Research Article

On $t$-Derivations of BCI-Algebras

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We introduce the notion of $t$-derivation of a BCI-algebra and investigate related properties. Moreover, we study $t$-derivations in a $p$-semisimple BCI-algebra and establish some results on $t$-derivations in a $p$-semisimple BCI-algebra.

1. Introduction

The notion of BCK-algebra was proposed by Imai and Iséki in 1966 [1]. In the same year, Iséki introduced the notion of a BCI-algebra [2], which is a generalization of a BCK-algebra. A series of interesting notions concerning BCI-algebras were introduced and studied, several papers have been written on various aspects of these algebras [3–5]. Recently, in the year 2004 [6], Jun and Xin have applied the notion of derivation in BCI-algebras which is defined in a way similar to the notion of derivation in rings and near-rings theory which was introduced by Posner in 1957 [7]. In fact, the notion of derivation in ring theory is quite old and plays a significant role in analysis, algebraic geometry and algebra.

After the work of Jun and Xin (2004) [6], many research articles have appeared on the derivations of BCI-algebras in different aspects as follows: in 2005 [8], Zhan and Liu have given the notion of $f$-derivation of BCI-algebras and studied $p$-semisimple BCI-algebras by using the idea of regular $f$-derivation in BCI-algebras. In 2006 [9], Abujabal and Al-Shehri have extended the results of BCI-algebras. Further, in the next year 2007 [10], they defined and studied the notion of left derivation of BCI-algebras and investigated some properties of left derivation in $p$-semisimple BCI-algebras. In 2009 [11], Öztürk and Çeven have defined the notion of derivation and generalized derivation determined by a derivation for a complicated subtraction algebra and discussed some related properties. Also, in 2009 [12], Öztürk et al. have introduced the notion of generalized derivation in BCI-algebras and established some results. Further, they have given the idea of torsion free BCI-algebra and
explored some properties. In 2010 [13], Al-Shehri has applied the notion of left-right (resp., right-left) derivation in BCI-algebra to B-algebra and obtained some of its properties. In 2011 [14], Ilbira et al. have studied the notion of left-right (resp., right-left) symmetric biderivation in BCI-algebras.

Motivated by a lot of work done on derivations of BCI-algebras and on derivations of other related abstract algebraic structures, in this paper we introduce the notion of \( t \)-derivations on BCI-algebras and obtain some of its related properties. Further, we characterize the notion of \( p \)-semisimple BCI-algebra \( X \) by using the notion of \( t \)-derivation and show that if \( d_t \) and \( d'_t \) are \( t \)-derivations on \( X \), then \( d_t \circ d'_t \) is also a \( t \)-derivation and \( d_t \circ d'_t = d'_t \circ d_t \). Finally, we prove that \( d_t \circ d'_t = d'_t \circ d_t \), where \( d_t \) and \( d'_t \) are \( t \)-derivations on a \( p \)-semisimple BCI-algebra.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

**Definition 2.1** (see [2]). Let \( X \) be a set with a binary operation \( "\ast" \) and a constant 0. Then \( (X, \ast, 0) \) is called a BCI algebra if the following axioms are satisfied for all \( x, y, z \in X \):

1. \( ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0 \),
2. \( (x \ast (x \ast y)) \ast y = 0 \),
3. \( x \ast x = 0 \),
4. \( x \ast y = 0 \) and \( y \ast x = 0 \) \( \Rightarrow \) \( x = y \).

Define a binary relation \( \leq \) on \( X \) by letting \( x \ast y = 0 \) if and only if \( x \leq y \). Then \( (X, \leq) \) is a partially ordered set. A BCI-algebra \( X \) satisfying \( 0 \leq x \) for all \( x \in X \), is called BCK-algebra (see [1]).

