Research Article

Convergence Theorems for a Common Point of Solutions of Equilibrium and Fixed Point of Relatively Nonexpansive Multivalued Mapping Problems

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We introduce an iterative process which converges strongly to a common point of set of solutions of equilibrium problem and set of fixed points of finite family of relatively nonexpansive multivalued mappings in Banach spaces.

1. Introduction

Let E be a real Banach space with dual E^* . The function $\phi : E \times E \rightarrow \mathbb{R}^+$, defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \text{for } x, y \in E, \quad (1.1)$$

is studied by Alber [1] and Reich [2], where J is the normalized duality mapping from E to 2^{E^*} defined by $Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that E is smooth if and only if J is single valued and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E . We note that in a Hilbert space H , J is the identity operator.

Let C be a nonempty closed convex subset of a Hilbert space H . It is well known that the metric projection of H onto C , $P_C : H \rightarrow C$, is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In

this direction, Alber [1] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of metric projection in Hilbert spaces.

Let C be a nonempty closed and convex subset of a reflexive, strictly convex and smooth Banach space E . The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_C : E \rightarrow C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x), y \in C\}. \quad (1.2)$$

Let C be a nonempty closed convex subset of a Banach space E . Let $T : C \rightarrow C$ be a single-valued mapping. An element $p \in C$ is called a *fixed point* of T if $T(p) = p$. The set of fixed points of T is denoted by $F(T)$. A point p in C is said to be an *asymptotic fixed point* of T (see [2]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$ and is called *relatively nonexpansive* if (A1) $F(T) \neq \emptyset$; (A2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(T)$ and (A3) $F(T) = \hat{F}(T)$.

Let C be a nonempty closed convex subset of a Banach space E and let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C , respectively. Let H be the Hausdorff metric on $CB(C)$ defined by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \quad (1.3)$$

for all $A, B \in CB(C)$, where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B .

Let $T : C \rightarrow CB(C)$ be a multivalued mapping. T is said to be a *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$, for $x, y \in C$. An element $p \in C$ is called a *fixed point* of T , if $p \in F(T)$, where $F(T) := \{p \in C : p \in T(p)\}$. A point $p \in C$ called an *asymptotic fixed point* of T , if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. T is said to be *relatively nonexpansive* if (B1) $F(T) \neq \emptyset$; (B2) $\phi(p, u) \leq \phi(p, x)$ for $x \in C$, $u \in Tx$, $p \in F(T)$ and (B3) $F(T) = \hat{F}(T)$, where $\hat{F}(T)$ is the set of asymptotic fixed points of T .

We remark that the class of relatively nonexpansive single-valued mappings is contained in a class of relatively nonexpansive multi-valued mappings. An example of relatively nonexpansive multi-valued mapping by Homaeipour and Razani [3] is given below.

Example 1.1. Let $I = [0, 1]$, $X = L^p(I)$, $1 < p < \infty$ and $C = \{f \in X : f(x) \geq 0, \text{ for all } x \in I\}$. Let $T : C \rightarrow CB(C)$ be defined by

$$T(f) = \begin{cases} \left\{ g \in C : f(x) - \frac{3}{4} \leq g(x) \leq f(x) - \frac{1}{4}, \forall x \in I \right\}, & \text{if } f(x) > 1, x \in I, \\ \{0\}, & \text{otherwise.} \end{cases} \quad (1.4)$$

It is shown in [3] that T is relatively nonexpansive multi-valued mapping which is not nonexpansive.

The study of fixed points for multi-valued nonexpansive mappings in relation to Hausdorff metric was introduced by Markin [4] (see also [5]). Since then a lot of activity in this area and fixed point theory for multi-valued nonexpansive mappings has been developed which has some nontrivial applications in pure and applied sciences including control theory, convex optimization, differential inclusion, and economics (see, e.g., [6] and references therein). Later, Lim [7] established the existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces.

It is well known that the normal *Mann's* iterative [8] algorithm has only weak convergence in an infinite-dimensional Hilbert space even for nonexpansive single-valued mappings. Consequently, in order to obtain strong convergence, one has to modify the normal Mann's iteration algorithm, the so called hybrid projection iteration method is such a modification. The hybrid projection iteration algorithm (HPIA) was introduced initially by Haugazeau [9] in 1968. For 40 years, (HPIA) has received rapid developments. For details, the readers are referred to papers [10–12] and the references therein.

