Research Article

Inequalities between Arithmetic-Geometric, Gini, and Toader Means

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We find the greatest values p_1, p_2 and least values q_1, q_2 such that the double inequalities $S_{p_1}(a, b) < M(a, b) < S_{q_1}(a, b)$ and $S_{p_2}(a, b) < T(a, b) < S_{q_2}(a, b)$ hold for all $a, b > 0$ with $a \neq b$ and present some new bounds for the complete elliptic integrals. Here $M(a, b)$, $T(a, b)$, and $S_p(a, b)$ are the arithmetic-geometric, Toader, and p th Gini means of two positive numbers a and b , respectively.

1. Introduction

For $p \in \mathbb{R}$ the p th Gini mean $S_p(a, b)$ and power mean $M_p(a, b)$ of two positive real numbers a and b are defined by

$$S_p(a, b) = \begin{cases} \left(\frac{a^{p-1} + b^{p-1}}{a + b} \right)^{1/(p-2)}, & p \neq 2, \\ (a^a b^b)^{1/(a+b)}, & p = 2, \end{cases} \quad (1.1)$$

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \quad (1.2)$$

respectively.

It is well known that $S_p(a, b)$ and $M_p(a, b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many means are special case of these means, for example,

$$\begin{aligned} S_1(a, b) &= M_1(a, b) = \frac{a+b}{2} = A(a, b) \text{ is the arithmetic mean,} \\ S_0(a, b) &= M_0(a, b) = \sqrt{ab} = G(a, b) \text{ is the geometric mean,} \\ M_{-1}(a, b) &= \frac{2ab}{a+b} = H(a, b) \text{ is the harmonic mean.} \end{aligned} \quad (1.3)$$

Recently, the Gini and power means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [1–7].

In [8], Toader introduced the Toader mean $T(a, b)$ of two positive numbers a and b as follows:

$$\begin{aligned} T(a, b) &= \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \\ &= \begin{cases} \frac{2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)}{\pi}, & a > b, \\ \frac{2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)}{\pi}, & a < b, \\ a, & a = b, \end{cases} \end{aligned} \quad (1.4)$$

where $\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{1/2} dt$, $r \in [0, 1]$, is the complete elliptic integrals of the second kind.

The classical arithmetic-geometric mean $M(a, b)$ of two positive number a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{aligned} a_0 &= a, & b_0 &= b, \\ a_{n+1} &= \frac{a_n + b_n}{2} = A(a_n, b_n), & b_{n+1} &= \sqrt{a_n b_n} = G(a_n, b_n). \end{aligned} \quad (1.5)$$

The Gauss identity [9] shows that

$$M(1, r) \mathcal{K}(\sqrt{1-r^2}) = \frac{\pi}{2} \quad (1.6)$$

for $r \in (0, 1)$, where $\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} dt$, $r \in [0, 1]$, is the complete elliptic integrals of the first kind.

Vuorinen [10] conjectured that

$$M_{3/2}(a, b) < T(a, b) \quad (1.7)$$

for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen in [11] and Barnard et al. in [12], respectively.

In [13], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a, b) < M_{\log 2 / \log(\pi/2)}(a, b) \quad (1.8)$$

for all $a, b > 0$ with $a \neq b$.

In [14–17], the authors proved that

$$M_0(a, b) = G(a, b) < M(a, b) < M_{1/2}(a, b), \quad (1.9)$$

$$L(a, b) < M(a, b) < \frac{\pi}{2} L(a, b) \quad (1.10)$$

for all $a, b > 0$ with $a \neq b$, where

$$L(a, b) = \begin{cases} \frac{a-b}{\log a - \log b}, & a \neq b, \\ a, & a = b, \end{cases} \quad (1.11)$$

denotes the classical logarithmic mean of two positive numbers a and b .

