} \quad (2.12)$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then, for $s \in [t_{i-1}, t_i]$, we have

$$\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh = \gamma x^{\gamma-1}(s). \quad (2.13)$$

Let

$$y(t) := 1 + \int_T^t \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^{\gamma-1}(\sigma(s))} \Delta s. \quad (2.14)$$

Hence, from (2.10), we get that

$$-\frac{r(t)x^\Delta(t)}{x^\gamma(t)} > y(t). \quad (2.15)$$

From (2.14) and (2.15), we get that

$$\begin{aligned} y^\Delta(t) &= \frac{r(t)\gamma \int_0^1 [x_h(t)]^{\gamma-1} dh [x^\Delta(t)]^2}{x^\gamma(t)x^\gamma(\sigma(t))} \\ &> y(t) \frac{\gamma \int_0^1 [x_h(t)]^{\gamma-1} dh [-x^\Delta(t)]}{x^\gamma(\sigma(t))}. \end{aligned} \quad (2.16)$$

Assume that $t = t_{i-1} < t_i = \sigma(t)$. From (2.16) and (2.12), we get that

$$\frac{y(\sigma(t)) - y(t)}{y(t)(\sigma(t) - t)} > \frac{x^\gamma(\sigma(t)) - x^\gamma(t)}{x(\sigma(t)) - x(t)} \frac{x(t) - x(\sigma(t))}{x^\gamma(\sigma(t))[\sigma(t) - t]}. \quad (2.17)$$

So,

$$\frac{y(\sigma(t))}{y(t)} > \frac{x^\gamma(t)}{x^\gamma(\sigma(t))}, \quad (2.18)$$

that is,

$$\frac{y(t_i)}{y(t_{i-1})} > \frac{x^\gamma(t_{i-1})}{x^\gamma(t_i)}. \quad (2.19)$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then, for $t \in (t_{i-1}, t_i]$, it follows from (2.16) and (2.13) that

$$\frac{y'(t)}{y(t)} > \frac{\gamma x^{\gamma-1}(t)[-x'(t)]}{x^\gamma(t)}, \quad (2.20)$$

that is,

$$(\ln y(t))' > -(\ln x^\gamma(t))'. \quad (2.21)$$

Integrating from t_{i-1} to t , we get that

$$\frac{y(t)}{y(t_{i-1})} > \frac{x^\gamma(t_{i-1})}{x^\gamma(t)}, \quad t \in (t_{i-1}, t_i]. \quad (2.22)$$

Let $T = t_{n_0}$, and let $t \in (T, \infty)_{\mathbb{T}}$. Then, there is an $n > n_0$ such that $t \in (t_{n-1}, t_n]_{\mathbb{T}}$. From (2.22) and (2.19), we get that

$$\frac{y(t)}{y(t_{n-1})} > \frac{x^\gamma(t_{n-1})}{x^\gamma(t)}, \quad \frac{y(t_{n-1})}{y(t_{n-2})} > \frac{x^\gamma(t_{n-2})}{x^\gamma(t_{n-1})}, \dots, \quad \frac{y(t_{n_0+1})}{y(t_{n_0})} > \frac{x^\gamma(t_{n_0})}{x^\gamma(t_{n_0+1})}. \quad (2.23)$$

Multiplying, we get that

$$\frac{y(t)}{y(t_{n_0})} > \frac{x^\gamma(t_{n_0})}{x^\gamma(t)}. \quad (2.24)$$

Using (2.15) again, we get

$$-\frac{r(t)x^\Delta(t)}{x^\gamma(t)} > y(t) > \frac{y(t_{n_0})x^\gamma(t_{n_0})}{x^\gamma(t)}. \quad (2.25)$$

If we set $L := y(t_{n_0})x^\gamma(t_{n_0})$, we get

$$x^\Delta(t) < -\frac{L}{r(t)}. \quad (2.26)$$

Integrating from T to t , we get that

$$x(t) - x(T) < -\int_T^t \frac{L}{r(s)} \Delta s \longrightarrow -\infty, \quad \text{as } t \longrightarrow \infty, \quad (2.27)$$

which contradicts $x(t) > 0$.

