Research Article

Ground-State Solutions for a Class of N-Laplacian Equation with Critical Growth

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We investigate the existence of ground-state solutions for a class of N-Laplacian equation with critical growth in $\mathbb{R}^N$. Our proof is based on a suitable Trudinger-Moser inequality, Pohozaev-Pucci-Serrin identity manifold, and mountain pass lemma.

1. Introduction

Consider the following N-Laplacian equation:

$$-\Delta_N u + |u|^{N-2} u = f(u), \quad \text{in } \mathbb{R}^N,$$

$$u > 0, \quad \text{in } \mathbb{R}^N, \quad u \in W^{1,N}(\mathbb{R}^N),$$

where $N \geq 2$. $\Delta_N u = \text{div}(|\nabla u|^{N-2} \nabla u)$ is the N-Laplacian, the nonlinear term $f(u)$ has critical growth.

The interest in these problems lies in that fact that the order of the Laplacian is the same as the dimension $N$ of the underlying space. The classical case of this problem that $N = 2$, and the the problem (1.1) reduces to

$$-\Delta u + u = f(u), \quad \text{in } \mathbb{R}^2,$$

has been treated by Atkinson and Peletier [1] and Berestycki and Lions [2]. They obtained the existence of ground-state solution which the nonlinear term $f(u)$ is subcritical growth.
Alves et al. [3] extend their results to the critical growth. As $N \neq 2$, do O and Medeiros [4] consider the following $N$-Laplacian equation problem:

$$-\Delta_N u = g(u), \quad \text{in } \mathbb{R}^N,$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ has a subcritical growth and obtain a mountain pass characterization of the ground-state solution for the problem (1.3). In the present paper, we will improve and complement some of the results cited above.

Assume the function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous and satisfies the following conditions:

\begin{enumerate}
  \item[(g_1)] $\lim_{s \to 0} (f(s)/s|s|^{N-2}) = 0$;
  \item[(g_2)] There exist constants $a_0, b_1, b_2 > 0$ such that $|f(s)| \leq b_1|s|^{N-1} + b_2[\exp(a_0|s|^{N/(N-1)}) - S_{N-2}(a_0,s)]$, where $S_{N-2}(a_0,s) = \sum_{k=0}^{N-2}(a_0^k/k!)|s|^{Nk/(N-1)}$;
  \item[(g_3)] There exist $\lambda > 0$ and $q > N$ such that $f(s) \geq \lambda s^{q-1}$, for every $s \geq 0$.
\end{enumerate}

**Remark 1.1.** Condition (g_2) implies that $f$ has a critical growth with critical exponent $a_0$.

Consider the energy functional $I : W^{1,N}(\mathbb{R}^N) \mapsto \mathbb{R}$

$$I(u) = \frac{1}{N} \int_{\mathbb{R}^N} \left( |\nabla u|^N + |u|^N \right) dx - \int_{\mathbb{R}^N} F(u) dx,$$

where $F(s) = \int_0^s f(t) dt$. By a ground-state solution, we mean a solution such $\omega \in W^{1,N}(\mathbb{R}^N)$ such that $I(\omega) \leq I(u)$ for every nontrivial solution $u$ of the problem (1.1). Let $C_q > 0$ denote the best constant of Sobolev embeddings:

$$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N),$$

for $q \in (N, +\infty)$, that is,

$$C_q \int_{\mathbb{R}^N} |u|^q dx \leq \int_{\mathbb{R}^N} \left( |\nabla u|^N + |u|^N \right) dx,$$

for all $u \in W^{1,N}(\mathbb{R}^N)$.

Now we state our main theorem in this paper.

**Theorem 1.2.** If $f$ satisfies $(g_1), (g_2)$, and $(g_3)$, with

$$\lambda > \left( \frac{q - N}{q} \right)^{(q-N)/N} C_q^{q/N},$$

then the problem (1.1) possesses a nontrivial ground-state solution.
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In this paper, we complement some results [4] from subcritical case to the critical case. Furthermore, the ground-state solution to the problem (1.1) is obtained without assuming that the function

$$\frac{f(s)}{|s|^{N-1}}$$

is increasing for $s > 0$ (see [5]), and the so-called Ambrosetti-Rabinowitz condition: there exists $\theta > N$, such that for all $x \in \mathbb{R}^N$,

$$0 < \theta F(x,u) \leq uf(x,u).$$

The paper is organized as follows. Section 2 contains some technical results which allows us to give a variational approach for our results. In Section 3, we prove our main results.