Very recently, Chu and Wang [18] and Guo and Qi [19] proved that

$$L_0(a, b) < T(a, b) < L_{1/4}(a, b) \quad (1.12)$$

for all $a, b > 0$ with $a \neq b$, and $L_0(a, b)$ and $L_{1/4}(a, b)$ are the best possible lower and upper Lehmer mean bounds for the Toader mean $T(a, b)$, respectively. Here, the p th Lehmer mean $L_p(a, b)$ of two positive numbers a and b is defined by $L_p(a, b) = (a^{p+1} + b^{p+1}) / (a^p + b^p)$.

The main purpose of this paper is to find the greatest values p_1, p_2 and least values q_1, q_2 such that the double inequalities $S_{p_1}(a, b) < M(a, b) < S_{q_1}(a, b)$ and $S_{p_2}(a, b) < T(a, b) < S_{q_2}(a, b)$ hold for all $a, b > 0$ with $a \neq b$ and present some new bounds for the complete elliptic integrals.

2. Preliminary Knowledge

Throughout this paper, we denote $r' = \sqrt{1 - r^2}$ for $r \in [0, 1]$.

For $0 < r < 1$, the following derivative formulas were presented in [9, Appendix E, pages 474–475]:

$$\begin{aligned} \frac{d\mathcal{K}(r)}{dr} &= \frac{\xi(r) - r'^2 \mathcal{K}(r)}{rr'^2}, & \frac{d\xi(r)}{dr} &= \frac{\xi(r) - \mathcal{K}(r)}{r}, \\ \frac{d[\xi(r) - r'^2 \mathcal{K}(r)]}{dr} &= r\mathcal{K}(r), & \frac{d[\mathcal{K}(r) - \xi(r)]}{dr} &= \frac{r\xi(r)}{r'^2}. \end{aligned} \quad (2.1)$$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \quad (2.2)$$

$$\xi\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\xi(r) - r'^2 \mathcal{K}(r)}{1+r}. \quad (2.3)$$

Lemma 2.1 can be found in [9, Theorem 3.21(7), (8), and (10), and Exercise 3.43(13) and (46)].

Lemma 2.1. (1) $r'^c \mathcal{K}(r)$ is strictly decreasing from $[0, 1)$ onto $(0, \pi/2]$ for $c \in [1/2, \infty)$;
 (2) $r'^c \xi(r)$ is strictly increasing on $(0, 1)$ if and only if $c \leq -1/2$ and strictly decreasing if and only if $c > 0$;
 (3) $\mathcal{K}(r) / \log(4/r')$ is strictly decreasing from $(0, 1)$ onto $(1, \pi / \log 16)$;
 (4) $2\xi(r) - r'^2 \mathcal{K}(r)$ is strictly increasing from $(0, 1)$ onto $(\pi/2, 2)$;
 (5) $[\xi(r) - r'^2 \mathcal{K}(r)] / [r^2 \mathcal{K}(r)]$ is strictly decreasing from $(0, 1)$ onto $(0, 1/2)$.

3. Main Results

Theorem 3.1. Inequality $S_{1/2}(a, b) < M(a, b) < S_1(a, b)$ holds for all $a, b > 0$ with $a \neq b$, and $S_{1/2}(a, b)$ and $S_1(a, b)$ are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean $M(a, b)$.

Proof. From (1.1) and (1.5) we clearly see that both $S_p(a, b)$ and $M(a, b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a = 1 > b$. Let $t = b$ and $r = (1 - t)/(1 + t)$. Then from (1.1) and (1.6) together with (2.2) we clearly see that

$$\begin{aligned} M(a, b) - S_{1/2}(a, b) &= \frac{\pi}{2\mathcal{K}(\sqrt{1-t^2})} - \left[\frac{(1+t)\sqrt{t}}{1+\sqrt{t}} \right]^{2/3} \\ &= \frac{\pi}{2(1+r)\mathcal{K}(r)} - \left[\frac{2\sqrt{1-r}}{(1+r)(\sqrt{1+r} + \sqrt{1-r})} \right]^{2/3} \\ &= \frac{1}{1+r} \left[\frac{\pi}{2\mathcal{K}(r)} - \left(\frac{2r'}{\sqrt{1+r} + \sqrt{1-r}} \right)^{2/3} \right]. \end{aligned} \quad (3.1)$$