In (2.5), letting $t \rightarrow \infty$, replacing T by τ , and denoting $\alpha = \lim_{t \rightarrow \infty} r(t)x^\Delta(t)/x^\gamma(t)$, we get that

$$\alpha + \int_\tau^\infty p(s) \Delta s + \int_\tau^\infty \frac{r(s)\gamma \int_0^1 (x_h(s))^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s = \frac{r(\tau)x^\Delta(\tau)}{x^\gamma(\tau)}. \quad (2.28)$$

We need to show that $\alpha \geq 0$.

Suppose that $\alpha < 0$. Then, there exists a large T_1 such that, for $t > T_1$, we have

$$\frac{r(t)x^\Delta(t)}{x^\gamma(t)} \leq \frac{\alpha}{2}. \quad (2.29)$$

So,

$$x^\Delta(t) \leq \frac{\alpha x^\gamma(t)}{2r(t)}. \quad (2.30)$$

Thus,

$$\begin{aligned} M(T_1) &= \int_{T_1}^\infty \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ &\geq -\frac{\alpha}{2} \int_{T_1}^\infty \frac{\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [-x^\Delta(s)]}{x^\gamma(\sigma(s))} \Delta s. \end{aligned} \quad (2.31)$$

Assume that $t = t_{i-1} < t_i = \sigma(t)$. From (2.12) and $x^\Delta(t) < 0$, we have

$$\begin{aligned}
 & \int_t^{\sigma(t)} \frac{\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh[-x^\Delta(s)]}{x^\gamma(\sigma(s))} \Delta s \\
 &= \frac{\gamma \int_0^1 [x_h(t)]^{\gamma-1} dh[-x^\Delta(t)](\sigma(t) - t)}{x^\gamma(\sigma(t))} \\
 &= \frac{x^\gamma(t) - x^\gamma(\sigma(t))}{x^\gamma(\sigma(t))} \\
 &\geq \int_{x^\gamma(\sigma(t))}^{x^\gamma(t)} \frac{1}{v} dv \\
 &= \ln \frac{x^\gamma(t)}{x^\gamma(\sigma(t))} \\
 &= \ln \frac{x^\gamma(t_{i-1})}{x^\gamma(t_i)}.
 \end{aligned} \tag{2.32}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, from (2.13) we have, for $t \in (t_{i-1}, t_i]$,

$$\begin{aligned}
 & \int_{t_{i-1}}^t \frac{\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh[-x^\Delta(s)]}{x^\gamma(\sigma(s))} \Delta s \\
 &= \int_{t_{i-1}}^t \frac{\gamma x^{\gamma-1}(s)[-x'(s)]}{x^\gamma(s)} ds \\
 &= \ln \frac{x^\gamma(t_{i-1})}{x^\gamma(t)}.
 \end{aligned} \tag{2.33}$$

From (2.31), (2.32), (2.33) and the additivity of the integral, it is easy to get

$$M(T_1) \geq -\frac{\alpha}{2} \lim_{u \rightarrow \infty} \ln \frac{x^\gamma(T_1)}{x^\gamma(u)}. \tag{2.34}$$

So, for large u , we have

$$\ln \frac{x^\gamma(T_1)}{x^\gamma(u)} \leq -\frac{2M(T_1)}{\alpha} + 1. \tag{2.35}$$

Thus,

$$x^\gamma(u) \geq x^\gamma(T_1) \exp\left(\frac{2M(T_1)}{\alpha} - 1\right). \tag{2.36}$$

By (2.30) and noticing that $\alpha < 0$, we get that

$$x^\Delta(u) \leq \frac{\alpha x^\gamma(T_1)}{2r(u)} \exp\left(\frac{2M(T_1)}{\alpha} - 1\right). \quad (2.37)$$

Integrating (2.37), we get that $x(u) \rightarrow -\infty$, which is a contradiction.

