Review Article

Some Properties and Identities of Bernoulli and Euler Polynomials Associated with p -adic Integral on $\mathbb{Z}_p$

D. S. Kim,¹ T. Kim,² D. V. Dolgy,³ S. H. Lee,⁴ and S.-H. Rim⁵

¹ Department of Mathematics, Sogang University, Seoul, Republic of Korea

² Department of Mathematics, Kwangwoon University, Seoul, Republic of Korea

³ Hanrimwon, Kwangwoon University, Seoul, Republic of Korea

⁴ Division of General Education, Kwangwoon University, Seoul, Republic of Korea

⁵ Department of Mathematics Education, Kyungpook National University, Taegu, Republic of Korea

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

Received 15 December 2011; Accepted 8 February 2012

Academic Editor: Ibrahim Sadek

Copyright © 2012 D. S. Kim et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate some properties and identities of Bernoulli and Euler polynomials. Further, we give some formulae on Bernoulli and Euler polynomials by using p -adic integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p -adic rational integers, the field of p -adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

For $f \in UD(\mathbb{Z}_p)$, the p -adic invariant integral on $\mathbb{Z}_p$ in the bosonic sense is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) d\mu(x) \quad (1.1)$$

(see [1, 2]). The fermionic p -adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x \quad (1.2)$$

(see [3]). As is well known, Bernoulli polynomials are defined by

$$\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.3)$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$, symbolically (see [1–19]). In the special case $x = 0$, $B_n(0) = B_n$ is called the n th Bernoulli number.

The Euler polynomials are also defined by the generating function as follows:

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.4)$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$, symbolically (see [1–19]). In the special case $x = 0$, $E_n(0) = E_n$ is called the n -th Euler number.

By (1.3) and (1.4), we easily see that

$$\begin{aligned} B_n(x) &= (B + x)^n = \sum_{l=0}^n \binom{n}{l} x^{n-l} B_l, \\ E_n(x) &= (E + x)^n = \sum_{l=0}^n \binom{n}{l} x^{n-l} E_l, \end{aligned} \quad (1.5)$$

where $\binom{n}{l} = n! / (n-l)! l! = n(n-1)(n-2) \cdots (n-l+1) / l!$ (see [14, 16, 19]).

The following properties of Bernoulli numbers and polynomials are well known (see [10, 11]).

For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$,

$$\sum_{j=0}^n \binom{n}{j} y^{n-j} \frac{B_{j+1}(x)}{j+1} = \frac{B_{n+1}(x+y) - y^{n+1}}{n+1}, \quad (1.6)$$

$$\sum_{j=0}^{[n/2]} \binom{n}{2j} y^{n-2j} \frac{B_{2j+1}(x)}{2j+1} = \frac{B_{n+1}(x+y) + (-1)^n B_{n+1}(x-y)}{2n+2}, \quad (1.7)$$

$$\sum_{j=1}^{[(n+1)/2]} \binom{n}{2j-1} y^{n+1-2j} \frac{B_{2j}(x)}{2j} = \frac{B_{n+1}(x+y) + (-1)^{n-1} B_{n+1}(x-y) - 2y^{n+1}}{2n+2}, \quad (1.8)$$

where $[\cdot]$ is Gauss' symbol.

First, we investigate some identities of Euler polynomials corresponding to (1.6), (1.7) and (1.8). From those identities, we derive some interesting identities and properties by using p -adic integral on $\mathbb{Z}_p$.

2. Some Identities of Bernoulli and Euler Polynomials

By (1.4), we get

$$E_k(x+y) = \sum_{j=0}^k \binom{k}{j} y^{k-j} E_j(x), \quad \text{for } \mathbb{Z}_+. \quad (2.1)$$

From (2.1), we note that

$$\begin{aligned} E_k(x+y) &= \sum_{j=0}^k \binom{k}{j} y^{k-j} E_j(x) \\ &= y^k + \sum_{j=1}^k \frac{k}{j} \binom{k-1}{j-1} y^{k-j} E_j(x). \end{aligned} \quad (2.2)$$

Thus, we have

$$\sum_{j=0}^{k-1} \binom{k-1}{j} y^{k-1-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_k(x+y) - y^k}{k}. \quad (2.3)$$

Replacing k by $k+1$ in (2.3), we obtain the following proposition.