Let

$$F(r) = \left[\frac{\pi}{2\mathcal{K}(r)} \right]^3 - \left(\frac{2r'}{\sqrt{1+r} + \sqrt{1-r}} \right)^2. \quad (3.2)$$

Then $F(r)$ can be rewritten as

$$F(r) = \left[\frac{\pi}{2\mathcal{K}(r)} \right]^3 - \frac{2r'^2}{1+r'} = \frac{2r'^2}{1+r'} F_1(r), \quad (3.3)$$

where

$$F_1(r) = \left(\frac{\pi}{2} \right)^3 \frac{1+r'}{2r'^2 \mathcal{K}(r)^3} - 1. \quad (3.4)$$

It is well known that the function $r \rightarrow \sqrt{r} + 1/\sqrt{r}$ is positive and strictly decreasing in $(0, 1)$. Then (3.4) and Lemma 2.1(1) lead to the conclusion that $F_1(r)$ is strictly increasing in $(0, 1)$, so that $F_1(r) > F_1(0) = 0$ for $r \in (0, 1)$.

Therefore, $M(a, b) > S_{1/2}(a, b)$ follows from (3.1)–(3.3).

On the other hand, $M(a, b) < S_1(a, b) = A(a, b)$ follows directly from (1.9).

Next, we prove that $S_{1/2}(a, b)$ and $S_1(a, b)$ are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean $M(a, b)$.

For any $0 < \varepsilon < 1/2$ and $0 < x < 1$, from (1.1), (1.6), and Lemma 2.1(3) we have

$$\begin{aligned} [M(1, 1-x)]^{3-2\varepsilon} - [S_{1/2+\varepsilon}(1, 1-x)]^{3-2\varepsilon} &= \left[\frac{\pi}{2 \int_0^{\pi/2} [1 - (2x - x^2)\sin^2 t]^{-1/2} dt} \right]^{3-2\varepsilon} \\ &\quad - \left[\frac{(2-x)(1-x)^{1/2-\varepsilon}}{1 + (1-x)^{1/2-\varepsilon}} \right]^2, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{M(1, x)}{S_{1-\varepsilon}(1, x)} &= \lim_{x \rightarrow 0} \left[\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} \mathcal{K}(\sqrt{1-x^2})} \left(\frac{1+x^\varepsilon}{1+x} \right)^{1/(1+\varepsilon)} \right] \\ &= \lim_{x \rightarrow 0} \frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} \mathcal{K}(\sqrt{1-x^2})} = \lim_{x \rightarrow 0} \frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} \log(4/x)} \frac{\log(4/x)}{\mathcal{K}(\sqrt{1-x^2})} \\ &= \lim_{x \rightarrow 0} \frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} \log(4/x)} = +\infty. \end{aligned} \quad (3.6)$$

Letting $x \rightarrow 0$ and making use of the Taylor expansion, one has

$$\begin{aligned}
& \left[\frac{\pi}{2 \int_0^{\pi/2} [1 - (2x - x^2) \sin^2 t]^{-1/2} dt} \right]^{3-2\varepsilon} - \left[\frac{(2-x)(1-x)^{1/2-\varepsilon}}{1 + (1-x)^{1/2-\varepsilon}} \right]^2 \\
&= 1 + \left(-\frac{3}{2} + \varepsilon \right) x + \frac{(2\varepsilon - 3)(4\varepsilon - 3)}{16} x^2 + o(x^2) \\
&\quad - \left[1 + \left(-\frac{3}{2} + \varepsilon \right) x + \frac{(2\varepsilon - 3)^2}{16} x^2 + o(x^2) \right] \\
&= -\frac{\varepsilon(3 - 2\varepsilon)}{8} x^2 + o(x^2).
\end{aligned} \tag{3.7}$$

