Research Article

Stability of Analytical and Numerical Solutions for Nonlinear Stochastic Delay Differential Equations with Jumps

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This paper is concerned with the stability of analytical and numerical solutions for nonlinear stochastic delay differential equations (SDDEs) with jumps. A sufficient condition for mean-square exponential stability of the exact solution is derived. Then, mean-square stability of the numerical solution is investigated. It is shown that the compensated stochastic $\theta$-methods inherit stability property of the exact solution. More precisely, the methods are mean-square stable for any stepsize $\Delta t = \tau/m$ when $1/2 \leq \theta \leq 1$, and they are exponentially mean-square stable if the stepsize $\Delta t \in (0, \Delta t_0)$ when $0 \leq \theta < 1$. Finally, some numerical experiments are given to illustrate the theoretical results.

1. Introduction

Models that incorporate jumps have become increasingly popular in finance and several areas of science and engineering. In particular, they are used in mathematical finance in order to simulate asset prices, interest rates, and volatilities \cite{1,2}. Jump models also arise in many other application areas and have proved successful at describing unexpected, abrupt changes of state \cite{3}. So, it is valuable to investigate the properties of the solutions of these problems.

As is well known, explicit solutions of stochastic differential equations can rarely be obtained. It is necessary to construct efficient numerical methods to solve these equations. In recent years, many researchers worked on the construction of numerical schemes for stochastic ordinary differential equations (SODEs) (see \cite{4,5}, and their references) and stochastic delay differential equations (SDDEs), see, for example, \cite{6–11} and references therein. For SODEs with jumps, the strong convergence and mean-square stability of some
semi-implicit numerical methods are investigated in [12–15]. A compensated split-step backward Euler method for SODEs with jumps is introduced in [12] and proved to satisfy a better stability property than the split-step backward Euler method.

For SDDEs with jumps, most of the existing work is concerned about convergence property of numerical methods, see, for example, [16–19]. There are few results on stability property, which motivates our work. In [20], Tan and Wang investigated the mean-square stability of the explicit Euler method for linear SDDEs with jumps. The aim of our paper is to investigate the mean-square stability of the compensated stochastic \( \theta \) methods for nonlinear SDDEs with jumps.

This paper is organized as follows. In Section 2, we obtain a stability result for the analytical solution of (2.1). In Section 3, the compensated stochastic \( \theta \) methods are constructed to solve problem (2.1). In Section 4, our main results will be stated and proved. It is shown that the compensated stochastic \( \theta \) methods inherit mean-square stability of the exact solution. More precisely, the methods are mean-square stable for any stepsize \( \Delta t = \tau / m \) when \( 1/2 \leq \theta \leq 1 \), and they are exponentially mean-square stable if the stepsize \( \Delta t \in (0, \Delta t_0) \) when \( 0 \leq \theta < 1 \). Moreover, when \( \theta = 1 \), the method is exponentially mean-square stable for every stepsize \( \Delta t = \tau / m \). Finally, some numerical experiments are reported to illustrate the theoretical results.

2. Stability of the Analytical Solution

Throughout this paper, we let \( C([−\tau, 0); \mathbb{R}^d) \) denote the family of continuous functions from \( [−\tau, 0] \) to \( \mathbb{R}^d \) equipped with the norm \( \| \varphi \| := \sup_{−\tau \leq s \leq 0} |\varphi(s)| \), where \( | \cdot | \) is the Euclidean norm in \( \mathbb{R}^d \). Denoted by \( C^b_{\mathcal{F}_0}([−\tau, 0]; \mathbb{R}^d) \) the family of all bounded, \( \mathcal{F}_0 \) measurable, \( C([−\tau, 0]; \mathbb{R}^d) \) valued stochastic variables. The inner product of \( x, y \) in \( \mathbb{R}^d \) is denoted by \( \langle x, y \rangle \). If \( A \) is a vector or matrix, its transpose is denoted by \( A^T \). If \( A \) is a matrix, its trace norm is denoted by \( |A| = \sqrt{\text{trace}(A^TA)} \).