Proposition 2.1. *For $k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^k \binom{k}{j} y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x+y) - y^{k+1}}{k+1}. \quad (2.4)$$

Let us replace y by $-y$ in Proposition 2.1. Then we have

$$\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x-y) - (-1)^{k+1} y^{k+1}}{k+1}. \quad (2.5)$$

Thus, we see that

$$\sum_{j=0}^k \binom{k}{j} (-1)^j y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{(-1)^k E_{k+1}(x-y) + y^{k+1}}{k+1}. \quad (2.6)$$

Therefore, adding (2.4) and (2.6), we obtain the following proposition.

Proposition 2.2. *For $k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{[k/2]} \binom{k}{2j} y^{k-2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{k+1}(x+y) + (-1)^k E_{k+1}(x-y)}{2k+2}. \quad (2.7)$$

From (2.2), we note that

$$\sum_{j=1}^k \frac{1}{j} \binom{k-1}{j-1} y^{k-j} (-1)^j E_j(x) = \frac{(-1)^k E_k(x-y) - y^k}{k}. \quad (2.8)$$

By (2.3) and (2.8), we get

$$\sum_{j=1}^{[k/2]} \frac{1}{2j} \binom{k-1}{2j-1} y^{k-2j} E_{2j}(x) = \frac{E_k(x+y) + (-1)^k E_k(x-y) - 2y^k}{2k}. \quad (2.9)$$

Therefore, replacing k by $k+1$, we obtain the following proposition.

Proposition 2.3. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=1}^{[(k+1)/2]} \binom{k}{2j-1} y^{k+1-2j} \frac{E_{2j}(x)}{2j} = \frac{E_{k+1}(x+y) + (-1)^{k+1} E_{k+1}(x-y) - 2y^{k+1}}{2k+2}. \quad (2.10)$$

Letting $y = 1$ in Proposition 2.1, we have

$$\sum_{j=0}^k \binom{k}{j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x+1) - 1}{k+1}, \quad (2.11)$$

$$\begin{aligned} E_{k+1}(x+1) &= \sum_{l=0}^{k+1} \binom{k+1}{l} (E+1)^l x^{k+1-l} \\ &= (2 - E_0) x^{k+1} - \sum_{l=1}^{k+1} \binom{k+1}{l} E_l x^{k+1-l} \\ &= 2x^{k+1} - \sum_{l=0}^{k+1} \binom{k+1}{l} E_l x^{k+1-l} = 2x^{k+1} - E_{k+1}(x). \end{aligned} \quad (2.12)$$

Therefore, by (2.11) and (2.12), we obtain the following corollary.

Corollary 2.4. *For $k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^k \binom{k}{j} \frac{E_{j+1}(x)}{j+1} = -\frac{E_{k+1}(x)}{k+1} + \frac{2x^{k+1} - 1}{k+1}. \quad (2.13)$$

Replacing y by 1 and k by $2k$ in Proposition 2.2, we have

$$\begin{aligned}
 \sum_{j=0}^k \binom{2k}{2j} \frac{E_{2j+1}(x)}{2j+1} &= \frac{E_{2k+1}(x+1) + E_{2k+1}(x-1)}{4k+2} \\
 &= \frac{E_{2k+1}(x+1) + E_{2k+1}(x) + E_{2k+1}(x) + E_{2k+1}(x-1)}{4k+2} - \frac{2E_{2k+1}(x)}{4k+2} \\
 &= \frac{2x^{2k+1} + 2(x-1)^{2k+1}}{4k+2} - \frac{E_{2k+1}(x)}{2k+1}.
 \end{aligned} \tag{2.14}$$

Therefore, by (2.14), we obtain the following corollary.

Corollary 2.5. *For $k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^k \binom{2k}{2j} \frac{E_{2j+1}(x)}{2j+1} = -\frac{E_{2k+1}(x)}{2k+1} + \frac{x^{2k+1} + (x-1)^{2k+1}}{2k+1}. \tag{2.15}$$

Replacing y by 1 and k by $2k$ in Proposition 2.3, we have

$$\begin{aligned}
 \sum_{j=1}^k \binom{2k}{2j-1} \frac{E_{2j}(x)}{2j} &= \frac{E_{2k+1}(x+1) - E_{2k+1}(x-1) - 2}{4k+2} \\
 &= \frac{(E_{2k+1}(x+1) + E_{2k+1}(x)) - (E_{2k+1}(x) + E_{2k+1}(x-1))}{4k+2} - \frac{1}{2k+1} \\
 &= \frac{2x^{2k+1} - 2(x-1)^{2k+1}}{4k+2} - \frac{1}{2k+1} \\
 &= \frac{x^{2k+1} - (x-1)^{2k+1}}{2k+1} - \frac{1}{2k+1}.
 \end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following corollary.