This completes the proof of the lemma. $\square$

Consider the second-order superlinear dynamic equation

$$\left[r(t)x^\Delta(t)\right]^\Delta + p(t)|x(\sigma(t))|^\gamma \operatorname{sgn} x(\sigma(t)) = 0, \quad \gamma > 1, \quad (2.38)$$

where $r(t) > 0$, $\int_T^\infty (1/r(s))\Delta s = \infty$,

$$P(t) = \lim_{\tau \rightarrow \infty} \int_t^\tau p(s)\Delta s \quad (2.39)$$

exists and is finite, and $P(t) \geq 0$ for $t \geq T$.

Theorem 2.2. Suppose that $\mathbb{T}$ satisfies Condition (H) and $P(t) \geq 0$ for $t \geq T$. Then each nonoscillatory solution $x(t)$ of (2.38) satisfies exactly one of the following three asymptotic properties:

$$\lim_{t \rightarrow \infty} x(t) = c \neq 0, \quad (2.40)$$

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\int_T^t \Delta s / r(s)} = 0, \quad \lim_{t \rightarrow \infty} x(t) = \pm\infty, \quad (2.41)$$

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\int_T^t \Delta s / r(s)} = c \neq 0. \quad (2.42)$$

Proof. Let $x(t)$ be a nonoscillatory solution of (2.38), say, $x(t) > 0$ for $t \geq T > 0$. From Lemma 2.1, it is known that $x(t)$ satisfies the equality

$$\begin{aligned} r(t)x^\Delta(t) &= \alpha x^\gamma(t) + P(t)x^\gamma(t) \\ &+ \gamma x^\gamma(t) \int_t^\infty \frac{r(s) \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \end{aligned} \quad (2.43)$$

for $t \geq T$. Therefore, we have

$$r(t)x^\Delta(t) \geq P(t)x^\gamma(t), \quad (2.44)$$

for $t \geq T$. Since $P(t) \geq 0$ for $t \geq T$, it follows that $x^\Delta(t) \geq 0$ for $t \geq T$. An integration by parts of (2.38) gives

$$\begin{aligned} r(u)x^\Delta(u) - P(u)x^\gamma(u) + \gamma \int_t^u P(s)x^\Delta(s) \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh \Delta s \\ = r(t)x^\Delta(t) - P(t)x^\gamma(t), \end{aligned} \quad (2.45)$$

where $u \geq t \geq T$. Let t be fixed. Since $P(s)x^\Delta(s) \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh$ is nonnegative, the integral term in (2.45) has a finite limit or diverges to ∞ as $t \rightarrow \infty$. If the latter case occurs, then $r(u)x^\Delta(u) - P(u)x^\gamma(u) \rightarrow -\infty$ as $u \rightarrow \infty$, which is a contradiction to (2.44). Thus, the former case occurs, that is,

$$\int_t^\infty P(s)x^\Delta(s) \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh \Delta s < \infty. \quad (2.46)$$

Define the function $k_1(t)$ as

$$k_1(t) = \int_t^\infty P(s)x^\Delta(s) \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh \Delta s < \infty \quad (2.47)$$

and the finite constant $\beta = \lim_{u \rightarrow \infty} [r(u)x^\Delta(u) - P(u)x^\gamma(u)]$. Then, equality (2.45) yields

$$r(t)x^\Delta(t) = \beta + P(t)x^\gamma(t) + \gamma k_1(t), \quad (2.48)$$

for $t \geq T$. Observe by (2.44) that $\beta \geq 0$. From (2.44) and (2.46), it follows that

$$\int_t^\infty \frac{P^2(s)x^\gamma(s)}{r(s)} \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh \Delta s \leq k_1(t) < \infty. \quad (2.49)$$

