Research Article

Strong Convergence of a Modified Extragradient Method to the Minimum-Norm Solution of Variational Inequalities

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We suggest and analyze a modified extragradient method for solving variational inequalities, which is convergent strongly to the minimum-norm solution of some variational inequality in an infinite-dimensional Hilbert space.

1. Introduction

Let $C$ be a closed convex subset of a real Hilbert space $H$. A mapping $A : C \rightarrow H$ is called $\alpha$-inverse-strongly monotone if there exists a positive real number $\alpha$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha \| Au - Av \|^2, \quad \forall u, v \in C. \quad (1.1)$$

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.2)$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. It is well known that variational inequality theory has emerged as an important tool in studying a wide
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class of obstacle, unilateral, and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems; see [1–36] and the references therein.

It is well known that variational inequalities are equivalent to the fixed point problem. This alternative formulation has been used to study the existence of a solution of the variational inequality as well as to develop several numerical methods. Using this equivalence, one can suggest the following iterative method.

**Algorithm 1.1.** For a given \( u_0 \in C \), calculate the approximate solution \( u_{n+1} \) by the iterative scheme

\[
    u_{n+1} = P_C [u_n - \lambda A u_n], \quad n = 0, 1, 2, \ldots
\]

It is well known that the convergence of Algorithm 1.1 requires that the operator \( A \) must be both strongly monotone and Lipschitz continuous. These restrict conditions rules out its applications in several important problems. To overcome these drawbacks, Korpelevič suggested in [8] an algorithm of the form

\[
    y_n = P_C [x_n - \lambda A x_n], \\
    x_{n+1} = P_C [x_n - \lambda A y_n], \quad n \geq 0.
\]

Noor [2] further suggested and analyzed the following new iterative methods for solving the variational inequality (1.2).

**Algorithm 1.2.** For a given \( u_0 \in C \), calculate the approximate solution \( u_{n+1} \) by the iterative scheme

\[
    w_n = P_C [u_n - \lambda A u_n], \\
    u_{n+1} = P_C [w_n - \lambda A w_n], \quad n = 0, 1, 2, \ldots
\]

which is known as the modified extragradient method. For the convergence analysis of Algorithm 1.2, see Noor [1, 2] and the references therein. We would like to point out that Algorithm 1.2 is quite different from the method of Korpelevič [8]. However, Algorithm 1.2 fails, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces.

In this paper, we suggest and consider a very simple modified extragradient method which is convergent strongly to the minimum-norm solution of variational inequality (1.2) in an infinite-dimensional Hilbert space. This new method includes the method of Noor [2] as a special case.
2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $C$ be a closed convex subset of $H$. It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}. \quad (2.1)$$

We denote $u_0$ by $P_C u$, where $P_C$ is called the metric projection of $H$ onto $C$. The metric projection $P_C$ of $H$ onto $C$ has the following basic properties:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$;
(ii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ for every $x, y \in H$;
(iii) $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H, y \in C$.

We need the following lemma for proving our main results.

Lemma 2.1 (see [15]). Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad (2.2)$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

1. $\sum_{n=1}^{\infty} \gamma_n = \infty$;
2. $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

3. Main Result

In this section we will state and prove our main result.

Theorem 3.1. Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A : C \to H$ be an $\alpha$-inverse-strongly monotone mapping. Suppose that $\text{VI}(C, A) \neq \emptyset$. For given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$y_n = P_C[(1 - \alpha_n)(x_n - \lambda Ax_n)],$$

$$x_{n+1} = P_C(y_n - \lambda Ay_n), \quad n \geq 0, \quad (3.1)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\lambda \in [a, b] \subset (0, 2\alpha)$ is a constant. Assume the following conditions are satisfied:

(C1) $\lim_{n \to \infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n \to \infty} (\alpha_{n+1} / \alpha_n) = 1$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $P_{\text{VI}(C, A)}(0)$ which is the minimum-norm element in $\text{VI}(C, A)$.
We will divide our detailed proofs into several conclusions.

