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Research Article

An Iterative Algorithm for the Least Squares Generalized Reflexive Solutions of the Matrix Equations $AXB = E, CXD = F$

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The generalized coupled Sylvester systems play a fundamental role in wide applications in several areas, such as stability theory, control theory, perturbation analysis, and some other fields of pure and applied mathematics. The iterative method is an important way to solve the generalized coupled Sylvester systems. In this paper, an iterative algorithm is constructed to solve the minimum Frobenius norm residual problem: $\left\|\left(\begin{smallmatrix} AXB \\ CXD \end{smallmatrix}\right) - \left(\begin{smallmatrix} E \\ F \end{smallmatrix}\right)\right\| = \min$ over generalized reflexive matrix X. For any initial generalized reflexive matrix X_1, by the iterative algorithm, the generalized reflexive solution X^* can be obtained within finite iterative steps in the absence of round-off errors, and the unique least-norm generalized reflexive solution X^* can also be derived when an appropriate initial iterative matrix is chosen. Furthermore, the unique optimal approximate solution $\widehat{X}$ to a given matrix X_0 in Frobenius norm can be derived by finding the least-norm generalized reflexive solution $\widetilde{X}^*$ of a new corresponding minimum Frobenius norm residual problem: $\min\left\|\left(\begin{smallmatrix} A\widetilde{X}B \\ C\widetilde{X}D \end{smallmatrix}\right) - \left(\begin{smallmatrix} \widetilde{E} \\ \widetilde{F} \end{smallmatrix}\right)\right\|$ with $\widetilde{E} = E - AX_0B$, $\widetilde{F} = F - CX_0D$. Finally, several numerical examples are given to illustrate that our iterative algorithm is effective.

1. Introduction

A matrix $P \in \mathcal{R}^{n\times n}$ is said to be a generalized reflection matrix if P satisfies that $P^T = P,\ P^2 = I$. Let $P \in \mathcal{R}^{m\times m}$ and $Q \in \mathcal{R}^{n\times n}$ be two generalized reflection matrices. A matrix $A \in \mathcal{R}^{m\times n}$ is called generalized reflexive (or generalized anti-reflexive) with respect to the matrix pair (P, Q) if $PAQ = A$(or $PAQ = -A$). The set of all m-by-n generalized reflexive matrices with respect to matrix pair (P, Q) is denoted by $\mathcal{R}_r^{m\times n}(P, Q)$. The generalized reflexive and anti-reflexive matrices have many special properties and usefulness in engineering and scientific computations [1–3].

In this paper, we will consider the minimum Frobenius norm residual problem and its optimal approximation problem as follows.

Problem 1. For given matrices $A \in \mathcal{R}^{p\times m},\ B \in \mathcal{R}^{n\times q},\ C \in \mathcal{R}^{s\times m}, D \in \mathcal{R}^{n\times t},\ E \in \mathcal{R}^{p\times q},\ F \in \mathcal{R}^{s\times t}$, find matrix $X \in \mathcal{R}_r^{m\times n}(P,Q)$ such that

$$\left\| \begin{pmatrix} AXB \\ CXD \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\| = \min . \tag{1.1}$$

Problem 2. Let S_E denote the set of the generalized reflexive solutions of Problem 1. For a given matrix $X_0 \in \mathcal{R}_r^{m\times n}(P,Q)$, find $\widehat{X} \in S_E$ such that

$$\left\| \widehat{X} - X_0 \right\| = \min_{X\in S_E} \|X - X_0\|. \tag{1.2}$$

Problem 1 plays a fundamental role in wide applications in several areas, such as Pole assignment, measurement feedback, and matrix programming problem.Liao and Lei [4] presented some examples to show a motivation for studying Problem 1. Problem 2 arises frequently in experimental design. Here the matrix X_0 may be a matrix obtained from experiments, but it may not satisfy the structural requirement (generalized reflexive) and/or spectral requirement (the solution of Problem 1). The best estimate $\widehat{X}$ is the matrix that satisfies both requirements and is the best approximation of $\widetilde{A}$ in the Frobenius norm.

Least-squares-based iterative algorithms are very important in system identification, parameter estimation, and signal processing, including the recursive least squares (RLS) and iterative least squares (ILS) methods for solving the solutions of some matrix equations, for example, the Lyapunov matrix equation, Sylvester matrix equations, and coupled matrix equations as well. Some related contributions in solving matrix equations and parameter identification/estimation should be mentioned in this paper. For example, novel gradient-based iterative (GI) method [5–9] and least-squares-based iterative methods [5, 9, 10] with highly computational efficiencies for solving (coupled) matrix equations are presented and have good stability performances, based on the hierarchical identification principle [11–13] which regards the unknown matrix as the system parameter matrix to be identified.

The explicit and numerical solutions of matrix equation pair $AXB = E,\ CXD = F$ have been addressed in a large body of the literature. Peng et al. [14] presented iterative methods to obtain the symmetric solutions of the matrix equation pair. Sheng and Chen [15] presented a finite iterative method when the matrix equation pair is consistent. Liao and Lei [4] presented an analytical expression of the least squares solution and an algorithm for the matrix equation pair with the minimum norm. Peng et al. [16] presented an efficient algorithm for the least squares reflexive solution. Dehghan and Hajarian [17] presented an iterative algorithm for solving a pair of the matrix equations over generalized centrosymmetric matrices. Cai and Chen [18] presented an iterative algorithm for the least squares bisymmetric solutions of the matrix equations. By applying the hierarchical identification principle,Kılıçman and Zhour [19] developed an iterative algorithm for obtaining the weighted least squares solution. Dehghan and Hajarian [20] constructed an iterative algorithm to solve the generalized coupled Sylvester matrix equations $(AXB + CYD, EXF + GYH) = (M, N)$ over generalized bisymmetric matrices. Wu et al. [21, 22] gave the finite iterative solutions to coupled Sylvester-conjugate matrix equations. Wu et al. [23] gave the finite iterative solutions

to a class of complex matrix equations with conjugate and transpose of the unknowns. Jonsson and Kågström [24, 25] proposed recursive block algorithms for solving the coupled Sylvester matrix equations and the generalized Sylvester and Lyapunov Matrix equations. Very recently, Huang et al. [26] presented a finite iterative algorithms for the one-sided and generalized coupled Sylvester matrix equations over generalized reflexive solutions. Yin et al. [27] presented a finite iterative algorithms for the two-sided and generalized coupled Sylvester matrix equations over reflexive solutions. For more studies on the matrix equations, we refer to [1–4, 16, 17, 28–40]. However, the problem of finding the least squares generalized reflexive solution of the matrix equation pair has not been solved.