In any BCI-algebra \( X \) for all \( x, y \in X \), the following properties hold:

1. \( (x \ast y) \ast z = (x \ast z) \ast y \).
2. \( x \ast 0 = x \).
3. \( (x \ast 0) \ast (y \ast z) \leq x \ast y \).
4. \( x \ast 0 = 0 \) implies \( x = 0 \).
5. \( x \leq y \) \( \Rightarrow \) \( x \ast z \leq y \ast z \) \( \text{and} \) \( z \ast y \leq z \ast x \). A BCI-algebra \( X \) is said to be associative if for all \( x, y, z \in X \), the following holds:
6. \( (x \ast y) \ast z = x \ast (y \ast z) \) [4]. Let \( X \) be a BCI-algebra, we denote \( X_e = \{ x \in X \mid 0 \leq x \} \), the BCK-part of \( X \) and by \( G(X) = \{ x \in X \mid 0 \ast x = x \} \), the BCI-G part of \( X \). If \( X_e = \{ 0 \} \), then \( X \) is called a \( p \)-semisimple BCI-algebra. In a \( p \)-semisimple BCI-algebra \( X \), the following properties hold:
7. \( x \ast (x \ast y) = y \).
8. \( x \ast (0 \ast y) = y \ast (0 \ast x) \).
9. \( x \ast y = 0 \) implies \( x = y \).
10. \( (x \ast z) \ast (y \ast z) = x \ast y \).
11. \( x \ast a = x \ast b \) implies \( a = b \) that is left cancelable.
12. \( a \ast x = b \ast x \) implies \( a = b \) that is right cancelable.
Definition 2.2 (see [6]). A subset \( S \) of a BCI-algebra \( X \) is called subalgebra of \( X \) if \( x \ast y \in S \) whenever \( x, y \in S \).

For a BCI-algebra \( X \), we denote \( x \land y = y \ast (y \ast x) \) for all \( x, y \in X \) [6]. For more details we refer to [3, 5, 6].

3. \( t \)-Derivations in a BCI-Algebra/p-Semisimple BCI-Algebra

The following definitions introduce the notion of \( t \)-derivation for a BCI-algebra.

Definition 3.1. Let \( X \) be a BCI-algebra. Then for any \( t \in X \), we define a self map \( d_t : X \to X \) by \( d_t(x) = x \ast t \) for all \( x \in X \).

Definition 3.2. Let \( X \) be a BCI-algebra. Then for any \( t \in X \), a self map \( d_t : X \to X \) is called a left-right \( t \)-derivation or \((l, r)\)-\( t \)-derivation of \( X \) if it satisfies the identity \( d_t(x \ast y) = (d_t(x) \ast y) \land (x \ast d_t(y)) \) for all \( x, y \in X \).

Similarly, we get the following.

Definition 3.3. Let \( X \) be a BCI-algebra. Then for any \( t \in X \), a self map \( d_t : X \to X \) is called a right-left \( t \)-derivation or \((r, l)\)-\( t \)-derivation of \( X \) if it satisfies the identity \( d_t(x \ast y) = (x \ast d_t(y)) \land (d_t(x) \ast y) \) for all \( x, y \in X \).

Moreover, if \( d_t \) is both a \((l, r)\)- and a \((r, l)\)-\( t \)-derivation on \( X \), we say that \( d_t \) is a \( t \)-derivation on \( X \).

Example 3.4. Let \( X = \{0, 1, 2\} \) be a BCI-algebra with the following Cayley table:

\[
\begin{array}{c|ccc}
  & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 0 \\
\end{array}
\] (3.1)

For any \( t \in X \), define a self map \( d_t : X \to X \) by \( d_t(x) = x \ast t \) for all \( x \in X \). Then it is easily checked that \( d_t \) is a \( t \)-derivation of \( X \).

Proposition 3.5. Let \( d_t \) be a self map of an associative BCI-algebra \( X \). Then \( d_t \) is a \((l, r)\)-\( t \)-derivation of \( X \).