**Proof.** Take \( x^* \in \text{VI}(C, A) \). First we need to use the following facts:

1. \( x^* = P_C(x^* - \lambda Ax^*) \) for all \( \lambda > 0 \); in particular,
   \[
   x^* = P_C[x^* - \lambda(1 - \alpha_n)Ax^*] = P_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda Ax^*)], \quad \forall n \geq 0; 
   \tag{3.2}
   \]

2. \( I - \lambda A \) is nonexpansive and for all \( x, y \in C \)
   \[
   \| (I - \lambda A)x - (I - \lambda A)y \|^2 \leq \| x - y \|^2 + \lambda(\lambda - 2\alpha_1)\|Ax - Ay\|^2. 
   \tag{3.3}
   \]

From (3.1), we have
\[
\begin{align*}
\| y_n - x^* \| & = \| P_C[(1 - \alpha_n)(x_n - \lambda Ax_n)] - P_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda Ax^*)] \| \\
& \leq \| \alpha_n(-x^*) + (1 - \alpha_n)[(x_n - \lambda Ax_n) - (x^* - \lambda Ax^*)] \| \\
& \leq \alpha_n\| x^* \| + (1 - \alpha_n)(I - \lambda A)x_n - (I - \lambda A)x^* \\
& \leq \alpha_n\| x^* \| + (1 - \alpha_n)\| x_n - x^* \|. 
\end{align*}
\]

Thus,
\[
\begin{align*}
\| x_{n+1} - x^* \| & = \| P_C(y_n - \lambda Ay_n) - P_C(x^* - \lambda Ax^*) \| \\
& \leq \| (y_n - \lambda Ay_n) - (x^* - \lambda Ax^*) \| \\
& \leq \| y_n - x^* \| \\
& \leq \alpha_n\| x^* \| + (1 - \alpha_n)\| x_n - x^* \| \\
& \leq \max\{\| x^* \|, \| x_0 - x^* \| \}. 
\end{align*}
\]

Therefore, \( \{ x_n \} \) is bounded and so are \( \{ y_n \}, \{ Ax_n \}, \text{ and } \{ Ay_n \} \).

From (3.1), we have
\[
\begin{align*}
\| x_{n+1} - x_n \| & = \| P_C(y_n - \lambda Ay_n) - P_C(y_{n-1} - \lambda Ay_{n-1}) \| \\
& \leq \| (y_n - \lambda Ay_n) - (y_{n-1} - \lambda Ay_{n-1}) \| \\
& \leq \| y_n - y_{n-1} \| \\
& = \| P_C[(1 - \alpha_n)(x_n - \lambda Ax_n)] - P_C[(1 - \alpha_{n-1})(x_{n-1} - \lambda Ax_{n-1})] \| \\
& \leq \| (1 - \alpha_n)[(I - \lambda A)x_n - (I - \lambda A)x_{n-1}] - (\alpha_n - \alpha_{n-1})(I - \lambda A)x_{n-1} \| \\
& \leq (1 - \alpha_n)(I - \lambda A)x_n - (I - \lambda A)x_{n-1} \| + |\alpha_n - \alpha_{n-1}|M, \\
& \leq (1 - \alpha_n)\| x_n - x_{n-1} \| + |\alpha_n - \alpha_{n-1}|M, 
\end{align*}
\]
where $M > 0$ is a constant such that $\sup_{n} \{ ||(I - \lambda A)x_n||, ||(I - \lambda A)x_n|| \} < M$. Hence, by Lemma 2.1, we obtain

$$\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0. \tag{3.7}$$

From (3.4), (3.5) and the convexity of the norm, we deduce

$$\|x_{n+1} - x^*\|^2 \leq \|\alpha_n (x^* - 1) - (x^* - \alpha_n A x_n)\|^2$$

$$\leq \alpha_n \|x^*\|^2 + (1 - \alpha_n) \| (I - \lambda A)x_n - (I - \lambda A)x^* \|^2$$

$$\leq \alpha_n \|x^*\|^2 + (1 - \alpha_n) \left[ \|x_n - x^*\|^2 + \lambda (\lambda - 2\alpha) \|A x_n - A x^*\|^2 \right]$$

$$\leq \alpha_n \|x^*\|^2 + \|x_n - x^*\|^2 + (1 - \alpha_n) a (b - 2\alpha) \|A x_n - A x^*\|^2. \tag{3.8}$$