We consider the nonlinear SDDEs with jumps in Itô’s sense of the form:

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dW(t) \\
&\quad + h(x(t^-), x(t^--\tau))dN(t), \quad t > 0,
\end{align*}
\]  

(2.1)

with initial data \( x(t) = \varphi(t), t \in [−\tau, 0] \) and \( \tau > 0 \) is a constant, \( x(t^-) \) denotes \( \lim_{s \to t^-} x(s) \), \( f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times m}, h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) are continuous functions, and \( \varphi(t) \in C^b_{\mathcal{F}_0}([−\tau, 0]; \mathbb{R}^d) \). \( W(t) \) is an \( m \)-dimensional Wiener process defined on the complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \) with a filtration satisfying the usual conditions (i.e., it is increasing and right continuous while \( \mathcal{F}_0 \) contains all \( P \)-null sets). \( N(t) \) is a scalar Poisson process with parameter \( \lambda \) defined on the same probability space. Assume that \( W(t) \) and \( N(t) \) are independent of \( \mathcal{F}_0 \). Moreover, we assume that \( \varphi(t) \) is \( \mathcal{F}_0 \)-measurable and right continuous with \( \mathbb{E}[\|\varphi\|^2] < \infty \). We also assume that \( f(0, 0) = 0, g(0, 0) = 0 \) and \( h(0, 0) = 0 \), so problem (2.1) admits a zero solution \( x(t) \equiv 0 \).

Definition 2.1 (see [21]). The zero solution of (2.1) is said to be \( p \)th moment exponentially stable if there is a pair of positive constants \( \lambda \) and \( C \) such that

\[
\mathbb{E}|x(t)|^p \leq C\|\varphi\|\exp^{-\lambda(t-t_0)}, \quad t \geq t_0,
\]  

(2.2)
Suppose that there are some constants $\alpha_i, \beta_i, \gamma_i, i = 1, 2$, such that

\begin{align}
\langle x_1 - x_2, f(x_1, y) - f(x_2, y) \rangle &\leq \alpha_i |x_1 - x_2|^2, \\
|f(x_1) - f(x_2)| &\leq \alpha_2 |y_1 - y_2|, \\
|g(x_1, y_1) - g(x_2, y_2)|^2 &\leq \beta_1 |x_1 - x_2|^2 + \beta_2 |y_1 - y_2|^2, \\
|h(x_1, y_1) - h(x_2, y_2)|^2 &\leq \gamma_1 |x_1 - x_2|^2 + \gamma_2 |y_1 - y_2|^2,
\end{align}

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$. If

$$\alpha = 2\alpha_1 + 2\alpha_2 + \beta_1 + \beta_2 + \lambda (1 + 2\gamma_1 + 2\gamma_2) < 0,$$

then the zero solution of (2.1) is exponentially mean-square stable.

Proof. Let $t \geq 0$, $\delta > 0$, it follows from Itô’s formula that

$$
|x(t + \delta)|^2 = |x(t)|^2 + \int_t^{t+\delta} \left( 2\langle x(s), f(x(s), x(s-\tau)) \rangle + |g(x(s), x(s-\tau))|^2 \right) ds \\
+ \int_t^{t+\delta} 2\langle x(s), g(x(s), x(s-\tau)) \rangle dW(s) \\
+ \int_t^{t+\delta} \left( 2\langle x(s^-), h(x(s^-), x(s^--\tau)) \rangle + |h(x(s^-), x(s^--\tau))|^2 \right) d\tilde{N}(s) \\
+ \lambda \int_t^{t+\delta} \left( 2\langle x(s), h(x(s), x(s-\tau)) \rangle + |h(x(s), x(s-\tau))|^2 \right) ds,
$$

where $\tilde{N}(t) = N(t) - \lambda t$. Taking expectation and using the properties of Itô integral give

$$
\mathbb{E}|x(t + \delta)|^2 = \mathbb{E}|x(t)|^2 + \mathbb{E} \int_t^{t+\delta} \left( 2\langle x(s), f(x(s), x(s-\tau)) \rangle + |g(x(s), x(s-\tau))|^2 \right) ds \\
+ \lambda \mathbb{E} \int_t^{t+\delta} \left( 2\langle x(s), h(x(s), x(s-\tau)) \rangle + |h(x(s), x(s-\tau))|^2 \right) ds.
$$
From (2.3) and (2.4), we have

\[
2\mathbb{E}\left(x(s), f(x(s), x(s - \tau))\right) = 2\mathbb{E}\left(x(s), f(x(s), x(s - \tau)) - f(0, x(s - \tau))\right) \\
+ 2\mathbb{E}\left(x(s), f(0, x(s - \tau))\right)
\leq 2\alpha_1 \mathbb{E}|x(s)|^2 + 2\alpha_2 \mathbb{E}|x(s)||x(s - \tau)|
\leq 2\alpha_1 \mathbb{E}|x(s)|^2 + 2\alpha_2 \sup_{s - \tau \leq t \leq s} \mathbb{E}|x(u)|^2. \tag{2.10}
\]