Proof. Let \( X \) be an associative BCI-algebra, then we have

\[
d_t(x \ast y) = (x \ast y) \ast t \\
= \{x \ast (y \ast t)\} \ast 0 \quad \text{by Property (6) and (2)} \\
= \{x \ast (y \ast t)\} \ast \{\{x \ast (y \ast t)\} \ast \{x \ast (y \ast t)\}\} \quad \text{by Property (iii)} \\
= \{x \ast (y \ast t)\} \ast \{\{x \ast (y \ast t)\} \ast \{(x \ast y) \ast t\}\} \quad \text{by Property (6)}
\]
Proposition 3.6. Let $d_t$ be a self map of an associative BCI-algebra $X$. Then, $d_t$ is a $(r,l)$-$t$-derivation of $X$.

Proof. Let $X$ be an associative BCI-algebra, then we have

\[
\begin{align*}
(d_t(x*y)) &= (x*y)*t \\
&= \{ (x*t)*y \} * 0 \quad \text{by Property (1) and (2)} \\
&= \{ (x*t)*y \} * \{ (x*y)*t \} \quad \text{(as $x*x = 0$)} \\
&= \{ (x*t)*y \} * \{ (x*y)*t \} \quad \text{by Property (1)} \\
&= \{ (x*t)*y \} * \{ (x*y)*t \} \quad \text{by Property (6)} \\
&= (x*(y*t)) \land (x*t*y) \quad \text{(as $y*(y*x) = x \land y$)} \\
&= (x*(x*y)) \land (d_t(x)*y). 
\end{align*}
\]

Combining Propositions 3.5 and 3.6, we get the following Theorem.

Theorem 3.7. Let $d_t$ be a self map of an associative BCI-algebra $X$. Then, $d_t$ is a $t$-derivation of $X$.

Definition 3.8. A self map $d_t$ of a BCI-algebra $X$ is said to be $t$-regular if $d_t(0) = 0$.

Example 3.9. Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

(i) For any $t \in X$, define a self map $d_t : X \to X$ by

\[
d_t(x) = x*t = \begin{cases} 
  b & \text{if } x = 0, a \\
  0 & \text{if } x = b. 
\end{cases}
\]

Then it is easily checked that $d_t$ is $(l,r)$ and $(r,l)$-$t$-derivations of $X$, which is not $t$-regular.
Proposition 3.10. Let $d_t$ be a self map of a BCI-algebra $X$. Then

(i) If $d_t$ is a $(l, r)$-derivation of $X$, then $d_t(x) = d_t(x) \wedge x$ for all $x \in X$.

(ii) If $d_t$ is a $(r, l)$-derivation of $X$, then $d_t(x) = x \wedge d_t(x)$ for all $x \in X$ if and only if $d_t$ is $t$-regular.

Proof of (i). Let $d_t$ be a $(l, r)$-t-derivation of $X$, then

$$d_t(x) = d_t(x \ast 0) = (d_t(x \ast 0) \wedge (x \ast d_t(0))) = d_t(x) \wedge (x \ast d_t(0)) = [x \ast d_t(0)] \ast ([x \ast d_t(0)] \ast d_t(x)) = [x \ast d_t(0)] \ast ([x \ast d_t(x)] \ast d_t(0)) \leq x \ast [x \ast d_t(x)] \text{ by Property (3)} = d_t(x) \wedge x.$$ 

But $d_t(x) \wedge x \leq d_t(x)$ is trivial so (i) holds. \(\square\)

Proof of (ii). Let $d_t$ be a $(r, l)$-t-derivation of $X$. If $d_t(x) = x \wedge d_t(x)$ then

$$d_t(0) = 0 \wedge d_t(0) = d_t(0) \ast \{d_t(0) \ast 0\} = d_t(0) \ast d_t(0) = 0$$

thereby implying $d_t$ is $t$-regular. Conversely, suppose that $d_t$ is $t$-regular, that is $d_t(0) = 0$, then we have

$$d_t(x) = d_t(x \ast 0) = (x \ast d_t(0)) \wedge (d_t(x) \ast 0) = (x \ast 0) \wedge d_t(x) = x \wedge d_t(x).$$

This completes the proof. \(\square\)
Theorem 3.11. Let $d_i$ be a $(l, r)$-t-derivation of a $p$-semisimple BCI-algebra $X$. Then the following hold:

(i) $d_i(0) = d_i(x) \ast x$ for all $x \in X$.
(ii) $d_i$ is one-one.
(iii) If $d_i$ is $t$-regular, then it is an identity map.
(iv) if there is an element $x \in X$ such that $d_i(x) = x$, then $d_i$ is identity map.
(v) if $x \leq y$, then $d_i(x) \leq d_i(y)$ for all $x, y \in X$.