Therefore, we have

$$(1 - \alpha_n) a (2\alpha - b) \|A x_n - A x^*\|^2 \leq \alpha_n \|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$

$$\leq \alpha_n \|x^*\|^2 + (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \times \|x_n - x_{n+1}\|. \tag{3.9}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain $\|A x_n - A x^*\| \to 0$ as $n \to \infty$. By the property (ii) of the metric projection $P_C$, we have

$$\|y_n - x^*\|^2 = \|P_C [(1 - \alpha_n)(x_n - \lambda A x_n) - P_C (x^* - \lambda A x^*)\|^2$$

$$\leq \langle [(1 - \alpha_n)(x_n - \lambda A x_n) - (x^* - \lambda A x^*)], y_n - x^* \rangle$$

$$= \frac{1}{2} \left\{ \| (x_n - \lambda A x_n) - (x^* - \lambda A x^*) - \alpha_n (I - \lambda A) x_n \|^2 + \| y_n - x^* \|^2$$

$$- \| (x_n - \lambda A x_n) - (x^* - \lambda A x^*) - (y_n - x^*) - \alpha_n (I - \lambda A) x_n \|^2 \right\}$$

$$\leq \frac{1}{2} \left\{ \| (x_n - \lambda A x_n) - (x^* - \lambda A x^*) \|^2 + \alpha_n M + \| y_n - x^* \|^2$$

$$- \| (x_n - y_n) - \lambda (A x_n - A x^*) - \alpha_n (I - \lambda A) x_n \|^2 \right\}$$

$$\leq \frac{1}{2} \left\{ \| x_n - x^* \|^2 + \alpha_n M + \| y_n - x^* \|^2 - \| x_n - y_n \|^2$$

$$+ 2 \lambda \langle x_n - y_n, A x_n - A x^* \rangle + 2 \alpha_n (I - \lambda A) x_n - y_n \rangle$$

$$- \| \lambda (A x_n - A x^*) + \alpha_n (I - \lambda A) x_n \|^2 \right\}$$

$$\leq \frac{1}{2} \left\{ \| x_n - x^* \|^2 + \alpha_n M + \| y_n - x^* \|^2 - \| x_n - y_n \|^2$$

$$+ 2 \lambda \| x_n - y_n \| \| A x_n - A x^* \| + 2 \alpha_n \| (I - \lambda A) x_n \| \| x_n - y_n \| \right\}. \tag{3.10}$$
It follows that
\[ \| y_n - x^* \|^2 \leq \| x_n - x^* \|^2 + \alpha_n M - \| x_n - y_n \|^2 \]
\[ + 2\lambda \| x_n - y_n \| \| Ax_n - Ax^* \| + 2\alpha_n \| (I - \lambda A)x_n \| \| x_n - y_n \|, \]
(3.11)
and hence
\[ \| x_{n+1} - x^* \|^2 \leq \| y_n - x^* \|^2 \]
\[ \leq \| x_n - x^* \|^2 + \alpha_n M - \| x_n - y_n \|^2 + 2\lambda \| x_n - y_n \| \| Ax_n - Ax^* \| \]
\[ + 2\alpha_n \| (I - \lambda A)x_n \| \| x_n - y_n \|, \]
(3.12)
which implies that
\[ \| x_n - y_n \|^2 \leq (\| x_n - x^* \| + \| x_{n+1} - x^* \|) \| x_{n+1} - x_n \| + \alpha_n M + 2\lambda \| x_n - y_n \| \| Ax_n - Ax^* \| \]
\[ + 2\alpha_n \| (I - \lambda A)x_n \| \| x_n - y_n \|. \]
(3.13)
Since \( \alpha_n \to 0 \), \( \| x_n - x_{n+1} \| \to 0 \), and \( \| Ax_n - Ax^* \| \to 0 \), we derive \( \| x_n - y_n \| \to 0 \).

Next we show that
\[ \limsup_{n \to \infty} \langle z_0, z_0 - y_n \rangle \leq 0, \]
(3.14)
where \( z_0 = P_{VI(C,A)}(0) \). To show it, we choose a subsequence \( \{ y_{n_i} \} \) of \( \{ y_n \} \) such that
\[ \limsup_{n \to \infty} \langle z_0, z_0 - y_n \rangle = \lim_{i \to \infty} \langle z_0, z_0 - y_{n_i} \rangle. \]
(3.15)
As \( \{ y_n \} \) is bounded, we have that a subsequence \( \{ y_{n_i} \} \) of \( \{ y_n \} \) converges weakly to \( z \).