Next, we define the function $k_2(t)$ by (noticing that $\gamma > 1$)

$$\begin{aligned} k_2(t) &= \int_t^\infty \frac{P^2(s)x^\gamma(s)}{r(s)} \int_0^1 [x(s) + h\mu(s)x^\Delta(s)]^{\gamma-1} dh \Delta s \\ &= \int_t^\infty \frac{P^2(s)x^\gamma(s)}{r(s)} \int_0^1 [(1-h)x(s) + hx(\sigma(s))]^{\gamma-1} dh \Delta s \\ &\geq \int_t^\infty \frac{P^2(s)x^\gamma(s)}{r(s)} \int_0^1 [(1-h)x(s) + hx(s)]^{\gamma-1} dh \Delta s \\ &= \int_t^\infty \frac{P^2(s)x^{2\gamma-1}(s)}{r(s)} \Delta s \end{aligned} \quad (2.50)$$

for $t \geq T$. Dividing (2.48) by $r(t)$ and integrating from T to t , we get

$$x(t) = x(T) + \int_T^t \frac{\beta}{r(s)} \Delta s + \int_T^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s + \gamma \int_T^t \frac{k_1(s)}{r(s)} \Delta s. \quad (2.51)$$

By Schwartz's inequality and the fact that $x^\Delta(t) \geq 0$ for $t \geq T$, the second integral term in (2.51) can be estimated as follows:

$$\begin{aligned} \int_T^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s &\leq \left(\int_T^t \frac{P^2(s)x^{2\gamma-1}(s)}{r(s)} \Delta s \right)^{1/2} \left(\int_T^t \frac{x(s)}{r(s)} \Delta s \right)^{1/2} \\ &\leq k_2^{1/2}(T) \left(\int_T^t \frac{\Delta s}{r(s)} \right)^{1/2} x^{1/2}(t), \end{aligned} \quad (2.52)$$

for $t \geq T$. From (2.51) and (2.52) and noticing that $k_1(t)$ is decreasing, we get that

$$x(t) \leq x(T) + \int_T^t \frac{\beta}{r(s)} \Delta s + k_2^{1/2}(T) \left(\int_T^t \frac{\Delta s}{r(s)} \right)^{1/2} x^{1/2}(t) + \gamma k_1(T) \int_T^t \frac{\Delta s}{r(s)}. \quad (2.53)$$

The above inequality may be regarded as a quadratic inequality in $x^{1/2}(t)$. Then, we have

$$x^{1/2}(t) \leq \frac{1}{2} k_2^{1/2}(T) \left(\int_T^t \frac{\Delta s}{r(s)} \right)^{1/2} + \frac{1}{2} D^{1/2}(t) \quad (2.54)$$

for $t \geq T$, where

$$D(t) = k_2(T) \int_T^t \frac{\Delta s}{r(s)} + 4 \left[x(T) + (\beta + \gamma k_1(T)) \int_T^t \frac{\Delta s}{r(s)} \right]. \quad (2.55)$$

It is obvious that $D(t) = O(\int_T^t \Delta s/r(s))$ as $t \rightarrow \infty$, and, consequently, there exists a positive constant m such that

$$x(t) \leq m \int_T^t \frac{\Delta s}{r(s)} \quad (2.56)$$

for $t \geq T$. Let $T_1 (\geq T)$ be an arbitrary number. It is clear that

$$0 \leq \int_T^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s = \int_T^{T_1} \frac{P(s)x^\gamma(s)}{r(s)} \Delta s + \int_{T_1}^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s \quad (2.57)$$

for $t \geq T_1$. Arguing as in (2.52), we find

$$\int_{T_1}^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s \leq \left[k_2(T_1)x(t) \int_{T_1}^t \frac{\Delta s}{r(s)} \right]^{1/2} \quad (2.58)$$

for $t \geq T_1$, which when combined with (2.56) yields

$$\begin{aligned} \int_{T_1}^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s &\leq \left[mk_2(T_1) \int_T^t \frac{\Delta s}{r(s)} \int_{T_1}^t \frac{\Delta s}{r(s)} \right]^{1/2} \\ &\leq [mk_2(T_1)]^{1/2} \int_T^t \frac{\Delta s}{r(s)}, \end{aligned} \quad (2.59)$$

for $t \geq T_1 \geq T$. Using (2.57) and (2.59) and noticing that $\int_T^\infty \Delta s/r(s) = \infty$, we obtain

$$0 \leq \limsup_{t \rightarrow \infty} \frac{1}{\int_T^t \Delta s/r(s)} \int_T^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s \leq [mk_2(T_1)]^{1/2}. \quad (2.60)$$