The following notations are also used in this paper. Let $\mathcal{R}^{m\times n}$ denote the set of all $m\times n$ real matrices. We denote by the superscript T the transpose of a matrix. In matrix space $\mathcal{R}^{m\times n}$, define inner product as $\langle A,B\rangle = \operatorname{trace}(B^TA)$ for all $A,B\in\mathcal{R}^{m\times n}$, and $\|A\|$ represents the Frobenius norm of A. $R(A)$ represents the column space of A. $\operatorname{vec}(\cdot)$ represents the vector operator, that is, $\operatorname{vec}(A) = (\mathbf{a}_1^T,\mathbf{a}_2^T,\ldots,\mathbf{a}_n^T)^T \in \mathcal{R}^{mn}$ for the matrix $A = (\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_n)\in\mathcal{R}^{m\times n}$, $\mathbf{a}_i\in\mathcal{R}^m$, $i=1,2,\ldots,n$. $A\otimes B$ stands for the Kronecker product of matrices A and B.

This paper is organized as follows. In Section 2, we will solve Problem 1 by constructing an iterative algorithm, that is, for an arbitrary initial matrix $X_1\in\mathcal{R}_r^{m\times n}(P,Q)$, we can obtain a solution $X^*\in\mathcal{R}_r^{m\times n}(P,Q)$ of Problem 1 within finite iterative steps in the absence of round-off errors. The convergence of the algorithm is also proved. Let $X_1 = A^THB^T + C^T\widehat{H}D^T + PA^THB^TQ + PC^T\widehat{H}D^TQ$, where $H\in\mathcal{R}^{p\times q},\widehat{H}\in\mathcal{R}^{s\times t}$ are arbitrary matrices, or more especially, let $X_1 = \mathbf{0}\in\mathcal{R}_r^{m\times n}(P,Q)$; we can obtain the unique least-norm solution X^* of Problem 1. Then in Section 3, we give the optimal approximate solution of Problem 2 by finding the least-norm generalized reflexive solution of a corresponding new minimum Frobenius norm residual problem. In Section 4, several numerical examples are given to illustrate the application of our iterative algorithm.

2. Solution of Problem 1

In this section, we firstly introduce some definitions, lemmas, and theorems which are required for solving Problem 1. Then we present an iterative algorithm to obtain the solution of Problem 1. We also prove that it is convergent. The following definitions and lemmas come from [41], which are needed for our derivation.

Definition 2.1. A set of matrices $S\in\mathcal{R}^{m\times n}$ is said to be convex if for X_1, $X_2\in S$ and $\alpha\in(0,1)$, $\alpha X_1+(1-\alpha)X_2\in S$. Let $\mathcal{R}_c$ denote a convex subset of $\mathcal{R}^{m\times n}$.

Definition 2.2. A matrix function $f:\mathcal{R}_c\to\mathcal{R}$ is said to be convex if

$$f(\alpha X_1+(1-\alpha)X_2)\le\alpha f(X_1)+(1-\alpha)f(X_2) \tag{2.1}$$

for $X_1,X_2\in\mathcal{R}_c$ and $\alpha\in(0,1)$.

Definition 2.3. Let $f:\mathcal{R}_c\to\mathcal{R}$ be a continuous and differentiable function. The gradient of f is defined as $\nabla f(X) = (\partial f(X)/\partial x_{ij})$.

Lemma 2.4. *Let $f : \mathcal{R}_c \to \mathcal{R}$ be a continuous and differentiable function. Then f is convex on $\mathcal{R}_c$ if and only if*

$$f(Y) \geq f(X) + \langle \nabla f(X), Y - X \rangle \tag{2.2}$$

for all $X, Y \in \mathcal{R}_c$.

Lemma 2.5. *Let $f : \mathcal{R}_c \to \mathcal{R}$ be a continuous and differentiable function, and there exists X^* in the interior of $\mathcal{R}_c$ such that $f(X^*) = \min_{X \in \mathcal{R}_c} f(X)$, then $\nabla f(X^*) = 0$.*

Note that the set $\mathcal{R}_r^{m\times n}(P,Q)$ is unbounded, open, and convex. Denote

$$F(X) = \left\| \begin{pmatrix} AXB \\ CXD \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\|^2, \tag{2.3}$$

then $F(X)$ is a continuous, differentiable, and convex function on $\mathcal{R}_r^{m\times n}(P,Q)$. Hence, by applying Lemmas 2.4 and 2.5, we obtain the following lemma.

Lemma 2.6. *Let $F(X)$ be defined by* (2.3)*, then there exists $X^* \in \mathcal{R}_r^{m\times n}(P,Q)$ if and only if $F(X^*) = \min_{X \in \mathcal{R}_r^{m\times n}(P,Q)} F(X)$, then $\nabla F(X^*) = 0$.*

From the Taylor series expansion, we have

$$F(X + \epsilon Y) = F(X) + \epsilon \langle \nabla F(X), Y \rangle + o(\epsilon), \quad \forall X, Y \in \mathcal{R}_r^{m\times n}(P,Q),\ \epsilon \in \mathcal{R}. \tag{2.4}$$