Similarly, (2.5) and (2.6) yield

\[
\mathbb{E}\left[g(x(s), x(s - \tau))\right] \leq (\beta_1 + \beta_2) \sup_{s - \tau \leq t \leq s} \mathbb{E}|x(u)|^2,
\]

\[
\mathbb{E}\left[h(x(s), x(s - \tau))\right] \leq (\gamma_1 + \gamma_2) \sup_{s - \tau \leq t \leq s} \mathbb{E}|x(u)|^2,
\]

\[
2\mathbb{E}(x(s), h(x(s), x(s - \tau))) \leq (1 + \gamma_1 + \gamma_2) \sup_{s - \tau \leq t \leq s} \mathbb{E}|x(u)|^2. \tag{2.11}
\]

Substituting (2.10)-(2.11) into (2.9) yields

\[
\mathbb{E}|x(t + \delta)|^2 \\
\leq \mathbb{E}|x(t)|^2 + \int_t^{t + \delta} \left(2\alpha_1 \mathbb{E}|x(s)|^2 + (2\alpha_2 + \beta_1 + \beta_2 + \lambda(1 + 2\gamma_1 + 2\gamma_2)) \sup_{s - \tau \leq t \leq s} \mathbb{E}|x(u)|^2\right) ds. \tag{2.12}
\]

Let \(v(t) = \mathbb{E}|x(t)|^2, \beta = 2\alpha_2 + \beta_1 + \beta_2 + \lambda(1 + 2\gamma_1 + 2\gamma_2),\) we have

\[
D^+ v(t) \leq 2\alpha_1 v(t) + \beta \sup_{t - \tau \leq u \leq t} v(u), \tag{2.13}
\]

where

\[
D^+ v(t) = \limsup_{\delta \searrow 0} \frac{v(t + \delta) - v(t)}{\delta}. \tag{2.14}
\]

Moreover, \(\alpha < 0\) implies \(-2\alpha_1 > \beta \geq 0.\) By Lemma 1.1 in [22], there exist positive constants \(\nu\) and \(k\) such that

\[
v(t) \leq ke^{-\nu t}, \quad t \geq 0. \tag{2.15}
\]

Hence, the theorem is proven. \(\square\)

Based on the above result, we are going to study the stability of numerical methods for (2.1) in the following sections.
3. Compensated Stochastic $\theta$ Methods for Nonlinear SDDEs with Jumps

Since the compensated Poisson process $\tilde{N}(t) = N(t) - \lambda t$ is a martingale satisfying the property
\[
\mathbb{E}(\tilde{N}(t) + s - \tilde{N}(t)) = 0, \quad \mathbb{E}\left|\tilde{N}(t + s) - \tilde{N}(t)\right|^2 = \lambda s, \quad t, s \geq 0,
\] (3.1)
we rewrite problem (2.1) in an equivalent form:
\[
dx(t) = \tilde{f}(x(t), x(t - \tau))dt + g(x(t), x(t - \tau))dW(t) + h(x(t), x(t - \tau))d\tilde{N}(t), \quad t > 0,
\] (3.2)
where $\tilde{f}(x, y)$ is defined as
\[
\tilde{f}(x, y) := f(x, y) + \lambda h(x, y).
\] (3.3)

Applying the stochastic $\theta$ methods to (3.2) leads to the following compensated stochastic $\theta$ methods:
\[
X_{n+1} = X_n + (1 - \theta)\tilde{f}(X_n, X_{n-m})\Delta t + \theta\tilde{f}(X_{n+1}, X_{n-m+1})\Delta t + g(X_n, X_{n-m})\Delta W_n + h(X_n, X_{n-m})\Delta \tilde{N}_n.
\] (3.4)

Here, $X_n$ denotes the approximation to $x(t_n)$, $\theta$ a parameter with $0 \leq \theta \leq 1$, $\Delta t := t_{n+1} - t_n$ is the stepsize which satisfies $\tau = m\Delta t$ for a positive integer $m$, $\Delta W_n := W(t_{n+1}) - W(t_n)$ and $\Delta \tilde{N}_n := \tilde{N}(t_{n+1}) - \tilde{N}(t_n)$. In particular,
\[
X_l = \varphi(l\Delta t), \quad l \leq 0.
\] (3.5)

Note that for $\theta > 0$, the numerical solutions in (3.4) are defined by implicit equations. However, due to the one-sided Lipschitz condition (2.3), (3.4) has a unique solution, with probability one, for all $\theta\Delta t(\alpha_1 + \lambda\sqrt{\gamma_1}) < 1$, see, for example, [23, Theorem 14.2] and (19) in [12].

Remark 3.1. Since $\alpha < 0$ implies $\alpha_1 + \lambda\sqrt{\gamma_1} < 0$, then the compensated stochastic $\theta$ methods (3.4) produce a well-defined, unique solution if the stability condition $\alpha < 0$ holds.