Since T_1 is arbitrary and $k_2(T_1) \rightarrow 0$ as $T_1 \rightarrow \infty$, letting $T_1 \rightarrow \infty$ in (2.60), we get

$$0 \leq \lim_{t \rightarrow \infty} \frac{1}{\int_T^t \Delta s/r(s)} \int_T^t \frac{P(s)x^\gamma(s)}{r(s)} \Delta s = 0. \quad (2.61)$$

Using L'Hospital's rule of time scale (see Theorem 1.119 of [2]), we have

$$\lim_{t \rightarrow \infty} \frac{\int_T^t (k_1(s)/r(s)) \Delta s}{\int_T^t \Delta s/r(s)} = \lim_{t \rightarrow \infty} k_1(t) = 0. \quad (2.62)$$

In view of (2.51), (2.61), and (2.62), we find $\lim_{t \rightarrow \infty} x(t)/\int_T^t \Delta s/r(s) = \beta$.

Recall that $x(t)$ is nondecreasing for $t \geq T$. Now, there are three cases to consider:

- (i) $\beta = 0$ and $x(t)$ is bounded above,
- (ii) $\beta = 0$ and $x(t)$ is unbounded,
- (iii) $\beta > 0$ (and hence $x(t)$ is unbounded).

Case (i) implies (2.40) with $c = \lim_{t \rightarrow \infty} x(t) > 0$, while case (iii) implies (2.42) with $c = \beta > 0$. It is also clear that case (ii) implies (2.41). This completes the proof. $\square$

The following lemma is from [1].

Lemma 2.3. Suppose that $\mathbb{T}$ satisfies Condition (H). $x(t) > 0$ is a solution of (1.1). Then, one has

$$\int_T^t \frac{x^\Delta(s)}{x^\gamma(\sigma(s))} \Delta s \leq \frac{x^{-\gamma+1}(T) - x^{-\gamma+1}(t)}{\gamma - 1} \leq \frac{x^{-\gamma+1}(T)}{\gamma - 1}. \quad (2.63)$$

Using Lemma 2.1, we can prove the following corollary.

Corollary 2.4. *Under the assumptions of Lemma 2.1, if $x(t)$ is a positive solution of (1.1) on $[T, \infty)$, then the integral equation*

$$\frac{r(t)x^\Delta(t)}{x^\gamma(\sigma(t))} = \alpha + P(t) + \int_{\sigma(t)}^{\infty} \frac{r(s) \left[\int_0^1 \gamma(x_h(s))^{\gamma-1} dh \right] [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \quad (2.64)$$

is satisfied for $t \geq T$, where $P(t) = \int_t^\infty p(s) \Delta s$, $x_h(s) = x(s) + h\mu(s)x^\Delta(s)$.

Proof. In the left side of (2.2), using

$$\begin{aligned} \frac{1}{x^\gamma(t)} &= \frac{1}{x^\gamma(\sigma(t))} - \left(\frac{1}{x^\gamma(t)} \right)^\Delta \mu(t) \\ &= \frac{1}{x^\gamma(\sigma(t))} + \frac{\int_0^1 \gamma(x_h(t))^{\gamma-1} dh x^\Delta(t)}{x^\gamma(t)x^\gamma(\sigma(t))} \mu(t) \end{aligned} \quad (2.65)$$

and using (2.2), (2.65), the additivity of the integral, and [2, Theorem 1.75], we have that