On the other hand, by the basic properties of Frobenius norm and the matrix inner product, we get the expression

$$\begin{aligned}
&F(X + \epsilon Y) \\
&\quad = \left\| \begin{pmatrix} A(X+\epsilon Y)B \\ C(X+\epsilon Y)D \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\|^2 \\
&\quad = \langle AXB - E + \epsilon AYB, AXB - E + \epsilon AYB \rangle + \langle CXD - F + \epsilon CYD, CXD - F + \epsilon CYD \rangle \\
&\quad = \langle AXB - E, AXB - E \rangle + \langle CXD - F, CXD - F \rangle \\
&\qquad + 2\epsilon \langle AXB - E, AYB \rangle + 2\epsilon \langle CXD - F, CYD \rangle + \epsilon^2 (\langle AYB, AYB \rangle + \langle CYD, CYD \rangle) \\
&\quad = F(X) + 2\epsilon \langle AXB - E, AYB \rangle + 2\epsilon \langle CXD - F, CYD \rangle + \epsilon^2 (\langle AYB, AYB \rangle + \langle CYD, CYD \rangle).
\end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned}
2\epsilon\langle AXB-E, AYB\rangle &= 2\epsilon\left\langle A^TAXBB^T - A^TEB^T, Y\right\rangle \\
&= 2\epsilon\left\langle \frac{A^TAXBB^T - A^TEB^T + P(A^TAXBB^T - A^TEB^T)Q}{2}, Y\right\rangle \\
&\quad + 2\epsilon\left\langle \frac{A^TAXBB^T - A^TEB^T - P(A^TAXBB^T - A^TEB^T)Q}{2}, Y\right\rangle \\
&= \epsilon\left\langle A^TAXBB^T - A^TEB^T + PA^TAXBB^TQ - PA^TEB^TQ, Y\right\rangle \\
&\quad + \epsilon\left\langle A^TAXBB^T - A^TEB^T - P\left(A^TAXBB^T - A^TEB^T\right)Q, Y\right\rangle \\
&= \epsilon\left\langle A^TAXBB^T - A^TEB^T + PA^TAXBB^TQ - PA^TEB^TQ, Y\right\rangle, \\
2\varepsilon\langle CXD-F, CYD\rangle &= 2\epsilon\left\langle C^TCXDD^T - C^TFD^T, Y\right\rangle \\
&= 2\epsilon\left\langle \frac{C^TCXDD^T - C^TFD^T + P(C^TCXDD^T - C^TFD^T)Q}{2}, Y\right\rangle \\
&\quad + 2\epsilon\left\langle \frac{C^TCXDD^T - C^TFD^T - P(C^TCXDD^T - C^TFD^T)Q}{2}, Y\right\rangle \\
&= \epsilon\left\langle C^TCXDD^T - C^TFD^T + PC^TCXDD^TQ - PC^TFD^TQ, Y\right\rangle \\
&\quad + \epsilon\left\langle C^TCXDD^T - C^TFD^T - P\left(C^TCXDD^T - C^TFD^T\right)Q, Y\right\rangle \\
&= \epsilon\left\langle C^TCXDD^T - C^TFD^T + PC^TCXDD^TQ - PC^TFD^TQ, Y\right\rangle.
\end{aligned} \tag{2.6}$$

Thus, we have

$$\begin{aligned}
F(X+\epsilon Y) = F(X) &+ \epsilon\left\langle A^TAXBB^T - A^TEB^T + PA^TAXBB^TQ - PA^TEB^TQ \right. \\
&\left. + C^TCXDD^T - C^TFD^T + PC^TCXDD^TQ - PC^TFD^TQ, Y\right\rangle \\
&+ \epsilon^2(\langle AYB, AYB\rangle + \langle CYD, CYD\rangle).
\end{aligned} \tag{2.7}$$

By comparing (2.4) with (2.7), we have

$$\begin{aligned}
\nabla F(X) = {} & A^TAXBB^T - A^TEB^T + PA^TAXBB^TQ - PA^TEB^TQ \\
& + C^TCXDD^T - C^TFD^T + PC^TCXDD^TQ - PC^TFD^TQ.
\end{aligned} \tag{2.8}$$

According to Lemma 2.6 and (2.8), we obtain the following theorem.

Theorem 2.7. *A matrix $X^* \in \mathcal{R}_r^{m\times n}(P,Q)$ is a solution of Problem 1 if and only if $\nabla F(X^*) = 0$.*

For the convenience of discussion, we adopt the following notations:

$$\begin{aligned} M(X) &= A^T AXBB^T + C^T CXDD^T + PA^T AXBB^T Q + PC^T CXDD^T Q, \\ N &= A^T EB^T + C^T FD^T + PA^T EB^T Q + PC^T FD^T Q, \\ G(X) &= -\nabla F(X) = N - M(X), \\ P_k &= G(X_k). \end{aligned} \tag{2.9}$$

The following algorithm is constructed to solve Problems 1 and 2.

Algorithm 2.8.

Step 1. Input matrices $A \in \mathcal{R}^{p\times m}$, $B \in \mathcal{R}^{n\times q}$, $C \in \mathcal{R}^{s\times m}$, $D \in \mathcal{R}^{n\times t}$, $E \in \mathcal{R}^{p\times q}$, $F \in \mathcal{R}^{s\times t}$, and two generalized reflection matrix $P \in \mathcal{R}^{m\times m}$, $Q \in \mathcal{R}^{n\times n}$;

Step 2. Choose an arbitrary matrix $X_1 \in \mathcal{R}_r^{m\times n}(P,Q)$. Compute

$$\begin{aligned} P_1 &= N - M(X_1), \\ Q_1 &= M(P_1), \\ k &:= 1. \end{aligned} \tag{2.10}$$

Step 3. If $P_1 = 0$, then stop. Else go to Step 4.

Step 4. Compute

$$\begin{aligned} X_{k+1} &= X_k + \frac{\|P_k\|^2}{\langle Q_k, M(P_k)\rangle} Q_k, \\ P_{k+1} &= P_k - \frac{\|P_k\|^2}{\langle Q_k, M(P_k)\rangle} M(Q_k), \\ Q_{k+1} &= P_{k+1} - \frac{\langle P_{k+1}, M(Q_k)\rangle}{\langle Q_k, M(Q_k)\rangle} Q_k. \end{aligned} \tag{2.11}$$

Step 5. If $P_{k+1} = 0$, then stop. Else, let $k := k+1$, and go to Step 4.

Remark 2.9. Obviously, it can be seen that $P_i \in \mathcal{R}_r^{m\times n}(P,Q)$, $Q_i \in \mathcal{R}_r^{m\times n}(P,Q)$, and $X_i \in \mathcal{R}_r^{m\times n}(P,Q)$, where $i = 1, 2, \ldots$.