Definition 3.2. For a given stepsize $\Delta t = \tau/m$, a numerical method on the nonlinear SDDEs with jumps (2.1) is said to be exponentially mean-square stable, if there exist positive constants $N$ and $\gamma$, such that the numerical solution $X_n$ produced by this method satisfies
\[
\mathbb{E}|X_n|^2 \leq N\|\varphi\|e^{-\gamma(t_n - t_0)},
\] (3.6)
for all initial data $\varphi \in C^1_F^b([-\tau, 0]; \mathbb{R}^d)$. 
Definition 3.3. For a given stepsize $\Delta t = \tau / m$, a numerical method on the nonlinear SDDEs with jumps (2.1) is said to be mean-square stable if the numerical solution $X_n$ produced by this method satisfies

$$
\lim_{n \to \infty} \mathbb{E}[X_n]^2 = 0. \quad (3.7)
$$

4. Stability Analysis of the Numerical Solutions

In this section, we study mean-square stability and exponentially mean-square stability of the compensated stochastic $\theta$ methods (3.4). Now, we present the main results of the paper.

Theorem 4.1. Suppose that (2.3)–(2.7) hold. If $1/2 \leq \theta \leq 1$, then the compensated stochastic $\theta$ methods are mean-square stable for every stepsize $\Delta t = \tau / m$.

Proof. It follows from (3.4) that

$$
\left| X_{n+1} - \theta \Delta t \tilde{f}(X_{n+1}, X_{n-m+1}) \right|^2 = \left| X_n - \theta \Delta t \tilde{f}(X_n, X_{n-m}) \right|^2 + 2\Delta t \left\langle X_n, \tilde{f}(X_n, X_{n-m}) \right\rangle \\
+ (\Delta t)^2 (1-2\theta) \left| \tilde{f}(X_n, X_{n-m}) \right|^2 + \left| \tilde{g}(X_n, X_{n-m}) \Delta W_n \right|^2 + \left| h(X_n, X_{n-m}) \Delta \tilde{N}_n \right|^2 + M_n, \quad (4.1)
$$

where

$$
M_n = 2 \left\langle X_n + (1-\theta) \Delta t \tilde{f}(X_n, X_{n-m}), \tilde{g}(X_n, X_{n-m}) \Delta W_n \right\rangle \\
+ 2 \left\langle X_n + (1-\theta) \Delta t \tilde{f}(X_n, X_{n-m}), h(X_n, X_{n-m}) \Delta \tilde{N}_n \right\rangle \\
+ 2 \left\langle \tilde{g}(X_n, X_{n-m}) \Delta W_n, h(X_n, X_{n-m}) \Delta \tilde{N}_n \right\rangle. \quad (4.2)
$$

Thus, for $1/2 \leq \theta \leq 1$, we have

$$
\left| X_{n+1} - \theta \Delta t \tilde{f}(X_{n+1}, X_{n-m+1}) \right|^2 \leq \left| X_n - \theta \Delta t \tilde{f}(X_n, X_{n-m}) \right|^2 + 2\Delta t \left\langle X_n, \tilde{f}(X_n, X_{n-m}) \right\rangle \\
+ \left| \tilde{g}(X_n, X_{n-m}) \Delta W_n \right|^2 + \left| h(X_n, X_{n-m}) \Delta \tilde{N}_n \right|^2 + M_n. \quad (4.3)
$$

It follows from (2.3), (2.4), and (2.6) that

$$
2 \left\langle X_n, \tilde{f}(X_n, X_{n-m}) \right\rangle = 2 \left\langle X_n, f(X_n, X_{n-m}) \right\rangle + 2\lambda \left( \left| X_{n-m} \right|^2 - \left| X_{n-m}^+ \right|^2 \right) \\
\leq 2\alpha_1 \left| X_n \right|^2 + \alpha_2 \left| \phi_n \right|^2 + \alpha_2 \left| X_{n-m} \right|^2 + \lambda \left( \left| X_{n-m} \right|^2 + \gamma_1 \left| X_n \right|^2 + \gamma_2 \left| X_{n-m} \right|^2 \right). \quad (4.4)
$$
Note that \( E(\Delta W_n) = 0, E(\Delta \tilde{N}_n) = 0 \) and \( E(\Delta \tilde{N}_n)^2 = \lambda \Delta t \). Furthermore, \( X_n \) and \( X_{n-m} \) are all \( \mathcal{F}_t \)-measurable. Therefore, we can easily obtain

\[
\mathbb{E}|g(X_n, X_{n-m})\Delta W_n|^2 = \Delta t \mathbb{E}|g(X_n, X_{n-m})|^2, \tag{4.5}
\]

\[
\mathbb{E}|h(X_n, X_{n-m})\Delta \tilde{N}_n|^2 = \lambda \Delta t \mathbb{E}|h(X_n, X_{n-m})|^2, \tag{4.6}
\]

\[
\mathbb{E}M_n = 0. \tag{4.7}
\]