2. The Variational Framework

For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^N)$ denotes the Lebesgue spaces with the norm $\|u\|_{L^p(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |u|^p \, dx)^{1/p}$, $W^{1,p}(\mathbb{R}^N)$ denotes the Sobolev spaces with the norm $\|u\|_{W^{1,p}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) \, dx)^{1/p}$. As $p = N$, we have the following version of Trudinger-Moser inequality.

**Lemma 2.1** (see [6]). If $N \geq 2$, $\alpha > 0$ and $u \in W^{1,N}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \left[ \exp \left( \alpha |u|^{N/(N-1)} \right) - S_{N-2}(\alpha,u) \right] \, dx < \infty. \quad (2.1)$$

Moreover, if $\|\nabla u\|_{L^N(\mathbb{R}^N)}^N \leq 1$, $\|u\|_{L^N(\mathbb{R}^N)} \leq M < \infty$ and $\alpha < \alpha_N$, then there exists a constant $C$, which depends only on $N$, $M$, and $\alpha$, such that

$$\int_{\mathbb{R}^N} \left[ \exp \left( \alpha |u|^{N/(N-1)} \right) - S_{N-2}(\alpha,u) \right] \, dx \leq C(N,M,\alpha), \quad (2.2)$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and $\omega^{1/(N-1)}$ is the measure of the unit sphere in $\mathbb{R}^N$.

In the sequel, since we seek positive solutions, and assume that $f(s) = 0$ for $s \leq 0$. Consider the following minimization problem:

$$\min \left\{ \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 0 \right\}, \quad (2.3)$$

where $g(s) = f(s) - s|s|^{N-2}$, $G(s) = \int_0^s g(t) \, dt = F(s) - (1/N)s^N$. Since the problem (1.1) is an autonomous problem, under the Schwarz symmetric process, we can minimize the problem...
(2.3) on the space $W^{1,N}_{rad}(\mathbb{R}^N)$, the subspace of $W^{1,N}(\mathbb{R}^N)$ formed by radially symmetric functions. Indeed, let $u^*$ be the Schwarz symmetrization of $u$, we have
\[
\int_{\mathbb{R}^N} G(u^*) dx = \int_{\mathbb{R}^N} G(u) dx, \quad \int_{\mathbb{R}^N} |\nabla u^*|^N dx \leq \int_{\mathbb{R}^N} |\nabla u|^N dx. \tag{2.4}
\]
Hence, we can minimize the problem (2.3) on the space $W^{1,N}_{rad}(\mathbb{R}^N)$ (see [7]). Now, we defined the following notations
\[
m = \inf \{ I(u) : u \text{ is nontrivial solution of the problem (1.1)} \}
\]
\[
A = \inf \left\{ \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx : \int_{\mathbb{R}^N} G(u) dx = 0 \right\}
\]
\[
b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),
\]
where $\Gamma = \{ \gamma \in C([0,1],W^{1,N}_{rad}(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0 \}$.

We recall that Pohozaev-Pucci-Serrin identity shows that any solutions $u$ of the problem (1.1) should satisfies the Pohozaev-Pucci-Serrin identity:
\[
(N - p) \int_{\mathbb{R}^N} |\nabla u|^p dx = Np \int_{\mathbb{R}^N} G(u) dx. \tag{2.6}
\]
Then, as $p = N$, we have $\int_{\mathbb{R}^N} G(u) dx = 0$.

Hence, we have the Pohozaev identity manifold:
\[
\mathcal{P} = \left\{ u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : (N - p) \int_{\mathbb{R}^N} |\nabla u|^p dx = Np \int_{\mathbb{R}^N} G(u) dx \right\}
\]
\[
= \left\{ u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} G(u) dx = 0 \right\}. \tag{2.7}
\]
So, we have
\[
A = \inf_{u \in \mathcal{P}} \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx, \quad m = \inf_{u \in \tau} I(u), \tag{2.8}
\]
where $\tau = \{ u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : I(u) = 0 \}$.