Lemma 2.10. *Suppose that $X \in R_r^{m\times n}(P,Q)$, $Y \in R_r^{m\times n}(P,Q)$, then*

$$\langle M(X), Y\rangle = \langle X, M(Y)\rangle. \tag{2.12}$$

Proof. One has

$$\begin{aligned}
&\langle M(X), Y\rangle \\
&\quad = \left\langle A^T AXBB^T + C^T CXDD^T + PA^T AXBB^T Q + PC^T CXDD^T Q, Y\right\rangle \\
&\quad = \left\langle A^T AXBB^T, Y\right\rangle + \left\langle C^T CXDD^T, Y\right\rangle + \left\langle PA^T AXBB^T Q, Y\right\rangle + \left\langle PC^T CXDD^T Q, Y\right\rangle \\
&\quad = \left\langle X, A^T AYBB^T\right\rangle + \left\langle X, C^T CYDD^T\right\rangle + \left\langle X, A^T APYQBB^T\right\rangle + \left\langle X, C^T CPYQDD^T\right\rangle \\
&\quad = \left\langle X, A^T AYBB^T\right\rangle + \left\langle X, C^T CYDD^T\right\rangle + \left\langle PXQ, A^T APYQBB^T\right\rangle + \left\langle PXQ, C^T CPYQDD^T\right\rangle \\
&\quad = \left\langle X, A^T AYBB^T\right\rangle + \left\langle X, C^T CYDD^T\right\rangle + \left\langle X, PA^T AYBB^T Q\right\rangle + \left\langle X, PC^T CYDD^T Q\right\rangle \\
&\quad = \left\langle X, A^T AYBB^T + C^T CYDD^T + PA^T AYBB^T Q + PC^T CYDD^T Q\right\rangle \\
&\quad = \langle X, M(Y)\rangle.
\end{aligned} \tag{2.13}$$

This completes the proof. □

Lemma 2.11. *For the sequences* $\{P_i\}$ *and* $\{Q_i\}$ *generated by Algorithm 2.8, if there exists a positive number* k *such that* $P_i \neq 0$ *for all* $i = 1, 2, \ldots, k$, *then*

$$\langle P_i, P_j\rangle = 0, \quad \langle Q_i, M(Q_j)\rangle = 0 \quad (i, j = 1, 2, \ldots, k, i \neq j). \tag{2.14}$$

Proof. Since $\langle A, B\rangle = \langle B, A\rangle$ holds for all matrices A and B in $\mathcal{R}^{m\times n}$, we only need prove that $\langle P_i, P_j\rangle = 0$, $\langle Q_i, M(Q_j)\rangle = 0$ for all $1 \leq i < j \leq k$. We prove the conclusion by induction and two steps are required.

Step 1. We will show that

$$\langle P_i, P_{i+1}\rangle = 0, \quad \langle Q_i, M(Q_{i+1})\rangle = 0, \quad i = 1, 2, \ldots, k-1. \tag{2.15}$$

To prove this conclusion, we also use induction.
For $i = 1$, by Algorithm 2.8 and Lemma 2.10, we have that

$$\begin{aligned}
\langle P_1, P_2\rangle = \langle P_2, P_1\rangle &= \left\langle P_1 - \frac{\|P_1\|^2}{\langle Q_1, M(P_1)\rangle} M(Q_1), P_1\right\rangle \\
&= \|P_1\|^2 - \frac{\|P_1\|^2}{\langle Q_1, M(P_1)\rangle}\langle M(Q_1), P_1\rangle = 0, \\
\langle Q_1, M(Q_2)\rangle = \langle Q_2, M(Q_1)\rangle &= \left\langle P_2 - \frac{\langle P_2, M(Q_1)\rangle}{\langle Q_1, M(Q_1)\rangle} Q_1, M(Q_1)\right\rangle \\
&= \langle P_2, M(Q_1)\rangle - \frac{\langle P_2, M(Q_1)\rangle}{\langle Q_1, M(Q_1)\rangle}\langle Q_1, M(Q_1)\rangle = 0.
\end{aligned} \tag{2.16}$$

Assume (2.15) holds for $i = s(1 < s < k)$. For $i = s+1$, by Lemma 2.10, we have

$$\begin{aligned}
\langle P_{s+1}, P_{s+2}\rangle &= \langle P_{s+2}, P_{s+1}\rangle = \left\langle P_{s+1} - \frac{\|P_{s+1}\|^2}{\langle Q_{s+1}, M(P_{s+1})\rangle} M(Q_{s+1}), P_{s+1}\right\rangle \\
&= \|P_{s+1}\|^2 - \frac{\|P_{s+1}\|^2}{\langle Q_{s+1}, M(P_{s+1})\rangle}\langle M(Q_{s+1}), P_{s+1}\rangle \\
&= \|P_{s+1}\|^2 - \frac{\|P_{s+1}\|^2}{\langle Q_{s+1}, M(P_{s+1})\rangle}\langle Q_{s+1}, M(P_{s+1})\rangle = 0, \\
\langle Q_{s+1}, M(Q_{s+2})\rangle &= \langle Q_{s+2}, M(Q_{s+1})\rangle = \left\langle P_{s+2} - \frac{\langle P_{s+2}, M(Q_{s+1})\rangle}{\langle Q_{s+1}, M(Q_{s+1})\rangle} Q_{s+1}, M(Q_{s+1})\right\rangle \\
&= \langle P_{s+2}, M(Q_{s+1})\rangle - \frac{\langle P_{s+2}, M(Q_{s+1})\rangle}{\langle Q_{s+1}, M(Q_{s+1})\rangle}\langle Q_{s+1}, M(Q_{s+1})\rangle = 0.
\end{aligned} \tag{2.17}$$

Hence, (2.15) holds for $i = s+1$. Therefore, (2.15) holds by the principle of induction.

