Research Article

A Modified Halpern’s Iterative Scheme for Solving Split Feasibility Problems

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The purpose of this paper is to introduce and study a modified Halpern’s iterative scheme for solving the split feasibility problem (SFP) in the setting of infinite-dimensional Hilbert spaces. Under suitable conditions a strong convergence theorem is established. The main result presented in this paper improves and extends some recent results done by Xu (Iterative methods for the split feasibility problem in infinite-dimensional Hilbert space, Inverse Problem 26 (2010) 105018) and some others.

1. Introduction

Let $C$ and $Q$ be nonempty-closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. Let $A$ be a linear-bounded operator from $H_1$ to $H_2$. The split feasibility problem (SFP) is finding a point $\tilde{x}$ satisfying the following property:

$$\tilde{x} \in C, \quad A\tilde{x} \in Q.$$  \hspace{1cm} (1.1)

The SFP was introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and medical image reconstruction [2], and very well-known iterative algorithms have been invented to solve it [2].

We use $\Gamma$ to denote the solution set of SFP:

$$\Gamma = \{\tilde{x} \in C : A\tilde{x} \in Q\},$$  \hspace{1cm} (1.2)
and assume that the SFP (1.1) is consistent (i.e., (1.1) has a solution) so that \( \Gamma \) is closed, convex, and nonempty, it is not hard to see that \( x \in C \) solves (1.1) if and only if it solves the following fixed point equation;

\[
x = P_C(I - \gamma A^*(I - P_Q)A)x, \quad x \in C,
\]

where \( P_C \) and \( P_Q \) are the (orthogonal) projections onto \( C \) and \( Q \), respectively, \( \gamma > 0 \) is any positive constant and \( A^* \) denotes the adjoint of \( A \). Moreover, for sufficiently small \( \gamma > 0 \), the operator \( P_C(I - \gamma A^*(I - P_Q)A) \) which defines the fixed point equation in (1.3) is nonexpansive.

To solve the SFP (1.1), Byrne [2] proposed his CQ algorithm (see also [3]) which generates a sequence \( \{x_n\} \) by

\[
x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \quad n \geq 0,
\]

where \( \gamma \in (0, 2/\lambda) \) with \( \lambda \) being the spectral radius of the operator \( A^*A \).

Very recently, Xu [4] has viewed the CQ algorithm for averaged mappings and applied Mann’s algorithm to solving the SFP, and he also proved that an averaged CQ algorithm is weakly convergent to a solution of the SFP.

In this paper, we also regard the CQ algorithm as a fixed point algorithm for averaged mappings and try to study the SFP by the following modified Halpern’s iterative scheme;

\[
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \xi A^*(I - P_Q)A)x_n, \quad n \geq 0,
\]

where \( \{\alpha_n\} \), \( \{\beta_n\} \), and \( \{\gamma_n\} \) are three sequences in \([0,1]\) satisfying \( \alpha_n + \beta_n + \gamma_n = 1 \). Furthermore, our result extends and improves the result of Xu [4] from weak to strong convergence theorems.

2. Preliminaries

Throughout the paper, we adopt the following notation.

Let \( x_n \) be a sequence and \( x \) be a point in a normed space \( X \). We use \( x_n \to x \) and \( x_n \rightharpoonup x \) to denote strong and weak convergence to \( x \) of the sequence \( \{x_n\} \), respectively. In addition, we use \( \omega^w(x_n) \) to denote the weak \( \omega \)-limit set of the sequence \( \{x_n\} \); namely,

\[
\omega^w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.
\]  

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \|\cdot\| \), respectively, and let \( K \) be a nonempty-closed convex subset of \( H \). For every point \( x \in H \), there exists a unique nearest point in \( K \), denoted by \( P_Kx \), such that

\[
\|x - P_Kx\| \leq \|x - y\|, \quad \forall y \in K,
\]
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$P_K$ is called the metric projection of $H$ onto $K$. It is well known that $P_K$ is a nonexpansive mapping of $H$ onto $K$ and satisfies

$$\langle x - y, P_K x - P_K y \rangle \geq \|P_K x - P_K y\|^2, \quad (2.3)$$

for every $x, y \in H$. Moreover, $P_K x$ is characterized by the following properties: $P_K x \in K$ and

$$\langle x - P_K x, y - P_K x \rangle \leq 0,$$

$$\|x - y\|^2 \geq \|x - P_K x\|^2 + \|y - P_K x\|^2, \quad (2.4)$$

for all $x \in H, y \in K$.