$$\begin{aligned} \frac{r(t)x^\Delta(t)}{x^\gamma(t)} &= \frac{r(t)x^\Delta(t)}{x^\gamma(\sigma(t))} + \frac{r(t) \int_0^1 \gamma(x_h(t))^{\gamma-1} dh [x^\Delta(t)]^2}{x^\gamma(t)x^\gamma(\sigma(t))} \mu(t) \\ &= \alpha + P(t) + \int_{\sigma(t)}^{\infty} \frac{r(s) \left[\int_0^1 \gamma(x_h(s))^{\gamma-1} dh \right] [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ &\quad + \int_t^{\sigma(t)} \frac{r(s) \left[\int_0^1 \gamma(x_h(s))^{\gamma-1} dh \right] [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ &= \alpha + P(t) + \int_{\sigma(t)}^{\infty} \frac{r(s) \left[\int_0^1 \gamma(x_h(s))^{\gamma-1} dh \right] [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ &\quad + \frac{r(t) \left[\int_0^1 \gamma(x_h(t))^{\gamma-1} dh \right] [x^\Delta(t)]^2}{x^\gamma(t)x^\gamma(\sigma(t))} \mu(t). \end{aligned} \quad (2.66)$$

From (2.66), we get (2.64). □

The following theorem can be regarded as a time scale version of [4, Theorem 1].

Theorem 2.5. *Suppose that $\mathbb{T}$ satisfies Condition (H), $r(t) > 0$ with $\int_T^\infty [r(t)]^{-1} \Delta t = \infty$, and suppose that $\lim_{t \rightarrow \infty} \int_T^t p(s) \Delta s$ exists and is finite. Let $P(t) = \int_t^\infty p(s) \Delta s$. Then, the superlinear dynamic equation (1.1) is oscillatory if*

$$\limsup_{t \rightarrow \infty} \int_T^t \frac{P(s)}{r(s)} \Delta s = \infty. \quad (2.67)$$

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) on $[T, \infty)$. Without loss of generality, assume that $x(t)$ is positive for $t \in [T, \infty)$. From Corollary 2.4, $x(t)$ satisfies the integral equation (2.64). Dropping the last integral term in (2.64), we have the inequality

$$\frac{r(t)x^\Delta(t)}{x^\gamma(\sigma(t))} \geq P(t). \quad (2.68)$$

Dividing (2.68) by $r(t)$, integrating from T to t , and using Lemma 2.3, we find

$$\frac{x^{-\gamma+1}(T)}{\gamma-1} \geq \int_T^t \frac{x^\Delta(s)}{x^\gamma(\sigma(s))} \Delta s \geq \int_T^t \frac{P(s)}{r(s)} \Delta s. \quad (2.69)$$

This contradicts (2.67), and so (1.1) is oscillatory. $\square$

Consider the second-order superlinear dynamic equation with forced term

$$\left[r(t)x^\Delta(t) \right]^\Delta + p(t)|x(\sigma(t))|^\gamma \operatorname{sgn} x(\sigma(t)) = h(t), \quad \gamma > 1, \quad (2.70)$$

where $r(t) > 0$, $\int_T^\infty (1/r(s)) \Delta s = \infty$, and

$$P(t) = \lim_{\tau \rightarrow \infty} \int_t^\tau p(s) \Delta s \quad (2.71)$$

exists and is finite.

Lemma 2.6. *Suppose that*

$$\int_T^\infty |h(s)| \Delta s < +\infty. \quad (2.72)$$

If $x(t)$ is a positive solution of (2.70) and $\liminf_{t \rightarrow \infty} x(t) > 0$, then

$$\int_T^\infty \frac{r(s) \gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s) x^\gamma(\sigma(s))} \Delta s < \infty, \quad (2.73)$$

$$\begin{aligned} \frac{r(t)x^\Delta(t)}{x^\gamma(\sigma(t))} &= \alpha + \int_t^\infty \left[p(s) - \frac{h(s)}{x^\gamma(\sigma(s))} \right] \Delta s \\ &\quad + \int_{\sigma(t)}^\infty \frac{r(s) \gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s) x^\gamma(\sigma(s))} \Delta s \end{aligned} \quad (2.74)$$

are satisfied for sufficiently large t , where $x_h(s) = x(s) + h\mu(s)x^\Delta(s)$, α is a nonnegative constant.