Corollary 2.6. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=1}^k \binom{2k}{2j-1} \frac{E_{2j}(x)}{2j} = \frac{x^{2k+1} - (x-1)^{2k+1}}{2k+1} - \frac{1}{2k+1}. \tag{2.17}$$

Replacing y by $1/2$ and k by $2k$ in Proposition 2.3, we get

$$\sum_{j=1}^k \binom{2k}{2j-1} \left(\frac{1}{2}\right)^{2k+1-2j} \frac{E_{2j}(x)}{2j} = \frac{E_{2k+1}(x+1/2) - E_{2k+1}(x-1/2) - 2^{-2k}}{4k+2}. \tag{2.18}$$

Thus, we have

$$\sum_{j=1}^k \binom{2k}{2j-1} 2^{2j} \frac{E_{2j}(x)}{2j} = \frac{2^{2k}(E_{2k+1}(x+1/2) - E_{2k+1}(x-1/2)) - 1}{2k+1}, \quad (2.19)$$

$$\begin{aligned} E_{2k+1}\left(x + \frac{1}{2}\right) &= E_{2k+1}\left(x - \frac{1}{2} + 1\right) = \sum_{l=0}^{2k+1} \binom{2k+1}{l} \left(x - \frac{1}{2}\right)^{2k+1-l} (E+1)^l \\ &= 2\left(x - \frac{1}{2}\right)^{2k+1} - \sum_{l=0}^{2k+1} \binom{2k+1}{l} \left(x - \frac{1}{2}\right)^{2k+1-l} E_l \\ &= 2\left(x - \frac{1}{2}\right)^{2k+1} - E_{2k+1}\left(x - \frac{1}{2}\right). \end{aligned} \quad (2.20)$$

Therefore, by (2.19) and (2.20), we obtain the following corollary.

Corollary 2.7. *For $k \in \mathbb{N}$, we have*

$$\sum_{j=1}^k \binom{2k}{2j-1} 2^{2j} \frac{E_{2j}(x)}{2j} = -\frac{2^{2k+1}E_{2k+1}(x-1/2)}{2k+1} + \frac{2^{2k+1}(x-1/2)^{2k+1}}{2k+1} - \frac{1}{2k+1}. \quad (2.21)$$

Replacing y by 1 and k by $2k+1$ in Proposition 2.2, we get

$$\begin{aligned} \sum_{j=0}^k \binom{2k+1}{2j} \frac{E_{2j+1}(x)}{2j+1} &= \frac{E_{2k+2}(x+1) - E_{2k+2}(x-1)}{4k+4} \\ &= \frac{(E_{2k+2}(x+1) + E_{2k+2}(x)) - (E_{2k+2}(x) + E_{2k+2}(x-1))}{4k+4} \\ &= \frac{2x^{2k+2} - 2(x-1)^{2k+2}}{4k+4} = \frac{x^{2k+2} - (x-1)^{2k+2}}{2k+2}. \end{aligned} \quad (2.22)$$

Therefore, by (2.22), we obtain the following corollary.

Corollary 2.8. *For $k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^k \binom{2k+1}{2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{x^{2k+2} - (x-1)^{2k+2}}{2k+2}. \quad (2.23)$$

Replacing k by $2k + 1$ and y by 1 in Proposition 2.3, we get

$$\begin{aligned}
 \sum_{j=1}^{k+1} \binom{2k+1}{2j-1} \frac{E_{2j}(x)}{2j} &= \frac{E_{2k+2}(x+1) + E_{2k+2}(x-1) - 2}{4k+4} \\
 &= \frac{(E_{2k+2}(x+1) + E_{2k+2}(x)) + (E_{2k+2}(x) + E_{2k+2}(x-1))}{4k+4} - \frac{E_{2k+2}(x) + 1}{2k+2} \\
 &= \frac{x^{2k+2} + (x-1)^{2k+2}}{2k+2} - \frac{E_{2k+2}(x) + 1}{2k+2}.
 \end{aligned} \tag{2.24}$$

Therefore, by (2.24), we obtain the following corollary.