Equations (3.5)–(3.7) imply that for any $1 < \varepsilon < 1/2$ there exist $\delta_1 = \delta_1(\varepsilon) \in (0, 1)$ and $\delta_2 = \delta_2(\varepsilon) \in (0, 1)$, such that $M(1, 1 - x) < S_{1/2+\varepsilon}(1, 1 - x)$ for $x \in (0, \delta_1)$ and $M(1, x) > S_{1-\varepsilon}(1, x)$ for $x \in (0, \delta_2)$. $\square$

Theorem 3.2. *Inequality $S_1(a, b) < T(a, b) < S_{3/2}(a, b)$ holds for all $a, b > 0$ with $a \neq b$, and $S_1(a, b)$ and $S_{3/2}(a, b)$ are the best possible lower and upper Gini mean bounds for the Toader mean $T(a, b)$.*

Proof. From (1.1) and (1.4) we clearly see that both $S_p(a, b)$ and $T(a, b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a = 1 > b$. Let $t = b$ and $r = (1 - t)/(1 + t)$. Then from (1.1), (1.4), and (2.3) we have

$$\begin{aligned}
\frac{T(a, b)}{S_{3/2}(a, b)} &= \frac{2}{\pi} \mathcal{E}(\sqrt{1 - t^2}) \cdot \left(\frac{1 + \sqrt{t}}{1 + t} \right)^2 \\
&= \frac{2}{\pi} \mathcal{E}\left(\frac{2\sqrt{r}}{1 + r}\right) \cdot (1 + r) \cdot \left(\frac{\sqrt{1 + r} + \sqrt{1 - r}}{2} \right)^2 \\
&= \frac{2}{\pi} [2\mathcal{E}(r) - r'^2 \mathcal{K}(r)] \cdot \left(\frac{\sqrt{1 + r} + \sqrt{1 - r}}{2} \right)^2 \\
&= \frac{1}{\pi} (1 + r') [2\mathcal{E}(r) - r'^2 \mathcal{K}(r)].
\end{aligned} \tag{3.8}$$

Let

$$G(r) = \frac{1}{\pi} (1 + r') [2\mathcal{E}(r) - r'^2 \mathcal{K}(r)]. \tag{3.9}$$

Then simple computations lead to

$$G(0) = 1, \quad (3.10)$$

$$\begin{aligned} G'(r) &= \frac{1}{\pi} \left[\left(-\frac{r}{r'} \right) (2\xi(r) - r'^2 \mathcal{K}(r)) + (1+r') \left(\frac{\xi(r) - r'^2 \mathcal{K}(r)}{r} \right) \right] \\ &= \frac{r'(1+r') [\xi(r) - r'^2 \mathcal{K}(r)] - r^2 [2\xi(r) - r'^2 \mathcal{K}(r)]}{\pi r r'} \\ &= \frac{r}{\pi r'} G_1(r), \end{aligned} \quad (3.11)$$

where

$$G_1(r) = (1+r') r' \mathcal{K}(r) \left[\frac{\xi(r) - r'^2 \mathcal{K}(r)}{r^2 \mathcal{K}(r)} \right] - [2\xi(r) - r'^2 \mathcal{K}(r)]. \quad (3.12)$$

It follows from (3.12) and Lemma 2.1(1), (4), and (5) that $G_1(r)$ is strictly decreasing from $(0, 1)$ onto $(-2, 0)$. Then (3.11) leads to the conclusion that $G'(r) < 0$ for $r \in (0, 1)$. Hence $G(r)$ is strictly decreasing in $(0, 1)$.

Therefore, $T(a, b) < S_{3/2}(a, b)$ follows from (3.8)–(3.10) together with the monotonicity of $G(r)$.

On the other hand, $T(a, b) > S_1(a, b) = A(a, b)$ follows directly from (1.7).

Next, we prove that $S_1(a, b)$ and $S_{3/2}(a, b)$ are the best possible lower and upper Gini mean bounds for the Toader mean $T(a, b)$.