Taking expectation on both sides of (4.3) and substituting (4.4)–(4.7) into (4.3), we have

\[
\mathbb{E}\left|X_{n+1} - \theta \Delta t \tilde{f}(X_{n+1}, X_{n-m+1})\right|^2 \\
\leq \mathbb{E}\left|X_n - \theta \Delta t \tilde{f}(X_n, X_{n-m})\right|^2 + \Delta t (2\alpha_1 + \alpha_2 + \beta_1 + \lambda + 2\lambda \gamma_1) \mathbb{E}|X_n|^2 \\
+ \Delta t (\alpha_2 + \beta_2 + 2\lambda \gamma_2) \mathbb{E}|X_{n-m}|^2. \tag{4.8}
\]

Consequently, by the recursion of inequality (4.8), we have

\[
\mathbb{E}\left|X_{n+1} - \theta \Delta t \tilde{f}(X_{n+1}, X_{n-m+1})\right|^2 \\
\leq \mathbb{E}\left|X_n - \theta \Delta t \tilde{f}(X_n, X_{n-m})\right|^2 \\
+ \Delta t (2\alpha_1 + \alpha_2 + \beta_1 + \lambda + 2\lambda \gamma_1) \sum_{j=n-1}^{n} \mathbb{E}|X_j|^2 + \Delta t (\alpha_2 + \beta_2 + 2\lambda \gamma_2) \sum_{j=n-1}^{n} \mathbb{E}|X_{j-m}|^2 \\
\leq \cdots \\
\leq \mathbb{E}\left|X_0 - \theta \Delta t \tilde{f}(X_0, X_{-m})\right|^2 \\
+ \Delta t (2\alpha_1 + \alpha_2 + \beta_1 + \lambda + 2\lambda \gamma_1) \sum_{j=0}^{n} \mathbb{E}|X_j|^2 + \Delta t (\alpha_2 + \beta_2 + 2\lambda \gamma_2) \sum_{j=0}^{n} \mathbb{E}|X_{j-m}|^2. \tag{4.9}
\]

Noting that \( \sum_{j=0}^{n} \mathbb{E}|X_{j-m}|^2 = \sum_{j=-m}^{n} \mathbb{E}|X_j|^2 + \sum_{j=-m}^{n} \mathbb{E}|X_j|^2 \) and \( \tau = m\Delta t \), we derive from (4.9) that

\[
\mathbb{E}\left|X_{n+1} - \theta \Delta t \tilde{f}(X_{n+1}, X_{n-m+1})\right|^2 \\
\leq \mathbb{E}\left|X_0 - \theta \Delta t \tilde{f}(X_0, X_{-m})\right|^2.
\]
+ \Delta t(2\alpha_1 + 2\alpha_2 + \beta_1 + \beta_2 + \lambda(1 + 2\gamma_1 + 2\gamma_2)) \sum_{j=0}^{n} \mathbb{E}|X_j|^2 + \tau(\alpha_2 + \beta_2 + 2\lambda \gamma_2) \\
\times \max_{-m \leq j \leq 0} \mathbb{E}|X_j|^2.
\end{align}

Rearranging (4.10) and using the notation $a$ in (2.7), we obtain

\begin{align}
\mathbb{E}|X_{n+1} - \theta \Delta t \tilde{f}(X_{n+1}, X_{n-m+1})|^2 - \alpha \Delta t \sum_{j=0}^{n} \mathbb{E}|X_j|^2 \\
\leq \mathbb{E}|X_0 - \theta \Delta t \tilde{f}(X_0, X_{-m})|^2 + (\alpha_2 + \beta_2 + 2\lambda \gamma_2) \tau \max_{-m \leq j \leq 0} \mathbb{E}|X_j|^2.
\end{align}

Since $\mathbb{E}\|\varphi\|^2 < \infty$ and $\alpha < 0$, we then derive that the series $\sum_{j=0}^{\infty} \mathbb{E}|X_j|^2$ is convergent, which implies $\lim_{n \to \infty} \mathbb{E}|X_n|^2 = 0$. Consequently, for $1/2 \leq \theta \leq 1$, the compensated stochastic $\theta$ methods are mean-square stable for any stepsize $\Delta t = \tau / m$.

In order to investigate the exponential stability of the numerical methods, we need the following lemma which is Theorem 1 in [24].