Corollary 2.9. For $k \in \mathbb{Z}_+$, we have

$$\sum_{j=1}^{k+1} \binom{2k+1}{2j-1} \frac{E_{2j}(x)}{2j} = \frac{x^{2k+2} + (x-1)^{2k+2}}{2k+2} - \frac{E_{2k+2}(x) + 1}{2k+2}. \tag{2.25}$$

Replacing k by $2k + 1$ and y by $1/2$ in Proposition 2.2, we have

$$\sum_{j=0}^k \binom{2k+1}{2j} \left(\frac{1}{2}\right)^{2k+1-2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{2k+2}(x+1/2) - E_{2k+2}(x-1/2)}{4k+4}. \tag{2.26}$$

Thus, by multiplying 2^{2k+1} on both sides, we get

$$\sum_{j=0}^k \binom{2k+1}{2j} 2^{2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{2^{2k+1} \{E_{2k+2}(x+1/2) - E_{2k+2}(x-1/2)\}}{4k+4}. \tag{2.27}$$

By (2.20) and (2.27), we see that

$$\begin{aligned}
 \sum_{j=0}^k \binom{2k+1}{2j} 2^{2j} \frac{E_{2j+1}(x)}{2j+1} &= \frac{2^{2k} (2(x-1/2)^{2k+2} - 2E_{2k+2}(x-1/2))}{2k+2} \\
 &= \frac{2^{2k} (x-1/2)^{2k+2} - 2^{2k} E_{2k+2}(x-1/2)}{k+1}.
 \end{aligned} \tag{2.28}$$

Therefore, by (2.28), we obtain the following corollary.

Corollary 2.10. For $k \in \mathbb{Z}_+$, we have

$$\sum_{j=0}^k \binom{2k+1}{2j} 2^{2j} \frac{E_{2j+1}(x)}{2j+1} = -\frac{2^{2k} E_{2k+2}(x-1/2)}{k+1} + \frac{2^{2k} (x-1/2)^{2k+2}}{k+1}. \tag{2.29}$$

From (1.6), we can derive the following equation:

$$\sum_{j=0}^{k-1} \binom{k}{j} \frac{B_{j+1}(x)}{j+1} = x^k - \frac{1}{k+1}, \quad \text{for } k \in \mathbb{N}. \quad (2.30)$$

Let us take the p -adic integral on both sides in (2.30) as follows: for $k \in \mathbb{N}$,

$$\begin{aligned} I_1 &= \sum_{j=0}^{k-1} \binom{k}{j} \int_{\mathbb{Z}_p} \frac{B_{j+1}(x)}{j+1} d\mu(x) = \sum_{j=0}^{k-1} \binom{k}{j} \frac{1}{j+1} \sum_{l=0}^{j+1} \binom{j+1}{l} B_{j+1-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} B_l. \end{aligned} \quad (2.31)$$

On the other hand,

$$I_1 = \int_{\mathbb{Z}_p} x^k d\mu(x) - \frac{1}{k+1} \int_{\mathbb{Z}_p} d\mu(x) = B_k - \frac{1}{k+1}. \quad (2.32)$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 2.11. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} B_l = B_k - \frac{1}{k+1}. \quad (2.33)$$

In (2.30), let us take the fermionic p -adic integral on both sides as follows:

$$\begin{aligned} I_2 &= \sum_{j=0}^{k-1} \binom{k}{j} \frac{1}{j+1} \int_{\mathbb{Z}_p} B_{j+1}(x) d\mu_{-1}(x) \\ &= \sum_{j=0}^{k-1} \frac{\binom{k}{j}}{j+1} \sum_{l=0}^{j+1} \binom{j+1}{l} B_{j+1-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} E_l. \end{aligned} \quad (2.34)$$

On the other hand

$$I_2 = \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) - \frac{1}{k+1} \int_{\mathbb{Z}_p} d\mu_{-1}(x) = E_k - \frac{1}{k+1}. \quad (2.35)$$

Therefore, by (2.34) and (2.35), we obtain the following theorem.

Theorem 2.12. For $k \in \mathbb{N}$, one has

$$\sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} E_l = E_k - \frac{1}{k+1}. \quad (2.36)$$

From (1.7), we can easily derive the following equation:

$$\sum_{j=0}^{k-1} \binom{2k}{2j} \frac{B_{2j+1}(x)}{2j+1} = \frac{x^{2k} - (x-1)^{2k}}{2}. \quad (2.37)$$

Let us take $\int_{\mathbb{Z}_p} d\mu(x)$ on both sides in (2.37). Then we have

$$\begin{aligned} I_3 &= \sum_{j=0}^{k-1} \binom{2k}{2j} \frac{1}{2j+1} \int_{\mathbb{Z}_p} B_{2j+1}(x) d\mu(x) \\ &= \sum_{j=0}^{k-1} \binom{2k}{2j} \frac{1}{2j+1} \sum_{l=0}^{2j+1} \binom{2j+1}{l} B_{2j+1-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} B_l. \end{aligned} \quad (2.38)$$