In 2003, Nakajo and Takahashi [12] proposed the following modification of the Mann iteration method for a nonexpansive single-valued mapping T in a Hilbert space H :

$$\begin{aligned} x_0 &\in C, \text{ chosen arbitrary,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \quad n \geq 0, \end{aligned} \tag{1.5}$$

where C is a closed convex subset of H , P_C denotes the metric projection from H onto C . They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}(x_0)$.

In spaces more general than Hilbert spaces, Matsushita and Takahashi [11] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive single-valued mapping T in a Banach space E :

$$\begin{aligned} x_0 &\in C, \text{ chosen arbitrary,} \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ C_n &= \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), \quad n \geq 0. \end{aligned} \tag{1.6}$$

They proved the following convergence theorem.

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E , let T be a relatively nonexpansive single-valued mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$.*

Suppose that $\{x_n\}$ is given by (1.6), where J is the duality mapping on E . If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from E onto $F(T)$.

Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for f is

$$\text{finding } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \quad (1.7)$$

The solution set of (1.7) is denoted by $EP(f)$.

If $f(x, y) = \langle Ax, y - x \rangle$, where $A : C \rightarrow C$ is a monotone mapping, then the problem (1.7) reduces to the system of variational inequality problem

$$\text{find an element } x^* \in C \text{ such that } \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \quad (1.8)$$

That is, the problem (1.8) is a special case of (1.7). The set of solutions of inequality (1.8) is denoted by $VI(C, A)$.

For solving the equilibrium problem for a bifunction $f : C \times C \rightarrow \mathbb{R}$, we assume that f satisfies the following conditions:

- (A1) $f(x, x) = 0$, for all $x \in C$,
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$,
- (A3) for each $x, y, z \in C$, $\lim_{t \rightarrow 0^+} f(tz + (1-t)x, y) \leq f(x, y)$,
- (A4) for each $x \in C$, $y \rightarrow f(x, y)$ is convex and lower semicontinuous.

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive or relatively nonexpansive single-valued mapping and the set of solutions of an equilibrium problems in the frame work of Hilbert spaces and Banach spaces, respectively: see, for instance, [2, 13–21] and the references therein.

In [22], Kumam introduced the following iterative scheme in a Hilbert space:

$$\begin{aligned} x_0 &\in H, \\ u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C, \\ w_n &= \alpha_n x_n + (1 - \alpha_n) T u_n, \\ C_n &= \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_{n+1} \cap Q_n}(x_0), \quad n \geq 0, \end{aligned} \quad (1.9)$$

for finding a common element of the set of fixed point of nonexpansive single-valued mapping T and set of solution of equilibrium problems.

In the case that E is a Banach space, Takahashi and Zembayashi [16] introduced the following iterative scheme which is called the shrinking projection method:

$$\begin{aligned}
 x_0 &\in C, \quad \text{chosen arbitrary,} \\
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\
 u_n &\in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\
 C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
 x_{n+1} &= \Pi_{C_{n+1}}(x_0), \quad n \geq 0,
 \end{aligned} \tag{1.10}$$

where J is the duality mapping on E , Π_C is the generalized projection from E onto C and T is relatively nonexpansive single-valued mapping. They proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed point of relatively nonexpansive single-valued mapping and set of solution of equilibrium problem under appropriate conditions.

We remark that the computation of x_{n+1} in (1.9) and (1.10) is not simple because of the involvement of computation of C_{n+1} from C_n for each $n \geq 0$.

More recently, Homaeipour and Razani [3] studied the following iterative scheme for a fixed point of relatively nonexpansive multi-valued mapping in uniformly convex and uniformly smooth Banach space E :

$$\begin{aligned}
 x_0 &\in C, \quad \text{chosen arbitrary,} \\
 x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \quad z_n \in Tx_n, \quad n \geq 0,
 \end{aligned} \tag{1.11}$$

where $\{\alpha_n\} \subset (0, 1)$ for all $n \geq 0$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. They proved that if J is weakly sequentially continuous then the sequence $\{x_n\}$ converges *weakly* to a fixed point of T . Furthermore, it is shown that the scheme converges strongly to a fixed point of T if *interior of $F(T)$ is nonempty*.

But it is worth mentioning that the convergence of the scheme is either *weak* or it requires that the *interior of $F(T)$ is nonempty*.