In what follows, we will show that $A$ is attained, and afterwards we prove that
\[
m = A = b, \tag{2.9}
\]
thereby proving that the problem (1.1) has a ground-state solution.
3. The Proof of Theorem 1.2

In this section, we prove that $A$ is attained, the equality (2.9) is satisfied. Hence the proof of Theorem 1.2 is obtained.

In the following, we consider the following minimax value:

$$ c = \inf_{0 \neq v \in W^{1,N}(\mathbb{R}^N)} \max_{t \geq 0} I(tv). \quad (3.1) $$

Now, we show a sufficient condition, on a sequence \( \{v_n\} \) to get a convergence like $F(v_n) \rightarrow F(v)$ in $L^1(\mathbb{R}^N)$.

**Lemma 3.1.** Assume that $f$ satisfies (g$_1$) and (g$_2$), and let $\{v_n\}$ be a sequence in $W^{1,N}_{rad}(\mathbb{R}^N)$ such that $\|\nabla v_n\|_{L^N(\mathbb{R}^N)} \leq 1$, $\|v_n\|_{L^N(\mathbb{R}^N)} \leq M < \infty$, then we have

$$ \int_{\mathbb{R}^N} F(v_n) dx \rightarrow \int_{\mathbb{R}^N} F(v) dx, \quad (3.2) $$

where $v_n \rightharpoonup v$ in $W^{1,N}(\mathbb{R}^N)$.

**Proof.** Without loss of generality, we assume that there exist $v \in W^{1,N}_{rad}(\mathbb{R}^N)$ such that

$$ v_n \rightharpoonup v, \quad \text{in} \; W^{1,N}_{rad}(\mathbb{R}^N), $$

$$ v_n \rightarrow v, \quad \text{a.e. in} \; \mathbb{R}^N. \quad (3.3) $$

Let $v^*$ is the Schwarz symmetrization of $v$, then we have

$$ \int_{\mathbb{R}^N} v \left[ \exp\left(\alpha_0 |v|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, v) \right] dx = \int_{\mathbb{R}^N} v^* \left[ \exp\left(\alpha_0 |v^*|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, v^*) \right] dx, $$

$$ \int_{\mathbb{R}^N} |v|^N dx = \int_{\mathbb{R}^N} |v^*|^N dx. \quad (3.4) $$

From (g$_1$), we obtain that for $\epsilon > 0$, there exists $\delta > 0$, such that

$$ f(s) \leq \epsilon |s|^{N-1}, \quad \text{for} \; |s| < \delta, \quad (3.5) $$

so, we have

$$ F(s) \leq \frac{\epsilon}{N} |s|^N, \quad \text{for} \; |s| < \delta. \quad (3.6) $$

From (g$_2$), we obtain

$$ F(s) \leq C_1 |s|^N + C_2 |s| \left[ \exp\left(\alpha_0 |s|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, u) \right], \quad \text{for} \; |s| \geq \delta. \quad (3.7) $$
There two estimates yield

$$F(s) \leq C_3|s|^N + C_2|s|\left[\exp\left(\alpha_0|s|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, u)\right], \quad \text{for } s > 0. \quad (3.8)$$

On one hand, from Lemma 2.1, we obtain that there exists a constant $C$, which depends only on $N$, $M$, and $\alpha$ such that

$$\int_{\mathbb{R}^N} \exp\left(\alpha|v_n|^N/(N-1)\right) \leq C. \quad (3.9)$$

When $\|\nabla v_n\|_{L^N(\mathbb{R}^N)}^N \leq 1$, $\|v_n\|_{L^N(\mathbb{R}^N)}^N \leq M < \infty$ and $\alpha < a_N$. Hence, we have

$$\int_{|x| \leq r} F(v_n) \leq C_3 \int_{|x| \leq r} |v_n|^N \, dx + C_2 \int_{|x| \leq r} |v_n^*| \left[\exp\left(\alpha_0|v_n|^N/(N-1)\right) - S_{N-2}(\alpha_0, v_n^*)\right] \, dx$$