Proof. The fact that $\liminf_{t \rightarrow \infty} x(t) > 0$ implies the existence of $t_1 \geq T$ and $m > 0$ such that $x(t) \geq m$ for $t \geq t_1$. Then, using (2.72), we find

$$\left| \int_{t_1}^t \frac{h(s)}{x^\gamma(\sigma(s))} \Delta s \right| \leq \int_{t_1}^t \left| \frac{h(s)}{x^\gamma(\sigma(s))} \right| \Delta s \leq \frac{1}{m^\gamma} \int_{t_1}^t |h(s)| \Delta s \leq M, \quad t \geq t_1, \quad (2.75)$$

where M is some finite positive constant.

So, $\lim_{\tau \rightarrow \infty} \int_t^\tau [p(s) - h(s)/x^\gamma(\sigma(s))] \Delta s$ exists and is finite.

Similar to the proof of Lemma 2.1 and Corollary 2.4, it is easy to know that (2.73) and (2.74) hold. $\square$

For subsequent results, we define

$$\Phi_0(t) = \int_t^\infty [p(s) - k|h(s)|] \Delta s, \quad t \geq T, \quad (2.76)$$

where k is a positive constant. It is noted that, if (2.71) and (2.72) hold, then $\Phi_0(t)$ is finite for any k . Assume that $\Phi_0(t) > 0$ for sufficiently large t . Define, for a positive integer n and a positive constant ρ , the following functions:

$$\begin{aligned} \Phi_1(t) &= \int_{\sigma(t)}^\infty \frac{[\Phi_0(s)]^2}{r(s)} \Delta s, \\ \Phi_{n+1}(t) &= \int_{\sigma(t)}^\infty \frac{[\Phi_0(s) + \rho \Phi_n(s)]^2}{r(s)} \Delta s. \end{aligned} \quad (2.77)$$

We introduce the following condition.

Condition (A). For every $\rho > 0$, there exists a positive integer N such that $\Phi_n(t)$ is finite for $n = 1, 2, \dots, N-1$ and $\Phi_N(t)$ is infinite.

Theorem 2.7. Suppose that (2.71), (2.72), and Condition (A) hold. Then, every solution $x(t)$ of (2.70) is either oscillatory or satisfies

$$\liminf_{t \rightarrow \infty} x(t) = 0. \quad (2.78)$$

Proof. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (2.70) and $\liminf_{t \rightarrow \infty} |x(t)| > 0$. Without loss of generality, let $x(t)$ be eventually positive. By Lemma 2.6, $x(t)$ satisfies (2.73) and (2.74). Further, there exist $t_1 \geq T$ and $m > 0$ such that $x(t) \geq m$ for $t \geq t_1$. Let

$$\Phi_0(t) = \int_t^\infty \left[p(s) - \frac{|h(s)|}{m^\gamma} \right] \Delta s. \quad (2.79)$$

Then, from (2.74) we find

$$\begin{aligned}\frac{r(t)x^\Delta(t)}{x^\gamma(\sigma(t))} &\geq \Phi_0(t) + \int_{\sigma(t)}^{\infty} \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ &\geq \Phi_0(t) > 0,\end{aligned}\quad (2.80)$$

for $t \geq t_1$. From (2.80), we get

$$x^\Delta(t) \geq \frac{\Phi_0(t)x^\gamma(\sigma(t))}{r(t)} > 0, \quad t \geq t_1. \quad (2.81)$$

Applying (2.81) and noticing that $\gamma > 1$, we find for $t \geq t_1$

$$\begin{aligned}\int_{\sigma(t)}^{\infty} \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ \geq \int_{\sigma(t)}^{\infty} \frac{\gamma [x(s)]^{\gamma-1} [\Phi_0(s)x^\gamma(\sigma(s))]^2}{r(s)x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ \geq \gamma m^{\gamma-1} \int_{\sigma(t)}^{\infty} \frac{[\Phi_0(s)]^2}{r(s)} \Delta s = \gamma m^{\gamma-1} \Phi_1(t).\end{aligned}\quad (2.82)$$

If $N = 1$ in Condition (A), then the right side of (2.82) is infinite. This is a contradiction to (2.73).