Step 2. Assume that $\langle P_i, P_s\rangle = 0,\ \langle Q_i, M(Q_s)\rangle = 0,\ i = 1, 2, \ldots, s-1$, then we show that

$$\langle P_i, P_{s+1}\rangle = 0, \quad \langle Q_i, M(Q_{s+1})\rangle = 0, \quad i = 1, 2, \ldots, s. \tag{2.18}$$

In fact, by Algorithm 2.8 we have

$$\begin{aligned}
\langle P_i, P_{s+1}\rangle &= \left\langle P_i, P_s - \frac{\|P_s\|^2}{\langle Q_s, M(P_s)\rangle} M(Q_s)\right\rangle = \langle P_i, P_s\rangle - \frac{\|P_s\|^2}{\langle Q_s, M(P_s)\rangle}\langle P_i, M(Q_s)\rangle \\
&= -\frac{\|P_s\|^2}{\langle Q_s, M(P_s)\rangle}\left\langle Q_i + \frac{\langle P_i, M(Q_{i-1})\rangle}{\langle Q_{i-1}, M(Q_{i-1})\rangle} Q_{i-1}, M(Q_s)\right\rangle \\
&= -\frac{\|P_s\|^2}{\langle Q_s, M(P_s)\rangle}\langle Q_i, M(Q_{i-1}\rangle + \frac{\|P_s\|^2}{\langle Q_s, M(P_s)\rangle}\frac{\langle P_i, M(Q_{i-1})\rangle}{\langle Q_{i-1}, M(Q_{i-1})\rangle}\langle Q_{i-1}, M(Q_s)\rangle \\
&= 0, \\
\langle Q_i, M(Q_{s+1})\rangle &= \langle M(Q_i), Q_{s+1}\rangle = \left\langle M(Q_i), P_{s+1} - \frac{\langle P_{s+1}, M(Q_s)\rangle}{\langle Q_s, M(Q_s)\rangle} Q_s\right\rangle \\
&= \langle M(Q_i), P_{s+1}\rangle - \frac{\langle P_{s+1}, M(Q_s)\rangle}{\langle Q_s, M(Q_s)\rangle}\langle M(Q_i), Q_s\rangle \\
&= \frac{\langle Q_i, M(P_i)\rangle}{\|P_i\|^2}\langle P_i - P_{i+1}, P_{s+1}\rangle \\
&= -\frac{\langle Q_i, M(P_i)\rangle}{\|P_i\|^2}\langle P_{i+1}, P_{s+1}\rangle \\
&= 0.
\end{aligned} \tag{2.19}$$

By the principle of induction, (2.14) is implied in Steps 1 and 2. This completes the proof. □

Lemma 2.12. *Assume that X^* is an arbitrary solution of Problem 1, then*

$$\langle X^* - X_k, M(Q_k)\rangle = 2\frac{\|P_k\|^2}{\|P_1\|^2}\left(\|AP_1B\|^2 + \|CP_1D\|^2\right), \quad k = 1, 2, \ldots, \tag{2.20}$$

where the sequences $\{X_k\}$, $\{P_k\}$, and $\{Q_k\}$ are generated by Algorithm 2.8.

Proof. First, by Algorithm 2.8, it is easy to verify that

$$-\frac{\langle P_{k+1}, M(Q_k)\rangle}{\langle Q_k, M(Q_k)\rangle} = \frac{\|P_{k+1}\|^2}{\|P_k\|^2}. \tag{2.21}$$

Thus

$$\begin{aligned}
\langle X^* - X_k, M(Q_k)\rangle &= \langle M(X^* - X_k), Q_k\rangle = \langle P_k, Q_k\rangle \\
&= \left\langle P_{k-1} - \frac{\|P_{k-1}\|^2}{\langle Q_{k-1}, M(P_{k-1})\rangle} M(Q_{k-1}), Q_k \right\rangle \\
&= \langle P_{k-1}, Q_k\rangle = \cdots = \langle P_1, Q_k\rangle \\
&= \left\langle P_1, P_k - \frac{\langle P_k, M(Q_{k-1})\rangle}{\langle Q_{k-1}, M(Q_{k-1})\rangle} Q_{k-1} \right\rangle \\
&= -\frac{\langle P_k, M(Q_{k-1})\rangle}{\langle Q_{k-1}, M(Q_{k-1})\rangle}\langle P_1, Q_{k-1}\rangle \\
&= \cdots = \left(-\frac{\langle P_k, M(Q_{k-1})\rangle}{\langle Q_{k-1}, M(Q_{k-1})\rangle}\right)\cdots\left(-\frac{\langle P_2, M(Q_1)\rangle}{\langle Q_1, M(Q_1)\rangle}\right)\langle P_1, Q_1\rangle \\
&= \frac{\|P_k\|^2}{\|P_{k-1}\|^2}\cdots\frac{\|P_2\|^2}{\|P_1\|^2}\langle P_1, M(P_1)\rangle \\
&= 2\frac{\|P_k\|^2}{\|P_1\|^2}\left(\|AP_1B\|^2 + \|CP_1D\|^2\right).
\end{aligned} \tag{2.22}$$

This complete the proof. □

Remark 2.13. Lemma 2.12 implies that if $P_i \neq 0$, then $M(Q_i) \neq 0$, thus $Q_i \neq 0$ $(i = 1, 2, \ldots)$.

Theorem 2.14. *For an arbitrary initial matrix $X_1 \in \mathcal{R}_r^{m\times n}(P, Q)$, a solution of Problem 1 can be obtained with finite iteration steps in the absence of round-off errors.*

Proof. If $P_i \neq 0$, $i = 1, 2, \ldots, mn$, by Lemma 2.12 we have $Q_i \neq 0$, $i = 1, 2, \ldots, mn$, then we can compute X_{mn+1}, P_{mn+1} by Algorithm 2.8.

By Lemma 2.11, we have

$$\begin{aligned}
\langle P_i, P_{mn+1}\rangle &= 0, \quad i = 1, 2, \ldots, mn, \\
\langle P_i, P_j\rangle &= 0, \quad i, j = 1, 2, \ldots, mn, \ i \neq j.
\end{aligned} \tag{2.23}$$

It can be seen that the set of $P_1, P_2, \ldots, P_{mn}$ is an orthogonal basis of the matrix space $\mathcal{R}_r^{m\times n}(P,Q)$, which implies that $P_{mn+1} = 0$, that is, X_{mn+1} is a solution of Problem 1. This completes the proof. □

To show the least-norm generalized reflexive solution of Problem 1, we first introduce the following result.