$$\leq C_3 M^N + C_2 \int_{|x| \leq r} |v_n^*| \exp\left(\alpha_0|v_n|^N/(N-1)\right) \, dx$$

$$\leq C_3 M^N + C_3 \left(\int_{|x| \leq r} |v_{n}^*|^\mu\right)^{1/\mu} \left(\int_{|x| \leq r} \exp\left(\beta_0|v_n|^N/(N-1)\right) \, dx\right)^{1/\beta} \quad (3.10)$$

$$\leq C_3 M^N + C_4 M \left(\int_{|x| \leq r} \exp\left(\beta_0|v_n|^N/(N-1)\right) \, dx\right)^{1/\beta}$$

$$\leq C_5,$$

where $C_i$ ($i = 2, 3, 4, 5$) are positive constants, the continuous imbedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^\mu(\mathbb{R}^N)$, $1/\mu + 1/\beta = 1$ and $\beta_0 < a_N$.

Then, by Dominated convergence theorem, we obtain

$$\int_{|x| \leq r} F(v_n) \, dx \rightarrow \int_{|x| \leq r} F(v) \, dx. \quad (3.11)$$

On the other hand,

$$\int_{|x| > r} F(v_n) \, dx \leq C_3 \int_{|x| > r} |v_n|^N \, dx + C_2 \int_{|x| > r} |v_n| \left[\exp\left(\alpha_0|v_n|^N/(N-1)\right) - S_{N-2}(\alpha_0, v_n)\right] \, dx$$

$$= \sum_{j=N-1}^{\infty} \frac{a_j^i}{j!} \int_{|x| > r} |v_n| \cdot |v_n|^{Nj/(N-1)} \, dx$$

$$= \sum_{j=N-1}^{\infty} \frac{a_j^i}{j!} \int_{|x| > r} |v_n^*| |v_n^*|^{Nj/(N-1)} \, dx,$$
where \( v_n^* \) is the Schwarz symmetrization of \( v_n \). Notice that the estimate

\[
\int_{|x|>r} \frac{1}{|x|^{1+Nj/(N-1)}} dx = \omega_{N-1} \int_r^\infty \frac{t^{N-1}}{t^{1+Nj/(N-1)}} dt
\]

\[
= \left( \frac{\omega_{N-1}}{N/(N-1) - N+1} \right) r^{N-1-Nj/(N-1)} \leq \frac{\omega_{N-1}}{r},
\]

(3.13)

for all \( j \geq N - 1 \), together with the Radial Lemma [4] leads to

\[
\sum_{j=N-1}^\infty \frac{a_0^j}{j!} \int_{|x|>r} |v_n^*|^j |v_n^*|^{Nj/(N-1)} dx
\]

\[
\leq M \left( \frac{N}{\omega_{N-1}} \right)^{1/N} \sum_{j=N-1}^\infty \frac{a_0^j}{j!} \left( \frac{N}{\omega_{N-1}} \right)^{j/(N-1)} M^{Nj/(N-1)} \int_{|x|>r} |x|^{-1-Nj/(N-1)} dx
\]

\[
\leq \frac{C(N)}{r}.
\]

Thus, given \( \delta > 0 \), there exists \( r > 0 \) such that

\[
\int_{|x|>r} |v_n|^N dx < \delta, \quad \int_{|x|>r} \left[ \exp \left( a_0 |v_n|^{N/(N-1)} \right) - S_{N-2}(a_0, v_n) \right] dx < \delta.
\]

(3.15)

Which implies that

\[
\int_{|x|>r} F(v_n) dx \leq C \delta, \quad \int_{|x|>r} F(v) dx \leq C \delta.
\]

(3.16)

Using the estimate

\[
\left| \int_{\mathbb{R}^N} (F(v_n) - F(v)) dx \right| \leq \left| \int_{|x| \leq r} (F(v_n) - F(v)) dx \right| + \left| \int_{|x| > r} (F(v_n) - F(v)) dx \right|,
\]

(3.17)

we get

\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^N} F(v_n) dx - \int_{\mathbb{R}^N} F(v) dx \right| \leq C \delta,
\]

(3.18)

Hence, we obtain that

\[
\int_{\mathbb{R}^N} F(v_n) dx \to \int_{\mathbb{R}^N} F(v) dx.
\]

(3.19)
**Lemma 3.2.** The numbers $A$ and $c$ satisfy the inequality $A \leq c$.