Proof of (i). Let $d_i$ be a $(l, r)$-t-derivation of a $p$-semisimple BCI-algebra $X$. Then for all $x \in X$, we have $x \ast x = 0$ and so

$$d_i(0) = d_i(x \ast x)$$

$$= (d_i(x) \ast x) \land (x \ast d_i(x))$$

$$= [x \ast d_i(x)] \ast [(x \ast d_i(x)) \ast d_i(x \ast x)]$$

(3.10)

$$= d_i(x \ast x) \text{ by property (7).}$$

Proof of (ii). Let $d_i(x) = d_i(y) \Rightarrow x \ast t = y \ast t$, then by property (12), we have $x = y$ and so $d_i$ is one-one.

Proof of (iii). Let $d_i$ be $t$-regular and $x \in X$. Then, $0 = d_i(0)$ so by the above part (i), we have $0 = d_i(x) \ast x$ and hence by property (9), we obtain $d_i(x) = x$ for all $x \in X$. Therefore, $d_i$ is the identity map.

Proof of (iv). It is trivial and follows from the above part (iii).

Proof of (v). Let $x \leq y$ implying $x \ast y = 0$. Now,

$$d_i(x) \ast d_i(y) = (x \ast t) \ast (y \ast t)$$

$$= x \ast y \text{ by property (10)}$$

(3.11)

$$= 0.$$

Therefore, $d_i(x) \leq d_i(y)$. This completes the proof.

Definition 3.12. Let $d_i$ be a $t$-derivation of a BCI-algebra $X$. Then, $d_i$ is said to be an isotone $t$-derivation if $x \leq y \Rightarrow d_i(x) \leq d_i(y)$ for all $x, y \in X$.

Example 3.13. In Example 3.9(ii), $d_i'$ is an isotone $t$-derivation, while in Example 3.9(i), $d_i$ is not an isotone $t$-derivation.

Proposition 3.14. Let $X$ be a BCI-algebra and $d_i$ be a $t$-derivation on $X$. Then for all $x, y \in X$, the following hold:

(i) If $d_i(x \land y) = d_i(x) \land d_i(y)$, then $d_i$ is an isotone $t$-derivation.
(ii) If $d_i(x \ast y) = d_i(x) \ast d_i(y)$, then $d_i$ is an isotone $t$-derivation.
Proof of (i). Let $d_i(x \land y) = d_i(x) \land d_i(y)$. If $x \leq y \implies x \land y = x$ for all $x, y \in X$. Therefore, we have

\[
\begin{align*}
d_i(x) &= d_i(x \land y) \\
&= d_i(x) \land d_i(y) \\
&\leq d_i(y).
\end{align*}
\]

Henceforth $d_i(x) \leq d_i(y)$ which implies that $d_i$ is an isotone $t$-derivation. \hfill \square

Proof of (ii). Let $d_i(x \ast y) = d_i(x) \ast d_i(y)$. If $x \leq y \implies x \ast y = 0$ for all $x, y \in X$. Therefore, we have

\[
\begin{align*}
d_i(x) &= d_i(x \ast 0) \\
&= d_i(x \ast (x \ast y)) \\
&= d_i(x) \ast d_i(x \ast y) \\
&= d_i(x) \ast \{d_i(x) \ast d_i(y)\} \\
&\leq d_i(y) \quad \text{by property (ii)}.
\end{align*}
\]

Thus, $d_i(x) \leq d_i(y)$. This completes the proof. \hfill \square

**Theorem 3.15.** Let $d_i$ be a $t$-regular $(r, l)$-$t$-derivation of a BCI-algebra $X$. Then, the following hold:

(i) $d_i(x) \leq x$ for all $x \in X$.