Lemma 2.15 (see [16, Lemma 2.7]). *Suppose that the minimum residual problem* $\|My - b\|$ =min *has a solution* $y^* \in R(M^T)$, *then* y^* *is the unique least Frobenius norm solution of the minimum residual problem.*

By Lemma 2.15, the following result can be obtained.

Theorem 2.16. *If one chooses the initial iterative matrix* $X_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q$, *where* $H \in \mathcal{R}^{p\times q}$, $\widehat{H} \in \mathcal{R}^{s\times t}$ *are arbitrary matrices, especially, let* $X_1 = \mathbf{0} \in \mathcal{R}_r^{m\times n}$, *one can obtain the unique least-norm generalized reflexive solution of Problem 1 within finite iteration steps in the absence of round-off errors by using Algorithm 2.8.*

Proof. By Algorithm 2.8 and Theorem 2.14, if we let $X_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q$, where $H \in \mathcal{R}^{p\times q}$, $\widehat{H} \in \mathcal{R}^{s\times t}$ are arbitrary matrices, we can obtain the solution X^* of Problem 1 within finite iteration steps in the absence of round-off errors, the solution X^* can be represented that

$$X^* = A^T G B^T + C^T \widehat{G} D^T + P A^T G B^T Q + P C^T \widehat{G} D^T Q. \tag{2.24}$$

In the sequel, we will prove that X^* is just the least-norm solution of Problem 1. Consider the following minimum residual problem

$$\min_{X\in\mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} AXB \\ CXD \\ APXQB \\ CPXQD \end{pmatrix} - \begin{pmatrix} E \\ F \\ E \\ F \end{pmatrix} \right\|. \tag{2.25}$$

Obviously, the solvability of Problem 1 is equivalent to that of the minimum residual problem (2.25), and the least-norm solution of Problem 1 must be the least-norm solution of the minimum residual problem (2.25).

In order to prove that X^* is the least-norm solution of Problem 1, it is enough to prove that X^* is the least-norm solution of the minimum residual problem (2.25). Denote $\mathrm{vec}(X) = x$, $\mathrm{vec}(X^*) = x^*$, $\mathrm{vec}(G) = g_1$, $\mathrm{vec}(\widehat{G}) = g_2$, $\mathrm{vec}(E) = e$, $\mathrm{vec}(F) = f$, then the minimum residual problem (2.25) is equivalent to the minimum residual problem as follows:

$$\min_{x\in\mathcal{R}^{mn}} \left\| \begin{pmatrix} B^T \otimes A \\ D^T \otimes C \\ (B^T Q) \otimes (AP) \\ (D^T Q) \otimes (CP) \end{pmatrix} x - \begin{pmatrix} e \\ f \\ e \\ f \end{pmatrix} \right\|. \tag{2.26}$$

Noting that

$$\begin{aligned}
x^* &= \operatorname{vec}\left(A^T G B^T + C^T \widehat{G} D^T + P A^T G B^T Q + P C^T \widehat{G} D^T Q\right) \\
&= \left(B \otimes A^T\right) g_1 + \left(D \otimes C^T\right) g_2 + \left((QB) \otimes \left(PA^T\right)\right) g_1 + \left((QD) \otimes \left(PC^T\right)\right) g_2 \\
&= \left(B \otimes A^T \quad D \otimes C^T \quad (QB) \otimes \left(PA^T\right) \quad (QD) \otimes \left(PC^T\right)\right) \begin{pmatrix} g_1 \\ g_2 \\ g_1 \\ g_2 \end{pmatrix} \\
&= \begin{pmatrix} B^T \otimes A \\ D^T \otimes C \\ (B^T Q) \otimes (AP) \\ (D^T Q) \otimes (CP) \end{pmatrix}^T \begin{pmatrix} g_1 \\ g_2 \\ g_1 \\ g_2 \end{pmatrix} \in R\left(\begin{pmatrix} B^T \otimes A \\ D^T \otimes C \\ (B^T Q) \otimes (AP) \\ (D^T Q) \otimes (CP) \end{pmatrix}^T\right),
\end{aligned} \tag{2.27}$$

by Lemma 2.15 we can see that x^* is the least-norm solution of the minimum residual problem (2.26). Since vector operator is isomorphic, X^* is the unique least-norm solution of the minimum residual problem (2.25); furthermore X^* is the unique least-norm solution of Problem 1. □

3. Solution of Problem 2

Since the solution set of Problem 1 is no empty, when $X \in S_E$, then

$$\min_{X \in \mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} AXB \\ CXD \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\| \Longleftrightarrow \min_{X \in \mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} A(X - X_0)B \\ C(X - X_0)D \end{pmatrix} - \begin{pmatrix} E - AX_0B \\ F - CX_0D \end{pmatrix} \right\|. \tag{3.1}$$

Let $\widetilde{X} = X - X_0$, $\widetilde{E} = E - AX_0B$, $\widetilde{F} = F - CX_0D$, then Problem 2 is equivalent to finding the least-norm generalized reflexive solution of a new corresponding minimum residual problem

$$\min_{\widetilde{X} \in \mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} A\widetilde{X}B \\ C\widetilde{X}D \end{pmatrix} - \begin{pmatrix} \widetilde{E} \\ \widetilde{F} \end{pmatrix} \right\|. \tag{3.2}$$

By using Algorithm 2.8, let initially iterative matrix $\widetilde{X}_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q$, or more especially, let $\widetilde{X}_1 = 0 \in R_r^{m\times n}(P,Q)$; we can obtain the unique least-norm generalized reflexive solution $\widetilde{X}^*$ of minimum residual problem (3.2), then we can obtain the generalized reflexive solution $\widehat{X}$ of Problem 2, and $\widehat{X}$ can be represented that $\widehat{X} = \widetilde{X}^* + X_0$.

4. Numerical Examples

In this section, we will show several numerical examples to illustrate our results. All the tests are performed by MATLAB 7.8.