For any $0 < \varepsilon < 1/2$ and $0 < x < 1$, from (1.1) and (1.4) one has

$$\begin{aligned} [T(1, 1-x)]^{1+2\varepsilon} - [S_{3/2-\varepsilon}(1, 1-x)]^{1+2\varepsilon} &= \left[\frac{2}{\pi} \int_0^{\pi/2} [1 - (2x - x^2) \sin^2 t]^{1/2} dt \right]^{1+2\varepsilon} \\ &\quad - \left[\frac{2-x}{1+(1-x)^{1/2-\varepsilon}} \right]^2, \end{aligned} \quad (3.13)$$

$$\lim_{x \rightarrow 0} \frac{T(1, x)}{S_{1+\varepsilon}(1, x)} = \lim_{x \rightarrow 0} \left[\frac{2}{\pi} \xi(\sqrt{1-x^2}) \left(\frac{1+x^\varepsilon}{1+x} \right)^{1/(1-\varepsilon)} \right] = \frac{2}{\pi} < 1. \quad (3.14)$$

Letting $x \rightarrow 0$ and making use of the Taylor expansion, we get

$$\begin{aligned}
 & \left[\frac{2}{\pi} \int_0^{\pi/2} \left[1 - (2x - x^2) \sin^2 t \right]^{1/2} dt \right]^{1+2\varepsilon} - \left[\frac{2-x}{1+(1-x)^{1/2-\varepsilon}} \right]^2 \\
 &= 1 - \left(\frac{1}{2} + \varepsilon \right) x + \frac{(2\varepsilon+1)(4\varepsilon+1)}{16} x^2 + o(x^2) \\
 & \quad - \left[1 - \left(\frac{1}{2} + \varepsilon \right) x + \frac{(2\varepsilon+1)^2}{16} x^2 + o(x^2) \right] \\
 &= \frac{\varepsilon(2\varepsilon+1)}{8} x^2 + o(x^2).
 \end{aligned} \tag{3.15}$$

Equations (3.13)–(3.15) imply that for any $0 < \varepsilon < 1/2$ there exist $\delta_3 = \delta_3(\varepsilon) \in (0, 1)$ and $\delta_4 = \delta_4(\varepsilon) \in (0, 1)$, such that $T(1, 1-x) > S_{3/2-\varepsilon}(1, 1-x)$ for $x \in (0, \delta_3)$ and $T(1, x) < S_{1+\varepsilon}(1, x)$ for $x \in (0, \delta_4)$. $\square$

4. Remarks and Corollaries

Remark 4.1. From (3.9) and Lemma 2.1(4) we clearly see that $G(1^-) = 2/\pi$. Then (3.8) and (3.9) together with the monotonicity of $G(r)$ lead to the conclusion that

$$\frac{2}{\pi} S_{3/2}(a, b) < T(a, b) \tag{4.1}$$

for all $a, b > 0$ with $a \neq b$.

Remark 4.2. We find that the lower bound $L(a, b)$ in (1.10) and the best possible lower Gini mean bound $S_{1/2}(a, b)$ in Theorem 3.1 are not comparable. In fact, from (1.1) and (1.11) we have

$$\begin{aligned}
 \lim_{x \rightarrow +\infty} \frac{S_{1/2}(1, x)}{L(1, x)} &= \lim_{x \rightarrow +\infty} \left[\frac{1+x^{-1}}{1+x^{-1/2}} \right]^{2/3} \frac{x^{2/3} \log x}{x-1} = \lim_{x \rightarrow +\infty} \frac{\log x}{x^{1/3} - x^{-2/3}} = 0, \\
 S_{1/2}(1, 1+x) - L(1, 1+x) &= 1 + \frac{1}{2}x - \frac{1}{16}x^2 + o(x^2) - \left[1 + \frac{1}{2}x - \frac{1}{12}x^2 + o(x^2) \right] \\
 &= \frac{1}{48}x^2 + o(x^2) \quad (x \rightarrow 0).
 \end{aligned} \tag{4.2}$$

Table 1: Comparison of $\mathcal{K}(r)$ with $H(r)$ for some $r \in (0, 1)$.

r	$\mathcal{K}(r)$	$H(r)$
0.1	1.574745561517...	1.574745561518...
0.2	1.586867847...	1.586867848...
0.3	1.608048620...	1.608048634...
0.4	1.639999866...	1.640000021...
0.5	1.685750355...	1.685751528...
0.6	1.750753803...	1.750760840...
0.7	1.845693998...	1.845732233...
0.8	1.995302778...	1.995519211...