Next, it follows from (2.80) and (2.82) that

$$\frac{r(t)x^\Delta(t)}{x^\gamma(\sigma(t))} \geq \Phi_0(t) + \gamma m^{\gamma-1} \Phi_1(t). \quad (2.83)$$

Using a similar technique and relations (2.83), we get

$$\begin{aligned}\int_{\sigma(t)}^{\infty} \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh [x^\Delta(s)]^2}{x^\gamma(s)x^\gamma(\sigma(s))} \Delta s \\ \geq \gamma m^{\gamma-1} \int_{\sigma(t)}^{\infty} \frac{[\Phi_0(s) + \gamma m^{\gamma-1} \Phi_1(s)]^2}{r(s)} \Delta s = \gamma m^{\gamma-1} \Phi_2(t), \quad t \geq t_1.\end{aligned}\quad (2.84)$$

If $N = 2$ in Condition (A), then the right side of (2.84) is infinite. This again contradicts (2.73).

A similar argument yields a contradiction for any integer $N > 2$. This completes the proof of the theorem. $\square$

Example 2.8. We have

$$\Delta^2 x(n) + \left(\frac{a}{n^{1+c}} + \frac{b(-1)^n}{n^c} \right) |x(n+1)|^\gamma \operatorname{sgn} x(n+1) = 0, \quad \gamma > 1, \quad (2.85)$$

where $b > 0$, $c > 1$, and $a/c > b/2$. It is easy to see that

$$\sum_{k=n}^{\infty} \frac{1}{k^{1+c}} \geq \int_n^{\infty} \frac{1}{t^{1+c}} dt = \frac{1}{cn^c}. \quad (2.86)$$

By [5], we have $\sum_{k=n}^{\infty} (-1)^k / k^c \sim (-1)^n / 2n^c$. So,

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{k^c} = [1 + o(1)] \frac{(-1)^n}{2n^c}. \quad (2.87)$$

Using (2.86) and (2.87), we get that, for large n ,

$$P(n) = \sum_{k=n}^{\infty} \left(\frac{a}{k^{1+c}} + \frac{b(-1)^k}{k^c} \right) \geq \frac{(a/c) + (b/2)[1 + o(1)](-1)^n}{n^c} > 0. \quad (2.88)$$

By Theorem 2.2, each nonoscillatory solution $x(t)$ of (2.85) satisfies exactly one of the following three asymptotic properties:

$$\begin{aligned} \lim_{n \rightarrow \infty} x(n) &= c \neq 0, \\ \lim_{n \rightarrow \infty} \frac{x(n)}{n} &= 0, \quad \lim_{n \rightarrow \infty} x(n) = \pm\infty, \\ \lim_{n \rightarrow \infty} \frac{x(n)}{n} &= c \neq 0. \end{aligned} \quad (2.89)$$

Acknowledgment

The paper is supported by the National Natural Science Foundation of China (no. 10971232).

References

- [1] B. Baoguo, L. Erbe, and A. Peterson, "Kiguradze-type oscillation theorems for second order superlinear dynamic equations on time scales," *Canadian Mathematical Bulletin*, vol. 54, no. 4, pp. 580–592, 2011.
- [2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
- [3] M. Naito, "Nonoscillatory solutions of second order differential equations with integrable coefficients," *Proceedings of the American Mathematical Society*, vol. 109, no. 3, pp. 769–774, 1990.
- [4] J. S. W. Wong, "Oscillation criteria for second order nonlinear differential equations with integrable coefficients," *Proceedings of the American Mathematical Society*, vol. 115, no. 2, pp. 389–395, 1992.
- [5] L. Erbe, J. Baoguo, and A. Peterson, "Nonoscillation for second order sublinear dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 232, no. 2, pp. 594–599, 2009.