*Proof.* For each $v \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$, since we only consider the nontrivial solutions of the problem (1.1), we divide them into two cases to consider.

**Case 1.** Let $v^+ = \max\{v, 0\} \neq 0$, we define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \int_{\mathbb{R}^N} G(tv)dx = \int_{\mathbb{R}^N} \left[ F(tv) - \frac{t^N v^N}{N} \right]dx. \quad (3.20)$$

By (g1), we obtain that there exists $\delta > 0$, $0 < c_0 < 1$ such that $|s| < \delta$, and

$$|f(s)| < c_0 |s|^{N-1}. \quad (3.21)$$

Hence

$$h(t) \leq \int_{\mathbb{R}^N} \int_0^{t^v} c_0 |s|^{N-1} dx - \frac{1}{N} \int_{\mathbb{R}^N} t^N v^N dx$$

$$= \frac{c_0}{N} \int_{\mathbb{R}^N} |tv|^N dx - \frac{1}{N} \int_{\mathbb{R}^N} |tv|^N dx, \quad (3.22)$$

we obtain that $h(t) < 0$ for $t$ small enough. On the other hand, by (g2), we obtain that $h(t) > 0$ for $t$ large enough. In this way, there exists $t_0 > 0$ such that $h(t_0) = 0$, That is, $t_0 v \in \mathcal{P}$. Hence

$$A \leq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla (t_0 v)|^N dx = I(t_0 v) \leq \max_{t \geq 0} I(tv). \quad (3.23)$$

**Case 2.** Let $v^+ = \max\{v, 0\} = 0$, since $f(s) = 0$ for all $s < 0$, we obtain

$$\max_{t \geq 0} I(tv) = +\infty. \quad (3.24)$$

As a consequence,

$$A \leq c. \quad (3.25)$$

Combining Cases 1 and 2, we obtain that $A \leq c$. \hfill \square

**Lemma 3.3.** The number $A$ defined by (2.8) is positive, that is, $A > 0$.

*Proof.* Clearly, $A \geq 0$. Assume by contradiction that $A = 0$ and let $\{u_n\}$ be a minimizing sequence in $W^{1,N}(\mathbb{R}^N)$ to $A$, that is,

$$\frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_n|^N dx \to A = 0 \quad \text{with} \quad \int_{\mathbb{R}^N} G(u_n)dx = 0. \quad (3.26)$$
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For each \( \lambda_n > 0 \), set \( v_n(x) = u_n(x/\lambda_n) \) satisfying

\[
\frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_n|^N dx = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_n|^N dx.
\]  

(3.27)

Similarly, we have

\[
\int_{\mathbb{R}^N} G(v_n)dx = \lambda_n^N \int_{\mathbb{R}^N} G(u_n)dx = 0,
\]

\[
\int_{\mathbb{R}^N} |v_n|^N dx = \lambda_n^N \int_{\mathbb{R}^N} |u_n|^N dx.
\]

(3.28)

We choose \( \lambda_n^N = 1/ \int_{\mathbb{R}^N} |u_n|^N dx \), so \( \int_{\mathbb{R}^N} |v_n|^N dx = 1 \). Then we get

\[
\frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_n|^N dx \rightarrow A = 0,
\]

\[
\int_{\mathbb{R}^N} |v_n|^N dx = 1, \quad \int_{\mathbb{R}^N} G(v_n)dx = 0.
\]

(3.29)

In what follows, we study in the space \( W^{1,N}_{\text{rad}}(\mathbb{R}^N) \). Firstly, we assume that there exists \( v \in W^{1,N}_{\text{rad}}(\mathbb{R}^N) \) such that \( v_n \rightharpoonup v \) in \( W^{1,N}_{\text{rad}}(\mathbb{R}^N) \).