On the other hand,

$$\begin{aligned} I_3 &= \frac{1}{2} \left(\int_{\mathbb{Z}_p} x^{2k} d\mu(x) - \int_{\mathbb{Z}_p} (x-1)^{2k} d\mu(x) \right) \\ &= \frac{1}{2} (B_{2k} - B_{2k}(-1)) = \frac{1}{2} (B_{2k} - B_{2k}(2)) \\ &= \frac{1}{2} (B_{2k} - (2k + \delta_{1,2k} + B_{2k})), \end{aligned} \quad (2.39)$$

where $\delta_{n,k}$ is a Kronecker symbol.

Therefore, by (2.38) and (2.39), we obtain the following theorem.

Theorem 2.13. For $k \in \mathbb{N}$, one has

$$\sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} B_l = -k. \quad (2.40)$$

Taking $\int_{\mathbb{Z}_p} d\mu_{-1}(x)$ on both sides in (2.37), we get

$$\begin{aligned} I_4 &= \sum_{j=0}^{k-1} \binom{2k}{2j} \frac{1}{2j+1} \sum_{l=0}^{2j+1} \binom{2j+1}{l} B_{2j+1-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} E_l. \end{aligned} \quad (2.41)$$

On the other hand

$$\begin{aligned} I_4 &= \frac{1}{2} \left(\int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) - \int_{\mathbb{Z}_p} (x-1)^{2k} d\mu_{-1}(x) \right) \\ &= \frac{1}{2} \left(\int_{\mathbb{Z}_p} x^{2k} d\mu_{-1}(x) - \int_{\mathbb{Z}_p} (x+2)^{2k} d\mu_{-1}(x) \right) \\ &= \frac{1}{2} (E_{2k} - E_{2k}(2)) = \frac{1}{2} \{E_{2k} - (2 + E_{2k} - 2\delta_{0,2k})\} \\ &= -1 + \delta_{0,k}. \end{aligned} \quad (2.42)$$

Therefore, by (2.41) and (2.42), we obtain the following theorem.

Theorem 2.14. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} E_l = -1. \quad (2.43)$$

From (1.8), we can also derive the following equation:

$$\sum_{j=1}^k \binom{2k}{2j-1} \frac{B_{2j}(x)}{2j} = \frac{x^{2k} + (x-1)^{2k}}{2} - \frac{1}{2k+1}. \quad (2.44)$$

Let us take the bosonic p -adic integral on both sides in (2.44). Then we get

$$\begin{aligned} I_5 &= \sum_{j=1}^k \binom{2k}{2j-1} \frac{1}{2j} \int_{\mathbb{Z}_p} B_{2j}(x) d\mu(x) \\ &= \sum_{j=1}^k \binom{2k}{2j-1} \frac{1}{2j} \sum_{l=0}^{2j} \binom{2j}{l} B_{2j-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{j=1}^k \sum_{l=0}^{2j} \frac{1}{2j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} B_l. \end{aligned} \quad (2.45)$$

On the other hand,

$$\begin{aligned}
 I_5 &= \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (-1+x)^{2k}) d\mu(x) - \frac{1}{2k+1} \int_{\mathbb{Z}_p} d\mu(x) \\
 &= \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (x+2)^{2k}) d\mu(x) - \frac{1}{2k+1} \\
 &= \frac{1}{2} (B_{2k} + B_{2k}(2)) - \frac{1}{2k+1} \\
 &= \frac{1}{2} (B_{2k} + 2k + B_{2k} + \delta_{1,2k}) - \frac{1}{2k+1}.
 \end{aligned} \tag{2.46}$$

Therefore, by (2.45) and (2.46), we obtain the following theorem.