Some important properties of projections are gathered in the following proposition.

**Proposition 2.1.** Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in K. \quad (2.5)$$

One also needs other sorts of nonlinear operators which are introduced below.

Let $T, A : H \to H$ be the nonlinear operators.

1. $T$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.
2. $T$ is firmly nonexpansive if $2T - I$ is nonexpansive. Equivalent, $T = (I + S)/2$, where $S : H \to H$ is nonexpansive. Alternatively, $T$ is firmly nonexpansive if and only if

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad x, y \in H. \quad (2.6)$$

3. $T$ is averaged if $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. In this case, one also says that $T$ is $\alpha$-averaged. A firmly nonexpansive mapping is $(1/2)$-averaged.
4. $A$ is monotone if $\langle Ax - Ay, x - y \rangle \geq 0$ for $x, y \in H$.
5. $A$ is $\beta$-strongly monotone, with $\beta > 0$, if

$$\langle x - y, Ax - Ay \rangle \geq \beta\|x - y\|^2, \quad x, y \in H. \quad (2.7)$$

6. $A$ is $\nu$-inverse strongly monotone ($\nu$-ism), with $\nu > 0$, if

$$\langle x - y, Ax - Ay \rangle \geq \nu\|Ax - Ay\|^2, \quad x, y \in H. \quad (2.8)$$

It is well known that both $P_K$ and $I - P_K$ are firmly nonexpansive and $(1/2)$-ism.

Denote by $\text{Fix}(T)$ the set of fixed points of a self-mapping $T$ defined on $H$, (i.e., $\text{Fix}(T) = x \in H : Tx = x$).
Proposition 2.2 (see [2, 5]). One has the following assertions.

1. $T$ is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.
2. If $T$ is $(\nu, \gamma)$-ism and $\gamma > 0$, then $\gamma T$ is $(\nu/\gamma)$-ism.
3. $T$ is averaged if and only if the complement $I - T$ is $\nu$-ism, for some $\nu > (1/2)$.
   Indeed, for $\alpha \in (0, 1)$, $T$ is $\alpha$-averaged if and only if $I - T$ is $(1/2\alpha)$-ism.
4. If $T_1$ is $\alpha_1$-averaged and $T_2$ is $\alpha_2$-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1T_2$ is $\alpha$-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.
5. If $T_1$ and $T_2$ are averaged and have a common fixed point, then $\text{Fix}(T_1T_2) = \text{Fix}(T_1) \cap \text{Fix}(T_2)$.

Lemma 2.3 (see [6]). Let $K$ be a nonempty-closed convex subset of a real Hilbert space $H$ and $T$ be nonexpansive mapping on $K$ with $\text{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in $K$ which converges weakly to $x$ and if $(I - T)x_n$ converges strongly to $y$, then $y = (I - T)x$. In particular, if $y = 0$, then $x \in \text{Fix}(T)$.

Lemma 2.4 (see [7]). Let $(E, \langle \cdot , \cdot \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, one has

$$
\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta \|x - y\|^2 - \alpha \gamma \|x - z\|^2 - \beta \gamma \|y - z\|^2.
$$

Lemma 2.5 (see [8]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$
a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n,
$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

1. $\sum_{n=0}^{\infty} \gamma_n = \infty$;
2. $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n \to \infty} a_n = 0$.

3. Main Result

Let $C$ be a nonempty closed and convex subset of a Hilbert space $H$. For any $u, x_0 \in C$, we define the sequence $\{x_n\}$ by

$$
x_{n+1} = a_n u + \beta_n x_n + \gamma_n P_C(I - \xi A^*(I - P_Q)A)x_n, \quad n \geq 0,
$$

where $\{a_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ and satisfy $\alpha_n + \beta_n + \gamma_n = 1$.

Theorem 3.1. Suppose that the SFP is consistent and $0 < \xi < (2/\|A\|^2)$. Let $\{x_n\}$ be a sequence defined as in (3.1). If the following assumptions are satisfied:
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(C1) \( \lim_{n \to \infty} \alpha_n = 0 \) but \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

(C2) \( \lim \sup_{n \to \infty} \beta_n < 1 \),

(C3) the sums \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| \), \( \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| \) and \( \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| \) are finite.