(ii) $d_i(x) \ast y \leq x \ast d_i(y)$ for all $x, y \in X$.

(iii) $d_i(x \ast y) = d_i(x) \ast y \leq d_i(x) \ast d_i(y)$ for all $x, y \in X$.

(iv) $\ker(d_i) := \{x \in X : d_i(x) = 0\}$ is a subalgebra of $X$.

Proof of (i). For any $x \in X$, we have $d_i(x) = d_i(x \ast 0) = (x \ast d_i(0)) \land (d_i(x) \ast 0) = (x \ast 0) \land (d_i(x) \ast 0) = x \land d_i(x) \leq x$. \hfill \square

Proof of (ii). Since $d_i(x) \leq x$ for all $x \in X$, then $d_i(x) \ast y \leq x \ast y \leq x \ast d_i(y)$ and hence the proof follows. \hfill \square

Proof of (iii). For any $x, y \in X$, we have

\[
\begin{align*}
d_i(x \ast y) &= (x \ast d_i(y)) \land (d_i(x) \ast y) \\
&= \{d_i(x) \ast y\} \ast \{d_i(x) \ast y\} \\
&= \{d_i(x) \ast y\} \ast 0 \\
&\leq d_i(x) \ast d_i(y).
\end{align*}
\]

\hfill \square
Proof of (iv). Let \( x, y \in \ker(d_i) \implies d_i(x) = 0 = d_i(y) \). From (iii), we have \( d_i(x \ast y) \leq d_i(x) \ast d_i(y) = 0 \ast 0 = 0 \) implying \( d_i(x \ast y) \leq 0 \) and so \( d_i(x \ast y) = 0 \). Therefore, \( x \ast y \in \ker(d_i) \). Consequently \( \ker(d_i) \) is a subalgebra of \( X \). This completes the proof. \( \square \)

**Definition 3.16.** Let \( X \) be a BCI-algebra and let \( d_i, d'_i \) be two self maps of \( X \). Then we define \( d_i \circ d'_i : X \to X \) by \( (d_i \circ d'_i)(x) = d_i(d'_i(x)) \) for all \( x \in X \).

**Example 3.17.** Let \( X = \{0, a, b\} \) be a BCI algebra which is given in Example 3.4. Let \( d_i \) and \( d'_i \) be two self maps on \( X \) as defined in Example 3.9(i) and Example 3.9(ii), respectively.

Now, define a self map \( d_i \circ d'_i : X \to X \) by

\[
(d_i \circ d'_i)(x) = \begin{cases} 
0 & \text{if } x = a, b \\
1 & \text{if } x = 0.
\end{cases}
\]  

(3.15)

Then, it is easily checked that \( (d_i \circ d'_i)(x) = d_i(d'_i(x)) \) for all \( x \in X \).

**Proposition 3.18.** Let \( X \) be a \( p \)-semisimple BCI-algebra \( X \) and let \( d_i, d'_i \) be \( (l, r) \)-\( t \)-derivations of \( X \). Then, \( d_i \circ d'_i \) is also a \( (l, r) \)-\( t \)-derivation of \( X \).

**Proof.** Let \( X \) be a \( p \)-semisimple BCI-algebra. \( d_i \) and \( d'_i \) are \( (l, r) \)-\( t \)-derivations of \( X \). Then for all \( x, y \in X \), we get

\[
(d_i \circ d'_i)(x \ast y) = d_i(d'_i(x \ast y)) \\
= d_i [(d'_i(x) \ast y) \land (x \ast d'_i(y))] \\
= d_i [(x \ast d'_i(y)) \ast [(x \ast d'_i(y)) \ast (d'_i(x) \ast y)]] \\
= d_i(d'_i(x) \ast y) \text{ by property (7)} \\
= \{x \ast d_i(d'_i(y))\} \ast \{[x \ast d_i(d'_i(y))] \ast [d_i(d'_i(x) \ast y)]\} \\
= \{d_i(d'_i(x) \ast y)\} \land [x \ast d_i(d'_i(y))] \\
= ((d_i \circ d'_i)(x) \ast y) \land (x \ast (d_i \circ d'_i)(y)).
\]  

(3.16)

Therefore, \( (d_i \circ d'_i) \) is a \( (l, r) \)-\( t \)-derivation of \( X \).