**Lemma 4.2** (see [24]). Suppose, for some fixed integer $N \geq 0$, that $t_n = t_0 + n\Delta t$ for some $\Delta t > 0$ and $\{v_n\}_{n=0}^\infty$ is a sequence of positive numbers that satisfies

\begin{equation}
\frac{v_{n+1} - v_n}{\Delta t} \leq -\alpha_{\Delta t} v_n + \beta_{\Delta t} \max_{j \in J} v_{n+j} \quad \text{for } n \in \mathbb{N}
\end{equation}

with $N = 0$ if $\beta_{\Delta t} = 0$, where $J := \{-N, \ldots, -1, 0\}$. If

\begin{equation}
0 \leq \beta_{\Delta t} < \alpha_{\Delta t}, \quad 0 < \alpha_{\Delta t} \Delta t < 1,
\end{equation}

then $v_n \leq \max_{j \in J} v_j \exp\{-\nu^*(t_n - t_0)\}$, where $\nu^* > 0$ is a constant.

Now, we present the result as follows.

**Theorem 4.3.** Suppose that (2.3)–(2.7) hold and the drift coefficient $f$ satisfies the linear growth condition, that is, there is a constant $D$ such that

\begin{equation}
|f(x, y)|^2 \leq D(|x|^2 + |y|^2).
\end{equation}

Define $\Delta t_1 = -\alpha / 2(1-\theta)^2(2D + \lambda^2 (\gamma_1 + \gamma_2))$, $\Delta t_2 = -(2\alpha_1 + \alpha_2 + \beta_1 + \lambda + 2\lambda \gamma_1) / 2(1-\theta)^2(D + \lambda^2 \gamma_1)$ and $\Delta t_3 = \inf \{\Delta t > 0 : P(\theta, \Delta t) < 0\}$, where $P(\theta, \Delta t) = 2(1-\theta)^2(D + \lambda^2 \gamma_1)(\Delta t)^2 + ((1-\theta)(2\alpha_1 + \alpha_2 + \lambda + \lambda \gamma_1) + \beta_1 + \lambda \gamma_1) \Delta t + 1$. If $0 \leq \theta < 1$, and the stepsize $\Delta t \in (0, \Delta t_0)$ with $\Delta t_0 = \min\{\Delta t_1, \Delta t_2, \Delta t_3\}$, then the compensated stochastic $\theta$ methods are exponentially mean-square stable.
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Proof. We derive from (3.4) that

\[ \left| X_{n+1} - \theta \Delta t \tilde{f}(X_{n+1}, X_{n-m+1}) \right|^2 = \left| X_n + (1 - \theta)\Delta t \tilde{f}(X_n, X_{n-m}) + g(X_n, X_{n-m}) \Delta W_n + h(X_n, X_{n-m}) \Delta \tilde{N}_n \right|^2. \]  

(4.15)

Hence, we have

\[ |X_{n+1}|^2 \leq |X_n|^2 + (1 - \theta)^2(\Delta t)^2 \left| \tilde{f}(X_n, X_{n-m}) \right|^2 + |g(X_n, X_{n-m}) \Delta W_n|^2 + |h(X_n, X_{n-m}) \Delta \tilde{N}_n|^2 \]

\[ + 2\theta \Delta t \left( X_{n+1}, \tilde{f}(X_{n+1}, X_{n-m+1}) \right) + 2(1 - \theta)\Delta t \left( X_n, \tilde{f}(X_n, X_{n-m}) \right) + M_n, \]

(4.16)

where \( M_n \) is defined as (4.2). From (2.6), (3.3), and (4.14), we obtain

\[ \left| \tilde{f}(X_n, X_{n-m}) \right|^2 = \left| f(X_n, X_{n-m}) + \lambda h(X_n, X_{n-m}) \right|^2 \]

\[ \leq 2 \left( D \left( |X_n|^2 + |X_{n-m}|^2 \right) + \lambda^2 \left( \gamma_1 |X_n|^2 + \gamma_2 |X_{n-m}|^2 \right) \right). \]  

(4.17)

Substituting (4.4)–(4.7) and (4.17) into (4.16), and taking expectation, we have

\[ \mathbb{E}|X_{n+1}|^2 \leq \mathbb{E}|X_n|^2 + 2(1 - \theta)^2(\Delta t)^2 \left( D + \lambda^2 \gamma_1 \right) \mathbb{E}|X_n|^2 + \left( D + \lambda^2 \gamma_2 \right) \mathbb{E}|X_{n-m}|^2 + \Delta t \left( 2\alpha_1 + \alpha_2 + \lambda \left( 1 + \gamma_1 \right) \right) \mathbb{E}|X_n|^2 + \left( 2\alpha_2 + \lambda \gamma_2 \right) \mathbb{E}|X_{n-m}|^2 \]

\[ + \theta \Delta t \left( 2\alpha_1 + \alpha_2 + \lambda \left( 1 + \gamma_1 \right) \right) \mathbb{E}|X_n|^2 + \left( 2\alpha_2 + \lambda \gamma_2 \right) \mathbb{E}|X_{n-m}|^2, \]  

