Research Article

Generalized Carleson Measure Spaces and Their Applications

Chin-Cheng Lin¹ and Kunchuan Wang²

¹ Department of Mathematics, National Central University, Chung-Li 320, Taiwan
² Department of Applied Mathematics, National Dong Hwa University, Hualien 970, Taiwan

Correspondence should be addressed to Chin-Cheng Lin, clin@math.ncu.edu.tw

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We introduce the generalized Carleson measure spaces $\text{CMO}^{p,q}$ that extend BMO. Using Frazier and Jawerth’s $\varphi$-transform and sequence spaces, we show that, for $\alpha \in \mathbb{R}$ and $0 < p \leq 1$, the duals of homogeneous Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_p$ for $1 < q < \infty$ and $0 < q \leq 1$ are $\text{CMO}^{\alpha,q}_{\dot{F}_p^{-\alpha,q}}$ and $\text{CMO}^{\alpha+\alpha/(p-1),\infty}_{\dot{F}_p^{-\alpha,q}}$ (for any $r \in \mathbb{R}$), respectively. As applications, we give the necessary and sufficient conditions for the boundedness of wavelet multipliers and paraproduct operators acting on homogeneous Triebel-Lizorkin spaces.

1. Introduction

In 1972, Fefferman and Stein [1] proved that the dual of $H^1$ is the BMO space. In 1990, Frazier and Jawerth [2, Theorem 5.13] generalized the above duality to homogeneous Triebel-Lizorkin spaces $F^{\alpha,q}_p$. More precisely, they showed that the dual of $F^{\alpha,q}_1$ is $F^{-\alpha,q}_{\infty}$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, where $q'$ is the conjugate index of $q$. Throughout the paper, $q'$ is interpreted as $q' = \frac{q}{q-1}$ whenever $0 < q \leq 1$, and $q' = \frac{q}{(q-1)}$ for $1 < q \leq \infty$. Note that $\dot{F}^{0,2}_1 = H^1$ and $\text{BMO} = F^{0,2}_{\infty}$. For $\alpha \in \mathbb{R}$, $0 < p < 1$, and $0 < q < \infty$, it is known (cf. [2–4]) that the dual of $\dot{F}^{\alpha,q}_p$ is $\dot{F}^{-\alpha,q}_{\infty}$. Here, we will give another characterization for the duals of $\dot{F}^{\alpha,q}_p$ in terms of the generalized Carleson measure spaces for $\alpha \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$.

We say that a cube $Q \subseteq \mathbb{R}^n$ is dyadic if $Q = Q_{j,k} = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 2^{-j}k_i \leq x_i < 2^{-j}(k_i+1), i = 1, 2, \ldots, n \}$ for some $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Denote by $\ell(Q) = 2^{-j}$ the side length of $Q$ and by $x_Q = 2^{-j}k$ the “left lower corner” of $Q$ when $Q = Q_{j,k}$. We use $\sup_P$ and $\sum_P$ to express the supremum and summation taken over all dyadic cubes $P$, respectively. Also, denote the summation taken over all dyadic cubes $Q$ contained in $P$ by $\sum_{Q \subseteq P}$. For any dyadic cubes $P$ and $Q$, either $P$ and $Q$ are nonoverlapping or one contains the other. For any
function $f$ defined on $\mathbb{R}^n$, $j \in \mathbb{Z}$, and dyadic cube $Q = Q_j$, set

\[
\begin{align*}
    f_Q(x) &= |Q|^{-1/2} f \left( \frac{x - x_Q}{\ell(Q)} \right) = 2^{jn/2} f \left( 2^j x - k \right), \\
    f_j(x) &= 2^{jn} f \left( 2^j x \right), \\
    \tilde{f}(x) &= f(-x).
\end{align*}
\]

It is clear that $\tilde{g}_j \ast f(x_Q) = |Q|^{-1/2} \langle f, \tilde{g}_Q \rangle$, where $\langle f, g \rangle$ denotes the paring in the usual sense for $g$ in a Fréchet space $X$ and $f$ in the dual of $X$.

Choose a fixed function $\varphi$ in Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, the collection of rapidly decreasing $C^\infty$ functions on $\mathbb{R}^n$, satisfying

\[
\begin{align*}
    \text{supp}(\hat{\varphi}) &\subseteq \left\{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \right\}, \\
    |\hat{\varphi}(\xi)| &\geq c > 0 \quad \text{if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}.
\end{align*}
\]