On one hand, since \( (1/N) \int_{\mathbb{R}^N} |\nabla v_n|^N \rightarrow A = 0 \), then \( \exists N_0 > 0 \), for all \( 0 < \epsilon < 1 \), when \( n > N_0 \), we have \( \int_{\mathbb{R}^N} |\nabla v_n|^N dx < \epsilon < 1 \) and we also know that \( \int_{\mathbb{R}^N} |v_n|^N dx = 1 \), so \( \|v_n\|_{L^N(\mathbb{R}^N)} \leq M < \infty \). From Lemma 3.1, we have

\[
\int_{\mathbb{R}^N} F(v_n)dx \rightarrow \int_{\mathbb{R}^N} F(v)dx.
\]  

(3.30)

Note that

\[
\int_{\mathbb{R}^N} G(v_n)dx = \int_{\mathbb{R}^N} \left( F(v_n) - \frac{1}{N} |v_n|^N \right) dx = 0,
\]  

(3.31)

so we have

\[
\int_{\mathbb{R}^N} F(v_n)dx = \frac{1}{N} \int_{\mathbb{R}^N} |v_n|^N dx = \frac{1}{N}.
\]

(3.32)

Hence, we have

\[
\int_{\mathbb{R}^N} F(v)dx = \frac{1}{N}.
\]  

(3.33)
It implies that \( v \neq 0 \). On the other hand, since \( v_n \rightharpoonup v \) in \( W^{1,N}_{\text{rad}}(\mathbb{R}^N) \), and the space \( W^{1,N}_{\text{rad}}(\mathbb{R}^N) \) is a reflexible Banach space, we have

\[
\lim_{n \to \infty} \inf \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_n|^N dx \geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^N dx \geq 0. \tag{3.34}
\]

Since

\[
\lim_{n \to \infty} \inf \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_n|^N dx = A = 0, \tag{3.35}
\]

we get

\[
\frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^N dx = 0. \tag{3.36}
\]

From which it follows that \( v = 0 \), we have an absurd. Hence, we have

\[
A > 0. \tag{3.37}
\]

\[\Box\]

**Lemma 3.4.** If \( \lambda > (q - N/q)/(q-N/C_q) \), then \( c < 1/N \).

**Proof.** From (g3), we have \( f(s) > \lambda s^{q-1} \), for all \( s \geq 0 \). Now we choose \( q \in W^{1,N}_{\text{rad}}(\mathbb{R}^N) \) such that

\[
q \geq 0, \quad \|q\|^N_q = C_q^{-1}, \quad \|q\|_{W^{1,N}(\mathbb{R}^N)} = 1. \tag{3.38}
\]

Hence, we have

\[
c \leq \max_{t \geq 0} I(tq) = \max_{t \geq 0} \left\{ \frac{1}{N} \int_{\mathbb{R}^N} \left( |\nabla (tq)|^N + |tq|^N \right) dx - \int_{\mathbb{R}^N} F(tq) dx \right\}
= \max_{t \geq 0} \left\{ \frac{t^N}{N} - \int_{\mathbb{R}^N} \int_{0}^{tq} f(s) ds \, dx \right\}
\leq \max_{t \geq 0} \left\{ \frac{t^N}{N} - \lambda \int_{\mathbb{R}^N} \int_{0}^{tq} s^{q-1} ds \, dx \right\}
= \max_{t \geq 0} \left\{ \frac{t^N}{N} - \frac{t^q}{q} \int_{\mathbb{R}^N} q^{q} \, dx \right\}. \tag{3.39}
\]