In this paper, motivated by Kumam [22], Takahashi and Zembayashi [16], and Homaeipour and Razani [3], we construct an iterative scheme which converges strongly to a common point of set of solutions of equilibrium problem and set of fixed points of finite family of relatively nonexpansive multi-valued mappings in Banach spaces. Our scheme does not involve computation of C_n and Q_n , for each $n \geq 0$, and the requirement that the interior of F is nonempty is dispensed with. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

2. Preliminaries

Let E be a normed linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau \right\}. \quad (2.1)$$

The space E is said to be *smooth* if $\rho_E(\tau) > 0$, for all $\tau > 0$ and E is called *uniformly smooth* if and only if $\lim_{t \rightarrow 0^+} (\rho_E(t)/t) = 0$.

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}. \quad (2.2)$$

E is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0$, for every $\epsilon \in (0, 2]$.

In the sequel, we will need the following results.

Lemma 2.1 (see [1]). *Let K be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then for all $y \in K$,*

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x). \quad (2.3)$$

We make use of the function $V : E \times E^* \rightarrow \mathbb{R}$, defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad \forall x \in E, x^* \in E^*, \quad (2.4)$$

studied by Alber [1]. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

Lemma 2.2 (see [1]). *Let E be reflexive strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \quad (2.5)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.3 (see [1]). *Let C be a convex subset of a real smooth Banach space E . Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall z \in C. \quad (2.6)$$

Lemma 2.4 (see [23]). Let E be a uniformly convex Banach space and $B_R(0)$ be a closed ball of E . Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N\|^2 \leq \sum_{i=1}^N \alpha_i \|x_i\|^2 - \alpha_i \alpha_j g(\|x_i - x_j\|), \quad (2.7)$$

for $i, j \in \{1, \dots, N\}$, $\alpha_i \in (0, 1)$ such that $\sum_{i=1}^N \alpha_i = 1$, and $x_i \in B_R(0) := \{x \in E : \|x\| \leq R\}$, for $i = 1, 2, \dots, N$.

Lemma 2.5 (see [24]). Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E . If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $x_n - y_n \rightarrow 0$, as $n \rightarrow \infty$.

Proposition 2.6 (see [3]). Let E be a strictly convex and smooth Banach space and C be a nonempty closed convex subset of E . Let $T : C \rightarrow N(C)$ be a relatively nonexpansive multi-valued mapping. Then $F(T)$ is closed and convex.

Lemma 2.7 (see [16]). Let C be a nonempty, closed and convex subset of a uniformly smooth, strictly convex and reflexive real Banach space E . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). For $r > 0$ and $x \in E$, define the mapping $F_r : E \rightarrow C$ as follows:

$$F_r x := \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}. \quad (2.8)$$

Then the following statements hold:

- (1) F_r is single-valued,
- (2) $F(F_r) = EP(f)$,
- (3) $\phi(q, F_r x) + \phi(F_r x, x) \leq \phi(q, x)$, for $q \in F(F_r)$,
- (4) $EP(f)$ is closed and convex.

Lemma 2.8 (see [25]). Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}. \quad (2.9)$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.9 (see [26]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \beta_n) a_n + \beta_n \delta_n, \quad n \geq n_0, \text{ for some } n_0 \in \mathbb{N}, \quad (2.10)$$

where $\{\beta_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Result

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space E with dual E^* . Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction. For the rest of this paper, $F_{r_n}x$ is a mapping defined as follows. For $x \in E$, let $F_{r_n} : E \rightarrow C$ be given by

$$F_{r_n}x := \left\{ z \in C : f(z, y) + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad (3.1)$$

where $\{r_n\}_{n \in \mathbb{N}} \subset [c_1, \infty)$, for some $c_1 > 0$.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E . Let $f : C \times C \rightarrow \mathbb{R}$, be a bifunction which satisfies conditions (A1)–(A4). Let $T_i : C \rightarrow CB(C)$, for $i = 1, 2, \dots, N$, be a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := \bigcap_{i=1}^N F(T_i) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily,} \\ w_n &= F_{r_n}x_n, \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n), \\ x_{n+1} &= J^{-1} \left(\beta_{n,0}Jw_n + \sum_{i=1}^N \beta_{n,i}Ju_{n,i} \right), \quad u_{n,i} \in T_i y_n, \quad n \geq 0, \end{aligned} \quad (3.2)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \dots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \dots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F .