Then \( \{x_n\} \) converges strongly to a solution of the SFP (1.1).

Proof. We firstly show that the sequence \( \{x_n\} \) is bounded. For our convenience, we take \( T := P_C(I - \xi A^*(I - P_Q)A) \). Then, for any \( x^* \in \Gamma \), we have \( Tx^* = x^* \). Now, we observe that

\[
\|x_{n+1} - x^*\| \leq \|\alpha_n u + \beta_n x_n + \gamma_n Tx_n - x^*\| \\
\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|Tx_n - x^*\|. \tag{3.2}
\]

Now, we note that the condition \( 0 < \xi < (2/\|A\|^2) \) implies that the operator \( P_C(I - \xi A^*(I - P_Q)A) \) is averaged. Since \( I - P_Q \) is firmly nonexpansive mappings and so is \( (1/2) \)-averaged, which is 1-ism. Also observe that \( A^*(I - P_Q)A \) is \( (1/\|A\|^2) \)-ism so that \( \xi A^*(I - P_Q)A \) is \( (1/\xi \|A\|^2) \)-ism. Further, from the fact that \( I - \xi A^*(I - P_Q)A \) is \( (\xi \|A\|^2/2) \)-averaged and \( P_C \) is \( (1/2) \)-averaged, we may obtain that \( P_C(I - \xi A^*(I - P_Q)A) \) is \( \chi \)-averaged, where

\[
\chi = \frac{1}{2} + \frac{\xi \|A\|^2}{2} - \frac{1}{2} \cdot \frac{\xi \|A\|^2}{2} = \frac{2 + \xi \|A\|^2}{4}. \tag{3.3}
\]

This implies that \( T = \chi I + (1 - \chi)S \), where \( \chi = (2 + \xi \|A\|^2/4) \in (0, 1) \) for some nonexpansive mappings \( S \). Note that \( T \) is also nonexpansive mappings. Hence, we have

\[
\|Tx_n - x^*\| = \|Tx_n - Tx^*\| \leq \|x_n - x^*\|. \tag{3.4}
\]

From the inequalities (3.2) and (3.4), we have

\[
\|x_{n+1} - x^*\| \leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|x_n - x^*\| \\
\leq \max \{\|u - x^*\|, \|x_n - x^*\|\}. \tag{3.5}
\]

Continuing inductively, we may obtain that the inequality

\[
\|x_{n+1} - x^*\| \leq \max \{\|u - x^*\|, \|x_0 - x^*\|\}, \tag{3.6}
\]

holds for all \( n \geq 0 \). So, \( \{x_n\} \) is bounded so does \( \{Tx_n\} \).
Next, we will show that \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \). Observe that

\[
\| x_{n+1} - x_n \| = \| (\alpha_n u + \beta_n x_n + \gamma_n T x_n) - (\alpha_{n-1} u + \beta_{n-1} x_{n-1} + \gamma_{n-1} T x_{n-1}) \|
\leq \| (\alpha_n u + \beta_n x_n + \gamma_n T x_n) - (\alpha_n u + \beta_n x_{n-1} + \gamma_n T x_{n-1}) \|
+ \| (\alpha_n u + \beta_n x_{n-1} + \gamma_n T x_{n-1}) - (\alpha_{n-1} u + \beta_{n-1} x_{n-1} + \gamma_{n-1} T x_{n-1}) \| 
\leq (1 - \alpha_n) \| x_n - x_{n-1} \| + \| \alpha_n - \alpha_{n-1} \| \| u \| + \| \beta_n - \beta_{n-1} \| \| x_{n-1} \| 
+ \| \gamma_n - \gamma_{n-1} \| \| T x_{n-1} \| .
\]

Since \( \{ x_n \} \) and \( \{ T x_n \} \) are bounded, there exists \( M = \sup(\| u \|, \| x_{n+1} \|, \| T x_{n-1} \|) > 0 \) such that

\[
\| x_{n+1} - x_n \| \leq (1 - \alpha_n) \| x_n - x_{n-1} \| + M (| \alpha_n - \alpha_{n-1} | + | \beta_n - \beta_{n-1} | + | \gamma_n - \gamma_{n-1} |).
\]

According to Lemma 2.5 and the condition (C3), we have \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \).