Theorem 2.15. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=1}^k \sum_{l=0}^{2j} \frac{1}{2^j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} B_l = B_{2k} + k - \frac{1}{2k+1}. \tag{2.47}$$

Now, let us consider the fermionic p -adic integral on both sides in (2.44):

$$\begin{aligned}
 I_6 &= \sum_{j=1}^k \binom{2k}{2j-1} \frac{1}{2^j} \sum_{l=0}^{2j} \binom{2j}{l} B_{2j-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\
 &= \sum_{j=1}^k \sum_{l=0}^{2j} \frac{1}{2^j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} E_l.
 \end{aligned} \tag{2.48}$$

On the other hand,

$$\begin{aligned}
 I_6 &= \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (x-1)^{2k}) d\mu_{-1}(x) - \frac{1}{2k+1} \int_{\mathbb{Z}_p} d\mu_{-1}(x) \\
 &= \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (x+2)^{2k}) d\mu_{-1}(x) - \frac{1}{2k+1} \\
 &= \frac{1}{2} (E_{2k} + E_{2k}(2)) - \frac{1}{2k+1} \\
 &= \frac{1}{2} (E_{2k} + (2 + E_{2k} - 2\delta_{0,2k})) - \frac{1}{2k+1} \\
 &= E_{2k} + 1 - \delta_{0,2k} - \frac{1}{2k+1} = \frac{2k}{2k+1}.
 \end{aligned} \tag{2.49}$$

Therefore, by (2.48) and (2.49), we obtain the following theorem.

Theorem 2.16. For $k \in \mathbb{N}$, one has

$$\sum_{j=1}^k \sum_{l=0}^{2j} \frac{1}{2^j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} E_l = \frac{2k}{2k+1}. \quad (2.50)$$

Acknowledgment

The first author was supported by National Research Foundation of Korea Grant funded by the Korean Government 2011-0002486.

References

- [1] T. Kim, “ q -Bernoulli numbers and polynomials associated with Gaussian binomial coefficients,” *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.
- [2] T. Kim, “ q -Volkenborn integration,” *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
- [3] T. Kim, “Symmetry of power sum polynomials and multivariate fermionic p -adic invariant integral on $\mathbb{Z}_p$,” *Russian Journal of Mathematical Physics*, vol. 16, no. 1, pp. 93–96, 2009.
- [4] S. Araci, D. Erdal, and D.-J. Kang, “Some new properties on the q -Genocchi numbers and polynomials associated with q -Bernstein polynomials,” *Honam Mathematical Journal*, vol. 33, no. 2, pp. 261–270, 2011.
- [5] A. Bayad, T. Kim, B. Lee, and S.-H. Rim, “Some identities on Bernstein polynomials associated with q -Euler polynomials,” *Abstract and Applied Analysis*, vol. 2011, Article ID 294715, 10 pages, 2011.
- [6] A. Bayad, “Modular properties of elliptic Bernoulli and Euler functions,” *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010.
- [7] L. Carlitz, “Some arithmetic properties of generalized Bernoulli numbers,” *Bulletin of the American Mathematical Society*, vol. 65, pp. 68–69, 1959.
- [8] L. Carlitz, “Note on the integral of the product of several Bernoulli polynomials,” *Journal of the London Mathematical Society*, vol. 34, pp. 361–363, 1959.
- [9] M. Cenkci, Y. Simsek, and V. Kurt, “Multiple two-variable p -adic q - L -function and its behavior at $s = 0$,” *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 447–459, 2008.
- [10] H. Cohen, *Number Theory. Vol. II. Analytic and Modern Tools*, vol. 240 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2007.
- [11] H. Cohen, *Number Theory. Vol. I. Tools and Diophantine Equations*, vol. 239 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2007.
- [12] D. Ding and J. Yang, “Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials,” *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 1, pp. 7–21, 2010.
- [13] T. Kim, “Some identities on the q -Euler polynomials of higher order and q -Stirling numbers by the fermionic p -adic integral on $\mathbb{Z}_p$,” *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.
- [14] H. Ozden, I. N. Cangul, and Y. Simsek, “Remarks on q -Bernoulli numbers associated with Daehee numbers,” *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.
- [15] H. Ozden, I. N. Cangul, and Y. Simsek, “Multivariate interpolation functions of higher-order q -Euler numbers and their applications,” *Abstract and Applied Analysis*, vol. 2008, Article ID 390857, 16 pages, 2008.
- [16] C. S. Ryoo, “Some identities of the twisted q -Euler numbers and polynomials associated with q -Bernstein polynomials,” *Proceedings of the Jangjeon Mathematical Society*, vol. 14, no. 2, pp. 239–248, 2011.
- [17] C. S. Ryoo, “Some relations between twisted q -Euler numbers and Bernstein polynomials,” *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 2, pp. 217–223, 2011.
- [18] Y. Simsek, “Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions,” *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 251–278, 2008.
- [19] Y. Simsek, “Theorems on twisted L -function and twisted Bernoulli numbers,” *Advanced Studies in Contemporary Mathematics*, vol. 11, no. 2, pp. 205–218, 2005.