Let \( K(t) = t^N/N - (\lambda t^q/q) \int_{\mathbb{R}^N} q^q \, dx \), then \( K(t) \) is continuous function, we have

\[
K'(t) = t^{N-1} - \lambda t^{q-1} \int_{\mathbb{R}^N} q^q \, dx = 0. \tag{3.40}
\]
By a simple calculation, when \( t_0 = (1/\lambda \int_{\mathbb{R}^N} q^q dx )^{1/(q-N)} \) > 0, we have

\[
\max_{t>0} K(t) = K(t_0) = \frac{1}{N} \left( \frac{1}{\lambda \int_{\mathbb{R}^N} q^q dx} \right)^{N/(q-N)} - \left( \frac{1}{q} \int_{\mathbb{R}^N} q^q dx \right) \left( \frac{1}{\lambda \int_{\mathbb{R}^N} q^q dx} \right)^{q/(q-N)}
\]

\[
= \frac{q-N}{Nq} \lambda^{-N/(q-N)} C_q^{(q/N)\cdot(-N/(q-N))} (q-N)^{-N/q} C_q^{C_q/q}\cdot(-N/(q-N))\cdot C_q^{q/(q-N)}
\]

\[
< \frac{q-N}{Nq} \left( \frac{q-N}{q} \right)^{(q-N)/N\cdot(-N/(q-N))} C_q^{(q/N)\cdot(-N/(q-N))} c^{q/(q-N)}
\]

\[
= \frac{1}{N}.
\]

Hence, we have

\[
c < \frac{1}{N}.
\]  

**Lemma 3.5.** The number \( A \) is attained, that is, there exists \( u \in W^{1,N}_{rad}(\mathbb{R}^N) \) such that \( A = \int_{\mathbb{R}^N} |\nabla u|^N dx \) and \( \int_{\mathbb{R}^N} G(u) dx = 0 \).

**Proof.** Let \( \{u_n\} \) be a minimizing sequence in \( W^{1,N}_{rad}(\mathbb{R}^N) \) for \( A \), that is,

\[
\frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_n|^N dx \rightarrow A \quad (n \rightarrow \infty), \quad \int_{\mathbb{R}^N} G(u_n) dx = 0.
\]  

(3.43)

Arguing as in Lemma 3.3, we assume that \( \int_{\mathbb{R}^N} |u_n|^N dx = 1 \). From (3.43), Lemmas 3.3 and 3.4, we obtain

\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^N dx = NA \leq Nc < 1.
\]  

(3.44)

From Lemma 3.1,

\[
\int_{\mathbb{R}^N} F(u_n) dx \rightarrow \int_{\mathbb{R}^N} F(u) dx,
\]

(3.45)

where \( u_n \rightharpoonup u \) in \( W^{1,N}(\mathbb{R}^N) \), as \( n \rightarrow \infty \).

By (3.43) and (3.45), we have

\[
\int_{\mathbb{R}^N} F(u_n) dx = \frac{1}{N} \int_{\mathbb{R}^N} |u_n|^N dx = \frac{1}{N},
\]

\[
\int_{\mathbb{R}^N} F(u) dx = \frac{1}{N}.
\]  

(3.46)
It implies that
\[ u \neq 0, \quad (3.47) \]
\[ \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx \leq \lim_{n \to \infty} \inf \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_n|^N dx = A, \quad (3.48) \]
\[ \int_{\mathbb{R}^N} |u|^N dx \leq \lim_{n \to \infty} \inf \int_{\mathbb{R}^N} |u_n|^N dx = 1. \quad (3.49) \]

From (3.48) and (3.49), we have
\[ \int_{\mathbb{R}^N} G(u)dx = \int_{\mathbb{R}^N} F(u)dx - \frac{1}{N} \int_{\mathbb{R}^N} |u|^N dx = \frac{1}{N} - \frac{1}{N} \int_{\mathbb{R}^N} |u|^N dx \geq 0. \quad (3.50) \]

If \( \int_{\mathbb{R}^N} G(u)dx \neq 0 \), from (3.50), we have \( \int_{\mathbb{R}^N} G(u)dx > 0 \). Consider the function \( h \) defined in Lemma 3.2 relative to the function:
\[ h(t) = \int_{\mathbb{R}^N} G(tu)dx. \quad (3.51) \]

We concludes that \( h(t) < 0 \) for \( t \) small enough. On the other hand, \( h(1) = \int_{\mathbb{R}^N} G(u)dx > 0 \). In this way, we obtain that there is \( t_0 \in (0, 1) \) such that \( h(t_0) = 0 \), that is,
\[ \int_{\mathbb{R}^N} G(t_0u)dx = 0. \quad (3.52) \]