(4.18)

which yields

\[ (1 - \theta \Delta t \left( 2\alpha_1 + \alpha_2 + \lambda \left( 1 + \gamma_1 \right) \right)) \mathbb{E}|X_{n+1}|^2 \]

\[ \leq \left( 1 - \theta \Delta t \left( 2\alpha_1 + \alpha_2 + \lambda \left( 1 + \gamma_1 \right) \right) \right) \mathbb{E}|X_n|^2 + \left( \left( 2\alpha_1 + \alpha_2 \right) + \beta_1 + \lambda + 2\gamma_1 \right) \Delta t \]

\[ + 2(1 - \theta)^2 \left( D + \lambda^2 \gamma_1 \right) (\Delta t)^2 \mathbb{E}|X_n|^2 \]

\[ + \left( \alpha_2 + \beta_2 + 2\lambda \gamma_2 \right) \Delta t + 2(1 - \theta)^2 \left( D + \lambda^2 \gamma_2 \right) (\Delta t)^2 \max_{n-m \leq n-m+1} \mathbb{E}|X_i|^2. \]  

(4.19)

Hence,

\[ \frac{\mathbb{E}|X_{n+1}|^2 - \mathbb{E}|X_n|^2}{\Delta t} \leq -A \mathbb{E}|X_n|^2 + B \max_{n-m \leq n-m+1} \mathbb{E}|X_i|^2, \]  

(4.20)
where

\[ A = \frac{2\alpha_1 + \alpha_2 + \beta_1 + \lambda + 2\lambda \gamma_1 + 2(1 - \theta)^2 (D + \lambda^2 \gamma_1)}{1 - \theta \Delta t (2\alpha_1 + \alpha_2 + \lambda (1 + \gamma_1))}, \]

\[ B = \frac{\alpha_2 + \beta_2 + 2\lambda \gamma_2 + 2(1 - \theta)^2 (D + \lambda^2 \gamma_2)}{1 - \theta \Delta t (2\alpha_1 + \alpha_2 + \lambda (1 + \gamma_1))}. \]  

(4.21)

By Lemma 4.2, we derive that the methods are exponentially mean-square stable if

\[ 0 \leq B < A, \quad 0 < A\Delta t < 1. \]  

(4.22)

That is,

\[ \Delta t < \frac{-\alpha}{2(1 - \theta)^2 (2D + \lambda^2 (\gamma_1 + \gamma_2))}, \]

\[ \Delta t < \frac{-2\alpha_1 + \alpha_2 + \beta_1 + \lambda + 2\lambda \gamma_1}{2(1 - \theta)^2 (D + \lambda^2 \gamma_1)}, \]

\[ P(\theta, \Delta t) > 0, \]  

(4.23)

where \( P(\theta, \Delta t) = 2(1 - \theta)^2 (D + \lambda^2 \gamma_1) (\Delta t)^2 + ((1 - \theta)(2\alpha_1 + \alpha_2 + \lambda + \lambda \gamma_1) + \beta_1 + \lambda \gamma_1) \Delta t + 1. \) Since \( P(\theta, 0) = 1, \) there must exist \( \Delta t_3 > 0 \) such that \( P(\theta, \Delta t) > 0 \) when \( \Delta t < \Delta t_3. \) On the other hand, if \( P(\theta, \Delta t) \geq 0 \) is always true, we then define \( \Delta t_3 \) as \( \infty. \) Therefore, let \( \Delta t_1 = -\alpha/2(1 - \theta)^2 (2D + \lambda^2 (\gamma_1 + \gamma_2)), \) \( \Delta t_2 = -(2\alpha_1 + \alpha_2 + \beta_1 + \lambda + 2\lambda \gamma_1) / 2(1 - \theta)^2 (D + \lambda^2 \gamma_1), \) \( \Delta t_3 = \inf \{ \Delta t > 0 : P(\theta, \Delta t) < 0 \} \) and \( \Delta t_0 = \min \{ \Delta t_1, \Delta t_2, \Delta t_3 \}, \) then (4.22) holds when \( \Delta t \in (0, \Delta t_0), \) which completes the proof of Theorem 4.3.

By the proof of Theorem 4.3, we can easily obtain the following result.