Example 4.1. Consider the generalized reflexive solution of Problem 1, where

$$A = \begin{pmatrix} 1 & 3 & -5 & 7 & -9 \\ 2 & 0 & 4 & 6 & -1 \\ 0 & -2 & 9 & 6 & -8 \\ 3 & 6 & 2 & 27 & -13 \\ -5 & 5 & -22 & -1 & -11 \\ 8 & 4 & -6 & -9 & -19 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 8 & -5 & 4 \\ -1 & 5 & 0 & -2 & 3 \\ 4 & -1 & 0 & 2 & 5 \\ 0 & 3 & 9 & 2 & -6 \\ -2 & 7 & -8 & 1 & 11 \end{pmatrix},$$

$$C = \begin{pmatrix} 6 & 32 & -5 & 7 & -9 \\ 2 & 10 & 4 & 6 & -11 \\ 9 & -12 & 9 & 3 & -8 \\ 13 & 6 & 4 & 27 & -15 \\ -5 & 15 & -22 & -13 & -11 \\ 2 & 9 & -6 & -9 & -19 \end{pmatrix}, \quad D = \begin{pmatrix} 7 & 1 & 8 & -6 & 14 \\ -4 & 5 & 0 & -2 & 3 \\ 3 & -12 & 0 & 8 & 25 \\ 1 & 6 & 9 & 4 & -6 \\ -5 & 8 & -2 & 9 & 17 \end{pmatrix},$$

$$E = \begin{pmatrix} 592 & -1191 & 1216 & -244 & -1331 \\ 305 & 431 & 1234 & -518 & 221 \\ 814 & -407 & 1668 & -1176 & 537 \\ 1434 & -179 & 4083 & -1374 & -808 \\ 242 & -3150 & -1362 & 1104 & -2848 \\ 423 & -2909 & 1441 & -182 & -3326 \end{pmatrix}, \quad F = \begin{pmatrix} -2882 & 2830 & 299 & 2291 & -4849 \\ 409 & 670 & 1090 & -783 & -793 \\ 3363 & -126 & 2979 & -3851 & 246 \\ 2632 & 173 & 4553 & -3709 & -100 \\ -1774 & -4534 & -4548 & 1256 & -6896 \\ 864 & -2512 & -1136 & -1633 & -5412 \end{pmatrix}. \tag{4.1}$$

Let

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

We will find the least squares generalized reflexive solution of the matrix equation pair $AXB = E$, $CXD = F$ by using Algorithm 2.8. Because of the influence of the error of calculation, $\|P_k\| = \|-\nabla F(X_k)\|$ is usually unequal to zero in the process of the iteration, where $k = 1, 2, \ldots$. For any chosen positive number ε, however small enough, for example, $\varepsilon = 1.0000e-8$, whenever $\|P_k\| < \varepsilon$, stop the iteration, and X_k is regarded to be the least squares generalized reflexive solution of the matrix equation pair $AXB = E$, $CXD = F$.

Let

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{4.3}$$

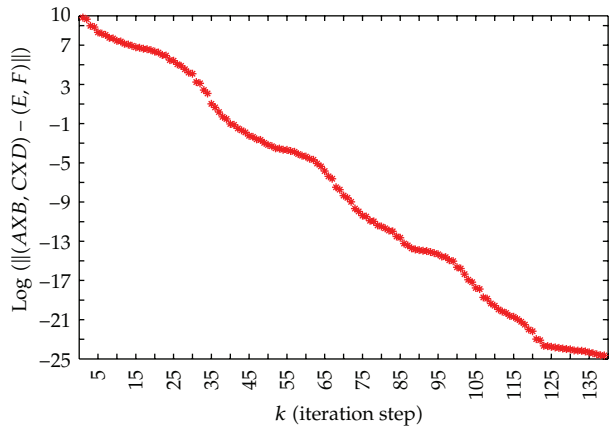

Figure 1: Convergence curve for the Frobenius norm of the residual for Example 4.1 with $X_1 = \mathbf{0}$.

by Algorithm 2.8, we have the unique least Frobenius norm generalized reflexive solution of Problem 1

$$X_{139} = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix},$$
$$\|P_{139}\| = 8.1258e-009 < \varepsilon,$$
$$\min_{X\in\mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} AXB \\ CXD \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\| = \left\| \begin{pmatrix} AX_{139}B \\ CX_{139}D \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\| = 1.9595e-011. \tag{4.4}$$

The convergence curve for the Frobenius norm of the residual is shown in Figure 1.

Example 4.2. Consider the least-norm generalized reflexive solution of the minimum residual problem in Example 4.1. Let

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 2 & 1 & 0 \\ -1 & 0 & -2 & -1 & 0 \end{pmatrix}, \qquad \widehat{H} = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 3 \\ 1 & -1 & 0 & -2 & 0 \\ 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 2 & 1 & 0 \\ -1 & 0 & -2 & 1 & 2 \end{pmatrix},$$
$$X_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q. \tag{4.5}$$

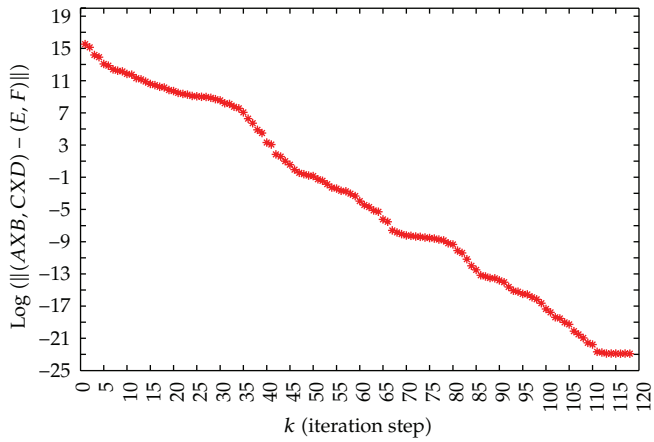

Figure 2: Convergence curve for the Frobenius norm of the residual for Example 4.2.

By using Algorithm 2.8, we have the unique least Frobenius norm generalized reflexive solution of Problem 1

$$X_{118} = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix},$$
$$\|P_{118}\| = 9.8093e-009 < \varepsilon,$$
$$\min_{X\in\mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} AXB \\ CXD \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\| = \left\| \begin{pmatrix} AX_{118}B \\ CX_{118}D \end{pmatrix} - \begin{pmatrix} E \\ F \end{pmatrix} \right\| = 1.1235e-010. \tag{4.6}$$

The convergence curve for the Frobenius norm of the residual is shown in Figure 2.