Similarly, we can prove the following. \( \square \)

**Proposition 3.19.** Let \( X \) be a \( p \)-semisimple BCI-algebra and let \( d_i, d'_i \) be \( (r, l) \)-\( t \)-derivations of \( X \). Then \( d_i \circ d'_i \) is also a \( (r, l) \)-\( t \)-derivation of \( X \).

Combining Propositions 3.18 and 3.19, we get the following.

**Theorem 3.20.** Let \( X \) be a \( p \)-semisimple BCI-algebra and let \( d_i, d'_i \) be \( t \)-derivations of \( X \). Then, \( d_i \circ d'_i \) is also a \( t \)-derivation of \( X \).

Now, we prove the following theorem.

**Theorem 3.21.** Let \( X \) be a \( p \)-semisimple BCI-algebra and let \( d_i, d'_i \) be \( t \)-derivations of \( X \). Then \( d_i \circ d'_i = d'_i \circ d_i \).
Proof. Let $X$ be a $p$-semisimple BCI-algebra. $d_t$ and $d'_t$, $t$-derivations of $X$. Suppose $d'_t$ is a \((l,r)\)-\(t\)-derivation, then for all $x, y \in X$, we have

\[
(d_t \circ d'_t)(x \ast y) = d_t(d'_t(x \ast y))
\]
\[
= d_t[(d'_t(x) \ast y) \land (x \ast d'_t(y))]
\]
\[
= d_t[(x \ast d'_t(y)) \ast ((x \ast d'_t(y)) \ast (d'_t(x) \ast y))]
\]
\[
= d_t(d'_t(x) \ast y) \quad \text{by property (7).}
\]

As $d_t$ is a \((r,l)\)-\(t\)-derivation, then

\[
= (d'_t(x) \ast d_t(y)) \land (d_t(d'_t(x)) \ast y)
\]
\[
= d'_t(x) \ast d_t(y).
\]

Again, if $d_t$ is a \((r,l)\)-\(t\)-derivation, then we have

\[
(d'_t \circ d_t)(x \ast y) = d'_t[d_t(x \ast y)]
\]
\[
= d'_t[(x \ast d_t(y)) \land (d_t(x) \ast y)]
\]
\[
= d'_t[x \ast d_t(y)] \quad \text{by property (7)}
\]

But $d'_t$ is a \((l,r)\)-\(t\)-derivation, then

\[
= (d'_t(x) \ast d_t(y)) \land (x \ast d'_t(d_t(y)))
\]
\[
= d'_t(x) \ast d_t(y).
\]

Therefore from (3.18) and (3.20), we obtain

\[
(d_t \circ d'_t)(x \ast y) = (d'_t \circ d_t)(x \ast y).
\]

By putting $y = 0$, we get

\[
(d_t \circ d'_t)(x) = (d'_t \circ d_t)(x) \quad \forall x \in X.
\]

Hence, $d_t \circ d'_t = d'_t \circ d_t$. This completes the proof. \(\Box\)

**Definition 3.22.** Let $X$ be a BCI-algebra and let $d_t, d'_t$ be two self maps of $X$. Then we define $d_t \ast d'_t : X \to X$ by $(d_t \ast d'_t)(x) = d_t(x) \ast d'_t(x)$ for all $x \in X$. 
Example 3.23. Let $X = \{0, a, b\}$ be a BCI algebra which is given in Example 3.4. Let $d_i$ and $d_i'$ be two self maps on $X$ as defined in Example 3.9(i) and Example 3.9(ii), respectively.

Now, define a self map $d_i * d_i' : X \rightarrow X$ by

$$(d_i * d_i')(x) = \begin{cases} 0 & \text{if } x = a, b \\ b & \text{if } x = 0. \end{cases} \quad (3.23)$$

Then, it is easily checked that $(d_i * d_i')(x) = d_i(x) * d_i'(x)$ for all $x \in X$.