Proof. Since F is nonempty closed and convex, put $x^* := \Pi_F w$. Now from (3.2), Lemma 2.7(3) and property of ϕ , we get that

$$\begin{aligned} \phi(x^*, y_n) &= \phi(x^*, \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n)) \\ &\leq \phi(x^*, J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n)) \\ &= \|x^*\|^2 - 2\langle x^*, \alpha_n Jw + (1 - \alpha_n)Jw_n \rangle + \|\alpha_n Jw + (1 - \alpha_n)Jw_n\|^2 \\ &\leq \|x^*\|^2 - 2\alpha_n \langle x^*, Jw \rangle - 2(1 - \alpha_n) \langle x^*, Jw_n \rangle \\ &\quad + \alpha_n \|w\|^2 + (1 - \alpha_n) \|w_n\|^2 \\ &\leq \alpha_n \phi(x^*, w) + (1 - \alpha_n) \phi(x^*, w_n) \\ &= \alpha_n \phi(x^*, w) + (1 - \alpha_n) \phi(x^*, F_{r_n}x_n) \\ &\leq \alpha_n \phi(x^*, w) + (1 - \alpha_n) \phi(x^*, x_n). \end{aligned} \quad (3.3)$$

Now, from (3.2), Lemma 2.7(3), relatively nonexpansiveness of T_i , property of ϕ and (3.3), we have that

$$\begin{aligned}
 \phi(x^*, x_{n+1}) &= \phi\left(x^*, J^{-1}\left(\beta_{n,0}Jw_n + \sum_{i=1}^N \beta_{n,i}Ju_{n,i}\right)\right) \\
 &\leq \beta_{n,0}\phi(x^*, w_n) + \sum_{i=1}^N \beta_{n,i}\phi(x^*, u_{n,i}) \\
 &= \beta_{n,0}\phi(x^*, F_{r_n}x_n) + \sum_{i=1}^N \beta_{n,i}\phi(x^*, u_{n,i}) \\
 &\leq \beta_{n,0}\phi(x^*, x_n) + (1 - \beta_{n,0})\phi(x^*, y_n) \\
 &\leq \beta_{n,0}\phi(x^*, x_n) + (1 - \beta_{n,0})[\alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, x_n)] \\
 &\leq \delta_n\phi(x^*, w) + (1 - \delta_n)\phi(x^*, x_n),
 \end{aligned} \tag{3.4}$$

where $\delta_n = (1 - \beta_{n,0})\alpha_n$. Thus, by induction,

$$\phi(x^*, x_{n+1}) \leq \max\{\phi(x^*, x_0), \phi(x^*, w)\}, \quad \forall n \geq 0, \tag{3.5}$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{w_n\}$ are bounded. Now let $z_n = J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n)$. Then we have that $y_n = \Pi_C z_n$. Using Lemma 2.2 and property of ϕ , we obtain that

$$\begin{aligned}
 \phi(x^*, y_n) &\leq \phi(x^*, z_n) = V(x^*, Jz_n) \\
 &\leq V(x^*, Jz_n - \alpha_n(Jw - Jx^*)) - 2\langle z_n - x^*, -\alpha_n(Jw - Jx^*) \rangle \\
 &= \phi\left(x^*, J^{-1}(\alpha_n Jx^* + (1 - \alpha_n)Jw_n) + 2\alpha_n\langle z_n - x^*, Jw - Jx^* \rangle\right) \\
 &\leq \alpha_n\phi(x^*, x^*) + (1 - \alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle z_n - x^*, Jw - Jx^* \rangle \\
 &= (1 - \alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle z_n - x^*, Jw - Jx^* \rangle \\
 &\leq (1 - \alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle z_n - x^*, Jw - Jx^* \rangle.
 \end{aligned} \tag{3.6}$$