Example 4.3. Let S_E denote the set of all generalized reflexive solutions of Problem 1 in Example 4.1. For a given matrix

$$X_0 = \begin{pmatrix} -3 & 3 & 1 & 1 & 1 \\ 0 & -7 & 1 & 6 & 10 \\ 10 & -9 & 0 & 9 & 10 \\ -1 & 1 & -1 & 3 & 3 \\ -10 & 6 & -1 & -7 & 0 \end{pmatrix} \in \mathcal{R}_r^{5\times 5}(P,Q), \tag{4.7}$$

we will find $\widehat{X} \in S_E$, such that

$$\left\| \widehat{X} - X_0 \right\| = \min_{X\in S_E} \|X - X_0\|, \tag{4.8}$$

that is, find the optimal approximate solution to the matrix X_0 in S_E.

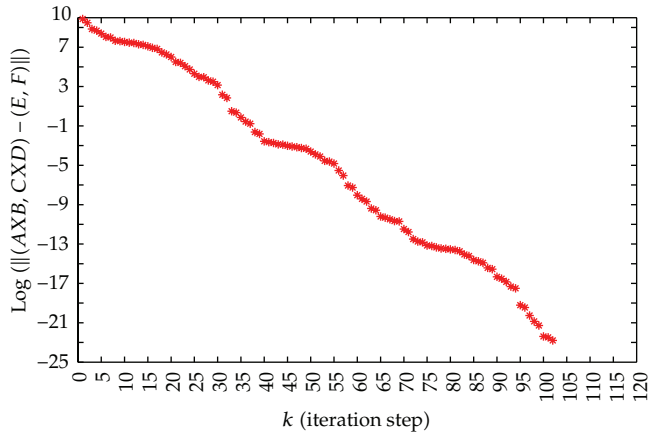

Figure 3: Convergence curve for the Frobenius norm of the residual for Example 4.3.

Let $\widetilde{X} = X - X_0$, $\widetilde{E} = E - AX_0B$, $\widetilde{F} = F - CX_0D$, by the method mentioned in Section 3, we can obtain the least-norm generalized reflexive solution $\widetilde{X}^*$ of the minimum residual problem (3.2) by choosing the initial iteration matrix $\widetilde{X}_1 = \mathbf{0}$, and $\widetilde{X}^*$ is such that

$$\widetilde{X}^*_{102} = \begin{pmatrix} 8.0000 & 0.0000 & -7.0000 & 11.0000 & -6.0000 \\ -11.0000 & 15.0000 & -2.0000 & 3.0000 & -3.0000 \\ 3.0000 & 5.0000 & -8.0000 & -5.0000 & 3.0000 \\ 6.0000 & 11.0000 & 7.0000 & 0.0000 & -8.0000 \\ 3.0000 & 3.0000 & 2.0000 & 15.0000 & 11.0000 \end{pmatrix},$$

$$\|P_{102}\| = 8.6456e - 009 < \varepsilon, \tag{4.9}$$

$$\widehat{X} = \widetilde{X}^*_{102} + X_0 = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix}.$$

The convergence curve for the Frobenius norm of the residual is shown in Figure 3.

5. Conclusion

This paper mainly solves the minimum Frobenius norm residual problem and its optimal approximate problem over generalized reflexive matrices by constructing an iterative algorithm. We solve the minimum Frobenius norm residual problem by constructing an iterative algorithm, that is, for an arbitrary initial matrix $X_1 \in \mathcal{R}_r^{m\times n}(P,Q)$, we obtain a solution $X^* \in \mathcal{R}_r^{m\times n}(P,Q)$ of Problem 1 within finite iterative steps in the absence of round-off errors. The convergence of the algorithm is also proved. Let $X_1 = A^THB^T + C^T\widehat{H}D^T + PA^THB^TQ + PC^T\widehat{H}D^TQ$, where $H \in \mathcal{R}^{p\times q}$, $\widehat{H} \in \mathcal{R}^{s\times t}$ are arbitrary matrices, or more especially, let $X_1 = \mathbf{0} \in \mathcal{R}_r^{m\times n}(P,Q)$; we obtain the unique least-norm solution X^* of the minimum Frobenius norm residual problem. Then we give the generalized reflexive solution

of the optimal approximate problem by finding the least-norm generalized reflexive solution of a corresponding new minimum Frobenius norm residual problem.

Several numerical examples are given to confirm our theoretical results. We can see that our iterative algorithm is effective. We also note that for the minimum Frobenius norm residual problem with large but not sparse matrices A, B, C, D, E, and F, Algorithm 2.8 may be terminated more than mn steps because of computational errors.

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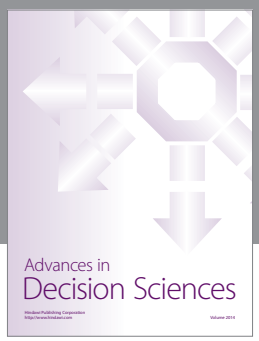

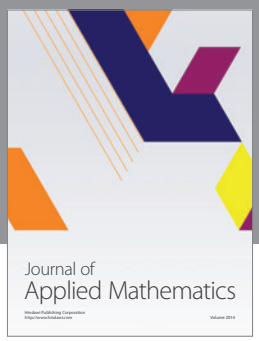

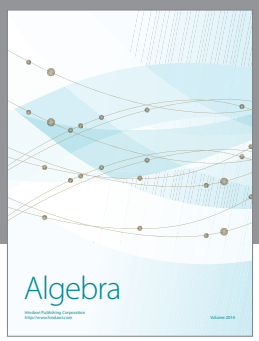

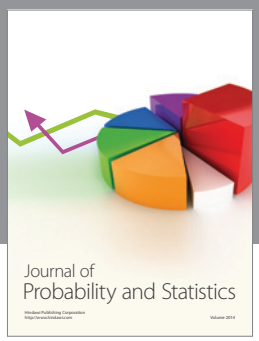

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