Next we show that \( z \in VI(C,A) \). We define a mapping \( T \) by
\[ Tv = \begin{cases} 
Av + N_C v, & v \in C, \\
\emptyset, & v \notin C.
\end{cases} \]
(3.16)
Then \( T \) is maximal monotone (see [16]). Let \( (v,w) \in G(T) \). Since \( w - Av \in N_C v \) and \( y_n \in C \), we have \( \langle v - y_n, w - Av \rangle \geq 0 \). On the other hand, from \( y_n = P_{C}[(1 - \alpha_n)(x_n - \lambda Ax_n)] \), we have
\[ \langle v - y_n, y_n - (1 - \alpha_n)(x_n - \lambda Ax_n) \rangle \geq 0, \]
(3.17)
that is,
\[ \left\langle v - y_n, \frac{y_n - x_n}{\lambda} + Ax_n + \frac{\alpha_n}{\lambda}(I - \lambda A)x_n \right\rangle \geq 0. \]
Therefore, we have
\[
\langle v - y_n, w \rangle \geq \langle v - y_n, Av \rangle
\]
\[
\geq \langle v - y_n, Av \rangle - \left( v - y_n, \frac{y_n - x_n}{\lambda} + Ax_n + \frac{\alpha_n}{\lambda} (I - \lambda A)x_n \right)
\]
\[
= \langle v - y_n, Av - Ax_n - \frac{y_n - x_n}{\lambda} - \frac{\alpha_n}{\lambda} (I - \lambda A)x_n \rangle
\]
\[
= \langle v - y_n, Av - Ay_n \rangle + \langle v - y_n, Ay_n - Ax_n \rangle
\]
\[
- \left( v - y_n, \frac{y_n - x_n}{\lambda} + \frac{\alpha_n}{\lambda} (I - \lambda A)x_n \right)
\]
\[
\geq \langle v - y_n, Ay_n - Ax_n \rangle - \left( v - y_n, \frac{y_n - x_n}{\lambda} + \frac{\alpha_n}{\lambda} (I - \lambda A)x_n \right).
\]
(3.19)

Noting that \( \alpha_n \to 0, \| y_n - x_n \| \to 0 \), and \( A \) is Lipschitz continuous, we obtain \( \langle v - z, w \rangle \geq 0 \). Since \( T \) is maximal monotone, we have \( z \in T^{-1}(0) \), and hence \( z \in VI(C, A) \). Therefore,
\[
\limsup_{n \to \infty} \langle z_0, z_0 - y_n \rangle = \lim_{i \to \infty} \langle z_0, z_0 - y_n \rangle = \langle z_0, z_0 - z \rangle \leq 0.
\]
(3.20)

Finally, we prove \( x_n \to z_0 \). By the property (ii) of metric projection \( P_C \), we have
\[
\| y_n - z_0 \|^2 = \| P_C[(1 - \alpha_n)(x_n - \lambda Ax_n)] - P_C[\alpha_n z_0 + (1 - \alpha_n)(z_0 - \lambda Az_0)] \|^2
\]
\[
\leq \langle \alpha_n(-z_0) + (1 - \alpha_n)(x_n - \lambda Ax_n) - (z_0 - \lambda Az_0), y_n - z_0 \rangle
\]
\[
\leq \alpha_n \langle z_0, z_0 - y_n \rangle + (1 - \alpha_n) \| (x_n - \lambda Ax_n) - (z_0 - \lambda Az_0) \| \| y_n - z_0 \|
\]
\[
\leq \alpha_n \langle z_0, z_0 - y_n \rangle + (1 - \alpha_n) \| x_n - z_0 \| \| y_n - z_0 \|
\]
\[
\leq \alpha_n \langle z_0, z_0 - y_n \rangle + \frac{1 - \alpha_n}{2} \left( \| x_n - z_0 \|^2 + \| y_n - z_0 \|^2 \right).
\]
(3.21)

Hence,
\[
\| y_n - z_0 \|^2 \leq (1 - \alpha_n) \| x_n - z_0 \|^2 + 2\alpha_n \langle z_0, z_0 - y_n \rangle.
\]
(3.22)

Therefore,
\[
\| x_{n+1} - z_0 \|^2 \leq \| y_n - z_0 \|^2 \leq (1 - \alpha_n) \| x_n - z_0 \|^2 + 2\alpha_n \langle z_0, z_0 - y_n \rangle.
\]
(3.23)

We apply Lemma 2.1 to the last inequality to deduce that \( x_0 \to z_0 \). This completes the proof.

\[ \Box \]

Remark 3.2. Our Algorithm (3.1) is similar to Noor’s modified extragradient method; see [2]. However, our algorithm has strong convergence in the setting of infinite-dimensional Hilbert spaces.
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