Hence, from (3.48),
\[ 0 < \frac{1}{N} \int_{\mathbb{R}^N} |\nabla (t_0u)|^N dx = \frac{1}{N} t_0^N \int_{\mathbb{R}^N} |\nabla u|^N dx \leq t_0^N A < A. \quad (3.53) \]

However, from (3.52), we have \( t_0u \in \mathcal{P} \). Hence, we obtain
\[ \frac{1}{N} \int_{\mathbb{R}^N} |\nabla (t_0u)|^N dx \geq A. \quad (3.54) \]

Which is contradictory with (3.53).

Thus, we obtain
\[ \int_{\mathbb{R}^N} G(u)dx = 0. \quad (3.55) \]

It implies \( u \in \mathcal{P} \) and
\[ \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx \geq A. \quad (3.56) \]
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From (3.48) and (3.56), we obtain that

$$\frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N \, dx = A, \quad (3.57)$$

with $\int_{\mathbb{R}^N} G(u) \, dx = 0$, $u \neq 0$.

We obtain that $A$ is attained. \hfill \Box

Proof of Theorem 1.2. From Lemma 3.5, there is $u \in W^{1,N}_{\text{rad}}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N \, dx = A, \quad \int_{\mathbb{R}^N} G(u) \, dx = 0. \quad (3.58)$$

we will prove that $m = b = A$.

By Lagrange multipliers, there exists $\rho \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^N} |\nabla u|^{N-2} \nabla u \nabla v \, dx = \rho \int_{\mathbb{R}^N} g(u) v \, dx, \quad (3.59)$$

for every $v \in W^{1,N}(\mathbb{R}^N)$.

Define the rescaled function $u_{\rho^{1/N}} = u(\rho^{-1/N} x)$, which is a nontrivial solution of (1.1) with

$$\int_{\mathbb{R}^N} |\nabla u_{\rho^{1/N}}|^N \, dx = \int_{\mathbb{R}^N} |\nabla u|^N \, dx, \quad (3.60)$$

$$\int_{\mathbb{R}^N} G(u_{\rho^{1/N}}) \, dx = \rho \int_{\mathbb{R}^N} G(u) \, dx = 0.$$

Thus, we have

$$m \leq I(u_{\rho^{1/N}}) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_{\rho^{1/N}}|^N \, dx - \int_{\mathbb{R}^N} G(u_{\rho^{1/N}}) \, dx = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N \, dx = A. \quad (3.61)$$

So, we have

$$m \leq A. \quad (3.62)$$

For each $\gamma \in \Gamma$, one has $\gamma([0,1]) \cap P \neq \emptyset$ from [4]. We obtain that there exists $t_0 \in [0,1]$ such that $\gamma(t_0) \in P$, that is, $\gamma(t_0)$ satisfies that $\int_{\mathbb{R}^N} G(\gamma(t_0)) \, dx = 0$ and then

$$A \leq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \gamma(t_0)|^N \, dx - \frac{1}{N} \int_{\mathbb{R}^N} G(\gamma(t_0)) \, dx = I(\gamma(t_0)). \quad (3.63)$$

Hence $A \leq I(\gamma(t_0)) \leq \max_{t \in [0,1]} I(\gamma(t))$ for every $\gamma \in \Gamma$, we obtain that

$$A \leq b. \quad (3.64)$$

From (3.62) and (3.64), we obtain that $m \leq A \leq b$. 
On the other hand, for every nontrivial solution $\omega \in W^{1,N}(\mathbb{R}^N)$ of the problem (1.1), there exists a path $\gamma_\omega \in \Gamma$ such that $\omega \in \gamma_\omega([0,1])$ and $\max_{t \in [0,1]} I(\gamma_\omega(t)) = I(\omega)$. Consequently, $b \leq I(\omega)$, $b \leq m$.

In conclusion, we obtain

$$m = A = b.$$  \hspace{1cm} (3.65)

Hence, the function $u_{\mu^{1/N}}$ is a ground-state solution of the problem (1.1).

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\section*{References}