Furthermore, from (3.2), Lemma 2.4, relatively nonexpansiveness of T_i , for each $i = 1, 2, \dots, N$, Lemma 2.7(3), and (3.6) we have that

$$\begin{aligned}
 \phi(x^*, x_{n+1}) &= \phi\left(x^*, J^{-1}\left(\beta_{n,0}Jw_n + \sum_{i=1}^N \beta_{n,i}Ju_{n,i}\right)\right) \\
 &\leq \beta_{n,0}\phi(x^*, w_n) + \sum_{i=1}^N \beta_{n,i}\phi(x^*, u_{n,i}) \\
 &\quad - \beta_{n,0}\beta_{n,i}g(\|Jw_n - Ju_{n,i}\|) \\
 &= \beta_{n,0}\phi(x^*, F_{r_n}x_n) + \sum_{i=1}^N \beta_{n,i}\phi(x^*, u_{n,i}) \\
 &\quad - \beta_{n,0}\beta_{n,i}g(\|Jw_n - Ju_{n,i}\|) \\
 &\leq \beta_{n,0}(\phi(x^*, x_n) - \phi(x_n, w_n)) + (1 - \beta_{n,0})\phi(x^*, y_n) \\
 &\quad - \beta_{n,0}\beta_{n,i}g(\|Jw_n - Ju_{n,i}\|) \leq \beta_{n,0}\phi(x^*, x_n) - \beta_{n,0}\phi(x_n, w_n) + (1 - \beta_{n,0}) \\
 &\quad \times [(1 - \alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle z_n - x^*, Jw - Jx^* \rangle] - \beta_{n,0}\beta_{n,i}g(\|Jw_n - Ju_{n,i}\|) \\
 &= (1 - \delta_n)\phi(x^*, x_n) + 2\delta_n\langle z_n - x^*, Jw - Jx^* \rangle \\
 &\quad - \beta_{n,0}\phi(x_n, w_n) - \beta_{n,0}\beta_{n,i}g(\|Jw_n - Ju_{n,i}\|),
 \end{aligned} \tag{3.7}$$

and hence

$$\phi(x^*, x_{n+1}) \leq (1 - \delta_n)\phi(x^*, x_n) + 2\delta_n\langle z_n - x^*, Jw - Jx^* \rangle, \tag{3.8}$$

where $\delta_n := \alpha_n(1 - \beta_{n,0})$, for all $n \in \mathbb{N}$. Note that δ_n satisfies $\lim_n \delta_n = 0$ and $\sum_{n=1}^\infty \delta_n = \infty$. $\square$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}$ is nonincreasing for all $n \geq n_0$. In this situation, $\{\phi(x^*, x_n)\}$ is then convergent. Then from (3.7), we have that $\phi(x_n, w_n) \rightarrow 0$ and hence Lemma 2.5 implies that

$$x_n - w_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.9}$$

Moreover, from (3.7), we have that $\beta_{n,0}\beta_{n,i}g(\|Jw_n - Ju_{n,i}\|) \rightarrow 0$, as $n \rightarrow \infty$, which implies by the property of g that $Jw_n - Ju_{n,i} \rightarrow 0$, as $n \rightarrow \infty$, for each $i \in \{1, 2, \dots, N\}$, and hence, since J^{-1} uniformly continuous on bounded sets, we obtain that

$$w_n - u_{n,i} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.10}$$

Furthermore, by Lemma 2.1, property of ϕ and the fact that $\alpha_n \rightarrow 0$, as $n \rightarrow \infty$, imply that

$$\begin{aligned}\phi(w_n, y_n) &= \phi(w_n, \Pi_C z_n) \leq \phi(w_n, z_n) \\ &= \phi\left(w_n, J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n)\right) \\ &\leq \alpha_n \phi(w_n, w) + (1 - \alpha_n) \phi(w_n, w_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty,\end{aligned}\tag{3.11}$$

and hence

$$w_n - y_n \rightarrow 0, \quad w_n - z_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.\tag{3.12}$$

Therefore, from (3.9), (3.10), and (3.12), we obtain that

$$x_n - z_n \rightarrow 0, \quad y_n - x_n \rightarrow 0, \quad \text{as } n \rightarrow \infty,\tag{3.13}$$

$$d(y_n, T_i y_n) \leq \|y_n - u_{n,i}\| \leq \|y_n - w_n\| + \|w_n - u_{n,i}\| \rightarrow 0,\tag{3.14}$$

as $n \rightarrow \infty$, for each $i \in \{1, 2, \dots, N\}$.

Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightarrow z$ and $\limsup_{n \rightarrow \infty} \langle z_n - x^*, Jw - Jx^* \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - x^*, Jw - Jx^* \rangle$. Then, from (3.12), (3.13), and the uniform continuity of J , we get that

$$x_{n_i}, w_{n_i}, y_{n_i} \rightarrow z, \quad Jx_n - Jw_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.\tag{3.15}$$

Now, we show that $z \in EP(f)$. But, from the definition of w_n and (A2) we note that

$$\frac{1}{r_{n_i}} \langle y - w_{n_i}, Jw_{n_i} - Jx_{n_i} \rangle \geq -f(w_{n_i}, y) \geq f(y, w_{n_i}), \quad \forall y \in C.\tag{3.16}$$

Letting $i \rightarrow \infty$, we have from (3.15) and (A4) that $f(y, z) \leq 0$, for all $y \in C$. Now, for $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $f(y_t, z) \leq 0$. So, from the convexity of the equilibrium bifunction $f(x, y)$ on the second variable y , we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, z) \leq tf(y_t, y),\tag{3.17}$$

and hence $f(y_t, y) \geq 0$. Now, letting $t \rightarrow 0$ and condition (A3), we obtain that $f(z, y) \geq 0$, for all $y \in C$, and hence $z \in EP(f)$.

Next, we show that $z \in \cap_{i=1}^N F(T_i)$. But, since each T_i satisfies condition (B3) we obtain from (3.13) and (3.15) that $z \in F(T_i)$, for each $i = 1, 2, \dots, N$, and hence $z \in \cap_{i=1}^N F(T_i)$. Thus, from the above discussions we obtain that $z \in F := \cap_{i=1}^N F(T_i) \cap EP(f)$. Therefore, by Lemma 2.3, we immediately obtain that $\limsup_{n \rightarrow \infty} \langle z_n - x^*, Jw - Jx^* \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - x^*, Jw - Jx^* \rangle = \langle z - x^*, Jw - Jx^* \rangle \leq 0$. It follows from (3.8) and Lemma 2.9 that $\phi(x^*, x_n) \rightarrow 0$, as $n \rightarrow \infty$. Consequently, $x_n \rightarrow x^*$ by Lemma 2.5.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1}) \quad (3.18)$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.8, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$, $\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1})$ and $\phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1})$, for all $k \in \mathbb{N}$. Now, from (3.7) and the fact that $\delta_n \rightarrow 0$, we have

$$\begin{aligned} & \beta_{m_k,0}\phi(x_{m_k}, w_{m_k}) + \beta_{m_k,0}\beta_{m_k,i}g(\|Jw_{m_k} - Ju_{m_k,i}\|) \\ & \leq (\phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1})) - \delta_{m_k}\phi(x^*, x_{m_k}) + 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^* \rangle, \end{aligned} \quad (3.19)$$

as $k \rightarrow \infty$. Thus, using the same proof of Case 1, we obtain that $x_{m_k} - w_{m_k} \rightarrow 0$ and $w_{m_k} - u_{m_k,i} \rightarrow 0$, as $k \rightarrow \infty$, for each $i = 1, 2, \dots, N$ and hence

$$\limsup_{n \rightarrow \infty} \langle z_{m_k} - x^*, Jw - Jx^* \rangle \leq 0. \quad (3.20)$$

Then from (3.8), we have that

$$\phi(x^*, x_{m_k+1}) \leq (1 - \delta_{m_k})\phi(x^*, x_{m_k}) + 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^* \rangle. \quad (3.21)$$

Since $\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1})$, (3.21) implies that

$$\begin{aligned} \delta_{m_k}\phi(x^*, x_{m_k}) & \leq \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) + 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^* \rangle \\ & \leq 2\delta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^* \rangle. \end{aligned} \quad (3.22)$$

In particular, since $\delta_{m_k} > 0$, we get

$$\phi(x^*, x_{m_k}) \leq 2\langle z_{m_k} - x^*, Jw - Jx^* \rangle. \quad (3.23)$$

Then, from (3.20), we obtain that $\phi(x^*, x_{m_k}) \rightarrow 0$, as $k \rightarrow \infty$. This together with (3.21) gives $\phi(x^*, x_{m_k+1}) \rightarrow 0$, as $k \rightarrow \infty$. But $\phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1})$ for all $k \in \mathbb{N}$, thus we obtain that $x_k \rightarrow x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to x^* and the proof is complete.

If in Theorem 3.1, we assume that $f(x, y) = \langle Ax, y - x \rangle$, for A continuous monotone mapping, then we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E . Let $A : C \rightarrow E^*$ be a continuous monotone mapping.*

Let $T_i : C \rightarrow CB(C)$, for $i = 1, 2, \dots, N$, be a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := \cap_{i=1}^N F(T_i) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily,} \\ w_n &\in C \quad \text{such that } \langle Aw_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, Jw_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n), \\ x_{n+1} &= J^{-1} \left(\beta_{n,0} Jw_n + \sum_{i=1}^N \beta_{n,i} Ju_{n,i} \right), \quad u_{n,i} \in T_i y_n, \quad n \geq 0, \end{aligned} \quad (3.24)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \dots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \dots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F .