Remark 4.3. The following two equations show that the best possible upper power mean bound $M_{\log 2 / \log(\pi/2)}(a, b)$ in (1.8) and the best possible upper Gini mean bound $S_{3/2}(a, b)$ in Theorem 3.2 are not comparable:

$$\lim_{x \rightarrow +\infty} \frac{S_{3/2}(1, x)}{M_{\log 2 / \log(\pi/2)}(1, x)} = 2^{\log(\pi/2) / \log 2} = \frac{\pi}{2},$$

$$\begin{aligned} M_{\log 2 / \log(\pi/2)}(1, 1+x) - S_{3/2}(1, 1+x) &= 1 + \frac{1}{2}x + \frac{1}{8} \left[\frac{\log 2}{\log(\pi/2)} - 1 \right] x^2 \\ &\quad + o(x^2) - \left[1 + \frac{1}{2}x + \frac{1}{16}x^2 + o(x^2) \right] \quad (4.3) \\ &= \frac{1}{16} \left[\frac{2 \log 2}{\log(\pi/2)} - 3 \right] x^2 + o(x^2) \\ &= 0.00436 \dots \times x^2 + o(x^2) \quad (x \rightarrow 0). \end{aligned}$$

From Theorem 3.1 we get an upper bound for the complete elliptic integrals of the first kind $\mathcal{K}(r)$ as follows.

Corollary 4.4. *Inequality*

$$\mathcal{K}(r) < \frac{\pi}{2} \left[\frac{1 + (1-r^2)^{1/4}}{(1 + \sqrt{1-r^2})(1-r^2)^{1/4}} \right]^{2/3} \quad (4.4)$$

holds for all $r \in (0, 1)$.

Remark 4.5. Computational and numerical experiments show that the upper bound in (4.4) for $\mathcal{K}(r)$ is very accurate for some $r \in (0, 1)$. In fact, if we let $H(r) = \pi[1 + (1-r^2)^{1/4}]^{2/3} / \{2[(1 + \sqrt{1-r^2})(1-r^2)^{1/4}]^{2/3}\}$, then we have Table 1 via elementary computation.

Table 2: Comparison of $\mathcal{E}(r)$ with $J(r)$ for some $r \in (0, 1)$.

r	$\mathcal{E}(r)$	$J(r)$
0.1	1.566861942021...	1.566861942028...
0.2	1.554968546...	1.554968548...
0.3	1.534833465...	1.534833516...
0.4	1.505941612...	1.505942206...
0.5	1.467462209...	1.467466484...
0.6	1.418083394...	1.418107161...
0.7	1.355661136...	1.355777213...
0.8	1.276349943...	1.276910677...

The following bounds for the complete elliptic integrals of the second kind $\mathcal{E}(r)$ follow from Theorem 3.2 and Remark 4.1.

Corollary 4.6. *Inequality*

$$\left[\frac{1 + \sqrt{1 - r^2}}{1 + (1 - r^2)^{1/4}} \right]^2 < E(r) < \frac{\pi}{2} \left[\frac{1 + \sqrt{1 - r^2}}{1 + (1 - r^2)^{1/4}} \right]^2 \quad (4.5)$$

holds for all $r \in (0, 1)$.

Remark 4.7. Computational and numerical experiments show that the upper bound in (4.5) for $\mathcal{E}(r)$ is very accurate for some $r \in (0, 1)$. In fact, if we let $J(r) = \pi[1 + \sqrt{1 - r^2}]^2 / \{2[1 + (1 - r^2)^{1/4}]^2\}$, then we have Table 2 via elementary computation.

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