We note that

\[
\| x_n - T x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T x_n \|
\leq \| x_n - x_{n+1} \| + \| \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n) T x_{n-1} - T x_n \|
= \| x_n - x_{n+1} \| + \| \alpha_n (u - T x_n) + \beta_n (x_n - T x_n) \|
\leq \| x_n - x_{n+1} \| + \alpha_n \| (u - T x_n) \| + \beta_n \| (x_n - T x_n) \|
= \frac{1}{1 - \beta_n} \| x_{n+1} - x_n \| + \frac{\alpha_n}{1 - \beta_n} \| u - T x_n \| .
\]

Consequently, by the condition (C1) and (C2), we also have \( \lim_{n \to \infty} \| x_n - T x_n \| = 0 \). Next, we will show that

\[
\limsup_{n \to \infty} \langle v - z_0, x_{n+1} - z_0 \rangle \leq 0, \quad \text{where } z_0 = P_I v .
\]

(3.10)

To show this, we can choose a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) such that

\[
\limsup_{n \to \infty} \langle v - z_0, T x_n - z_0 \rangle = \lim_{k \to \infty} \langle v - z_0, T x_{n_k} - z_0 \rangle .
\]

(3.11)

As \( \{ x_n \} \) is bounded, there exists a subsequence \( \{ x_{n_k} \} \) which converges weakly to \( z \). We may assume without loss of generality that \( x_{n_k} \to z \). Since \( \| T x_n - x_n \| \to 0 \), we obtain \( T x_{n_k} \to z \) as \( k \to \infty \). By Lemma 2.3, we obtain that \( z \in \text{Fix}(T) = \Gamma \).
Remark 3.3. Theorem 3.1 and Corollary 3.2 extend and improve the result of Xu to strong convergence theorems by using the modified Halpern’s iterative scheme.

Therefore, we compute

\[
\limsup_{n \to \infty} \langle v - z_0, x_n - z_0 \rangle = \limsup_{n \to \infty} \langle v - z_0, T x_n - z_0 \rangle \\
= \lim_{k \to \infty} \langle v - z_0, T x_{n_k} - z_0 \rangle \\
= \langle v - z_0, z - z_0 \rangle \leq 0.
\]

(3.12)

Therefore, we compute

\[
\|x_{n+1} - z_0\|^2 = \langle \alpha_n v + \beta_n x_n + \gamma_n T x_n - z_0, x_{n+1} - z_0 \rangle \\
= \alpha_n \langle v - z_0, x_{n+1} - z_0 \rangle + \beta_n \langle x_n - z_0, x_{n+1} - z_0 \rangle + \gamma_n \langle T x_n - z_0, x_{n+1} - z_0 \rangle \\
\leq \alpha_n \langle v - z_0, x_{n+1} - z_0 \rangle + \frac{1}{2} \beta_n \left( \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 \right) \\
+ \frac{1}{2} \gamma_n \left( \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 \right) \\
\leq \alpha_n \langle v - z_0, x_{n+1} - z_0 \rangle + \frac{1}{2} (1 - \alpha_n) \left( \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 \right)
\]

(3.13)

which implies that

\[
\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n) \left( \|x_n - z_0\|^2 \right) + 2 \alpha_n \langle v - z_0, x_{n+1} - z_0 \rangle.
\]

(3.14)

Finally, by (3.12), (3.14), and Lemma 2.5, we conclude that \( \{x_n\} \) converges to \( z_0 \). This completes the proof. \( \square \)

Letting \( \beta_n \equiv 0 \) of iterative scheme (3.1) in Theorem 3.1, then we obtain the following corollary.

**Corollary 3.2.** For any \( u, x_0 \in C \), one defines the sequence \( \{x_n\} \) by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C} \left( I - \xi A^{*} (I - P_{Q}) A \right) x_n, \quad n \geq 0,
\]

(3.15)

where \( \{\alpha_n\} \) is a sequence in \([0, 1]\). Suppose that the SFP is consistent and \( 0 < \xi < (2/\|A\|^2) \).

Let \( \{x_n\} \) be defined as in (3.15). If the following assumptions are satisfied:

1. \( \lim_{n \to \infty} \alpha_n = 0 \) but \( \sum_{n=1}^{\infty} \alpha_n = \infty \),
2. \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \).

Then \( \{x_n\} \) converges to a solution of the SFP (1.1).

**Remark 3.3.** Theorem 3.1 and Corollary 3.2 extend and improve the result of Xu [4] from weak to strong convergence theorems by using the modified Halpern’s iterative scheme.
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