Proof. Let $f(x, y) = \langle Ax, y - x \rangle$. Since A is monotone and continuous, we get that a bifunction f satisfies conditions (A1)–(A4). Thus, the conclusion follows from Theorem 3.1. $\square$

If in Theorem 3.1, we assume that $N = 1$, then we get the following theorem.

Corollary 3.3. Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex real Banach space E . Let $f : C \times C \rightarrow \mathbb{R}$, be a bifunction which satisfies conditions (A1)–(A4). Let $T : C \rightarrow CB(C)$ be a relatively nonexpansive multi-valued mapping. Assume that $F := F(T) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily,} \\ w_n &= F_{r_n} x_n, \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n), \\ x_{n+1} &= J^{-1}(\beta_n Jw_n + (1 - \beta_n)Ju_n), \quad u_n \in Ty_n, \quad n \geq 0, \end{aligned} \quad (3.25)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0, 1)$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F .

Proof. The proof follows from Theorem 3.1 with $N = 1$. $\square$

If in Theorem 3.1, we assume that $f \equiv 0$, we get the following corollary.

Corollary 3.4. Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E . Let $T_i : C \rightarrow CB(C)$, for $i = 1, 2, \dots, N$, be a finite family of relatively

nonexpansive multi-valued mappings. Assume that $F := \cap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily,} \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jx_n), \\ x_{n+1} &= J^{-1} \left(\beta_{n,0} Jx_n + \sum_{i=1}^N \beta_{n,i} Ju_{n,i} \right), \quad u_{n,i} \in T_i y_n, \quad n \geq 0, \end{aligned} \quad (3.26)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \dots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \dots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F .

If in Theorem 3.1, we assume that each T_i , $i = 1, 2, \dots, N$ is single valued, we get the following corollary.

Corollary 3.5. Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E . Let $f : C \times C \rightarrow \mathbb{R}$, be a bifunction which satisfies conditions (A1)–(A4). Let $T_i : C \rightarrow C$, for $i = 1, 2, \dots, N$, be a finite family of relatively nonexpansive single-valued mappings. Assume that $F := \cap_{i=1}^N F(T_i) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily,} \\ w_n &= F_{r_n} x_n, \\ y_n &= \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n), \\ x_{n+1} &= J^{-1} \left(\beta_{n,0} Jw_n + \sum_{i=1}^N \beta_{n,i} JT_i y_n \right), \quad n \geq 0, \end{aligned} \quad (3.27)$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \dots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \dots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F .

If $E = H$, a real Hilbert space, then E is uniformly convex and uniformly smooth real Banach space. In this case, $J = I$, identity map on H and $\Pi_C = P_C$, projection mapping from H onto C . Thus, the following corollary holds.

Corollary 3.6. Let C be a nonempty, closed, and convex subset of a Hilbert space H . Let $f : C \times C \rightarrow \mathbb{R}$, be a bifunction which satisfies conditions (A1)–(A4). Let $T_i : C \rightarrow CB(C)$, for $i = 1, 2, \dots, N$, be

a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := \cap_{i=1}^N F(T_i) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_0 &= w \in C, \quad \text{chosen arbitrarily,} \\ w_n &= F_{r_n} x_n, \\ y_n &= P_C(\alpha_n w + (1 - \alpha_n) w_n), \\ x_{n+1} &= \beta_{n,0} w_n + \sum_{i=1}^N \beta_{n,i} u_{n,i}, \quad u_{n,i} \in T_i y_n, \quad n \geq 0, \end{aligned} \tag{3.28}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \dots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \dots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of F .

Remark 3.7. (1) Theorem 3.1 improves and extends the corresponding results of Kumam [22] and Takahashi and Zembayashi [16] in the sense that either our scheme does not require computation of C_{n+1} , for each $n \geq 1$, or the space considered is more general.

(2) Theorem 3.1 improves the corresponding results of Homaeipour and Razani [3] in the sense that our convergence is strong and the requirement that the interior of F is nonempty is dispensed with.

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