Theorem 3.24. Let $X$ be a p-semisimple BCI-algebra and let $d_i, d_i'$ be t-derivations of $X$. Then $d_i * d_i' = d_i' * d_i$.

Proof. Let $X$ be a p-semisimple BCI-algebra. $d_i$ and $d_i'$, t-derivations of $X$.

Since $d_i'$ is a $(r, l)$-t-derivation of $X$, then for all $x, y \in X$, we have

$$(d_i \circ d_i')(x * y) = d_i[d_i'(x * y)]$$

$$= d_i[(x * d_i'(y)) \wedge (d_i'(x) * y)] \quad (3.24)$$

$$= d_i[x * d_i'(y)] \quad \text{by property (7).}$$

But $d_i$ is a $(l, r)$-t-derivation, so

$$= (d_i(x) * d_i'(y)) \wedge (x * d_i(d_i'(y)))$$

$$= d_i(x) * d_i'(y). \quad (3.25)$$

Again, if $d_i'$ is a $(l, r)$-t-derivation of $X$, then for all $x, y \in X$, we have

$$(d_i \circ d_i')(x * y) = d_i[d_i'(x * y)]$$

$$= d_i[(d_i'(x) * y) \wedge (x * d_i'(y))]$$

$$= d_i[(x * d_i'(y)) \ast ((x * d_i'(y)) \ast (d_i'(x) * y))] \quad (3.26)$$

$$= d_i(d_i'(x) * y) \quad \text{by property (7).}$$

As $d_i$ is a $(r, l)$-t-derivation, then

$$= (d_i'(x) * d_i'(y)) \wedge (d_i(d_i'(x)) * y)$$

$$= d_i'(x) * d_i(y). \quad (3.27)$$
Henceforth from (3.25) and (3.27), we conclude

$$d_t(x) \ast d'_t(y) = d'_t(x) \ast d_t(y)$$  \hspace{1cm} (3.28)

By putting $y = x$, we get

$$d_t(x) \ast d'_t(x) = d'_t(x) \ast d_t(x)$$

$$ (d_t \ast d'_t)(x) = (d'_t \ast d_t)(x) \hspace{1cm} \forall x \in X. \hspace{1cm} (3.29)$$

Hence, $d_t \ast d'_t = d'_t \ast d_t$. This completes the proof. \hfill \Box

### 4. Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. The theory of derivations of algebraic structures is a direct descendant of the development of classical Galois theory (namely, Suzuki [15] and Van der Put and Singer [16, 17]) and the theory of invariants. An extensive and deep theory has been developed for derivations in algebraic structures viz. BCI-algebras, C*-algebras, commutative Banach algebras and Galois theory of linear differential equations (see, e.g., Jun and Xin [6], Ara and Mathieu [18], Bonsall and Duncan [19], Murphy [20] and Villena [21] where further references can be found). It plays a significant role in functional analysis; algebraic geometry; algebra and linear differential equations.

In the present paper, we have considered the notion of $t$-derivations in BCI-algebras and investigated the useful properties of the $t$-derivations in BCI-algebras. Finally, we investigated the notion of $t$-derivations in a $p$-semisimple BCI-algebra and established some results on $t$-derivations in a $p$-semisimple BCI-algebra. In our opinion, these definitions and main results can be similarly extended to other algebraic systems such as subtraction algebras [11], B-algebras [13], MV-algebras [22], $d$-algebras, Q-algebras and so forth. In future we can study the notion of $t$-derivations on various algebraic structures which may have a lot of applications in different branches of theoretical physics, engineering and computer science. It is our hope that this work would serve as a foundation for the further study in the theory of derivations of BCK/BCI-algebras.

In our future study of $t$-derivations in BCI-algebras, may be the following topics should be considered:

1. to find the generalized $t$-derivations of BCI-algebras,
2. to find more results in $t$-derivations of BCI-algebras and its applications,
3. to find the $t$-derivations of B-algebras, Q-algebras, subtraction algebras, $d$-algebra and so forth.

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