**Theorem 4.4.** Suppose that (2.3)–(2.7) hold. If \( \theta = 1, \) then the compensated stochastic \( \theta \)-method is exponentially mean-square stable for every stepsize \( \Delta t = \tau / m. \)

### 5. Numerical Examples

The purpose of this section is to illustrate our theoretical results presented in the previous section by numerical experiments. We first consider the following nonlinear scalar SDDEs with jumps:

\[ dx(t) = \left( -4x(t) - x^3(t) + x(t - 1) \right) dt + \sin(x(t - 1))dW(t) - x(t^-)dN(t), \quad t > 0, \]

\[ x(t) = 1, \quad t \in [-1, 0], \]  

(5.1)

where \( N(t) \) is a scalar Poisson process with parameter \( \lambda = 1. \) In this case, (2.3)–(2.6) are satisfied with \( \alpha_1 = -4, \alpha_2 = 1, \beta_1 = 0, \beta_2 = 1, \gamma_1 = 1, \gamma_2 = 0, \) and \( \tau = 1. \) So we have \( \alpha = -2 \) in (2.7), which guarantees mean-square stability of the zero solution of (5.1) by Theorem 2.2.
The following numerical experiments will show how the parameter $\theta$ and the stepsize $\Delta t$ influence the mean-square stability of the compensated stochastic $\theta$ methods. We simulate the expectation of $|X_n|^2$ by using 1000 trajectories, that is,

$$
\mathbb{E}|X_n|^2 \approx \frac{1}{1000} \sum_{i=1}^{1000} |X_n(\omega_i)|^2.
$$

(5.2)

Theorem 4.1 shows that the compensated stochastic $\theta$ methods are mean-square stable for every stepsize $\Delta t = \tau/m$ when $1/2 \leq \theta \leq 1$. In Figure 1, we use (3.4) to solve (5.1) and choose the parameter $\theta$ with different values 0.5 and 0.8, and we take the stepsize $\Delta t = 1/8, 1/4, 1/2$, and 1, respectively. We can find that the compensated stochastic $\theta$ methods are mean-square stable with these stepizes.

Theorem 4.3 shows that the compensated stochastic $\theta$ methods are exponentially mean-square stable if the stepsize $\Delta t \in (0, \Delta t_0)$ when $0 \leq \theta < 1$. Now, we consider the following nonlinear scalar SDDEs with jumps:

$$
dx(t) = -3x(t)dt + \sin(x(t-1))dW(t) - x(t-1)dN(t), \quad t > 0,
$$

$$
x(t) = 1, \quad t \in [-1, 0],
$$

(5.3)

where $N(t)$ is a scalar Poisson process with parameter $\lambda = 1$. (2.3)–(2.7) and (4.14) are satisfied with $\alpha_1 = -3$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 1$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\tau = 1$, $D = 9$, and $\alpha = -2$. Therefore, the zero solution of (5.3) is exponentially mean-square stable. By Theorem 4.3, we calculate $\Delta t_1 = 1/19(1-\theta)^2$, $\Delta t_2 = 5/18(1-\theta)^2$, and $P(\theta, \Delta t) = 18(1-\theta)^2(\Delta t)^2 - 5(1-\theta)\Delta t + 1$. It is easy to see that $P(\theta, \Delta t) > 0$ for every $\Delta t > 0$, then we get $\Delta t_3 = \infty$. Therefore, we obtain $\Delta t_0 = \min\{\Delta t_1, \Delta t_2, \Delta t_3\} = 1/19(1-\theta)^2$, which implies that the methods applied to (5.3) have less restrictions on the stepsize as the value of $\theta$ increases. Now, we use (3.4) to solve (5.3) and choose the parameter $\theta = 0$ and $\theta = 0.2$, and we take the stepsize $\Delta t = 1/20, 1/8, 1/2$, and
1, respectively. By Theorem 4.3, we compute that $\Delta t_0 = 0.0526$ when $\theta = 0$ and $\Delta t_0 = 0.0822$ when $\theta = 0.2$. Figure 2 indicates that both methods are exponentially mean-square stable if the stepsize $\Delta t = 0.05$, which is well selected in $\Delta t \in (0, \Delta t_0)$. It also shows that the methods maybe stable when the stepsize is bigger than $\Delta t_0$, since both methods are stable when stepsize $\Delta t = 0.125$, but they are not stable for stepsize $\Delta t = 1$. This indicates that the restriction of the stepsize $\Delta t_0$ in Theorem 4.3 is not theoretical optimal. From Figure 2, we also can find that the methods behave better stability when the value of $\theta$ increases and the stepsize $\Delta t$ decreases.

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**References**