For $\alpha \in \mathbb{R}$ and $0 < p, q \leq +\infty$, we say that $f$ belongs to the homogeneous Triebel-Lizorkin space $F^{\alpha,q}_p$ if $f \in \mathcal{S}'/\mathcal{D}'$, the tempered distributions modulo polynomials, satisfies

\[
\|f\|_{F^{\alpha,q}_p} := \left\{ \left\{ \sum_{k \in \mathbb{Z}} (2^{ka} |\varphi_k \ast f|)^q \right\}^{1/q} \right\}^{1/p} < \infty \quad \text{for } 0 < p < \infty,
\]

\[
\sup_{p} \left\{ |p|^{-1} \int_{p = \log_2 \ell(p)} \sum_{k = -\infty}^{\infty} (2^{ka} |\varphi_k \ast f(x)|)^q dx \right\}^{1/q} < \infty \quad \text{for } p = \infty.
\]

When $0 < p < \infty$ and $q = \infty$, the above $\ell^1$-norm is modified to be the supremum norm as usual, and $F^{\alpha,\infty}_\infty$ is defined to be $B^{\alpha,\infty}_\infty$, which is

\[
\|f\|_{F^{\alpha,\infty}_\infty} := \sup_{k \in \mathbb{Z}} \sup_{x \in Q, \ell(Q) = 2^{-k}} 2^{ka} |\varphi_k \ast f(x)| \approx \sup_Q |Q|^{-(\alpha/n)-(1/2)} \langle f, \varphi_Q \rangle < \infty.
\]

We now introduce a new space $\text{CMO}^{\alpha,q}_r$ as follows.

**Definition 1.1.** Let $\varphi \in \mathcal{S}$ satisfy (1.2). For $\alpha, r \in \mathbb{R}$ and $0 < q \leq \infty$, the generalized Carleson measure spaces $\text{CMO}^{\alpha,q}_r$ is the collection of all $f \in \mathcal{S}'/\mathcal{D}'$ satisfying $\|f\|_{\text{CMO}^{\alpha,q}_r} < \infty$, where

\[
\|f\|_{\text{CMO}^{\alpha,q}_r} := \left\{ \sup_{p} \left\{ |p|^{-r} \int_{Q \subset \mathcal{P}} \left( |Q|^{-(\alpha/n)-(1/2)} |\langle f, \varphi_Q \rangle \chi_Q(x)\right)^q dx \right\}^{1/q} \right\}^{1/p}, \quad 0 < q < \infty,
\]

\[
\sup_{Q \subset \mathcal{P}} |Q|^{-(\alpha/n)-(1/2)} |\langle f, \varphi_Q \rangle| = \sup_{Q} |Q|^{-(\alpha/n)-(1/2)} |\langle f, \varphi_Q \rangle|, \quad q = \infty,
\]

and $\chi_Q$ denotes the characteristic function of $Q$. 
Remark 1.2. By definition, we immediately have $\text{CMO}^{\alpha,0,0} = \mathcal{F}^{\alpha,0,0}$ for $\alpha, r \in \mathbb{R}$, and it is easy to check $\text{CMO}^{\alpha,0,q} = \{0\}$ for $r < 0$ and $0 < q < \infty$. Note that the zero element in $\text{CMO}^{\alpha,0,q}$ means the class of polynomials. Also note that $\text{CMO}^{\alpha,0,q} = \mathcal{F}^{\alpha,0,q}$ with equivalent norms for $\alpha \in \mathbb{R}$ and $0 < q < \infty$. It follows from Proposition 3.3 that $\text{CMO}^{\alpha,q}_{1} = \mathcal{F}^{\alpha,q}_{1}$ for $\alpha \in \mathbb{R}$ and $0 < q < \infty$. In particular, $\text{CMO}^{0,2}_{1} = \text{BMO}$, and hence the spaces $\text{CMO}^{\alpha,0,q}_{r}$ generalize BMO.

Remark 1.3. For a dyadic cube $P$, denote by $k_{P} = -\log_{2}\mathcal{L}(P)$; that is, $k_{P}$ is the integer so that $\mathcal{L}(P) = 2^{-k_{P}}$. In [5, 6], Yang and Yuan introduced the so-called “unified and generalized” Triebel-Lizorkin-type spaces $\mathcal{F}^{\alpha,q}_{\tau}$ with four parameters by

$$
\|f\|_{\mathcal{F}^{\alpha,q}_{\tau}} := \sup_{P} |P|^{-\tau} \left\{ \int_{P} \left[ \sum_{k \geq k_{P}} \left( 2^{k\alpha} |\varphi_{k} * f(x)| \right)^{q} \right]^{p/q} dx \right\}^{1/p} < \infty,
$$

for $\alpha, \tau \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$, and $f \in \mathcal{S}'/\mathcal{D}$. Note that in [5] the space $\mathcal{F}^{\alpha,q}_{\tau}$ was defined for $\tau \in [0, \infty)$, $p \in (1, \infty)$, and $q \in [1, \infty]$. It follows from [6, Theorem 3.1] that

$$
\|f\|_{\mathcal{F}^{\alpha,q}_{\tau}} \approx \sup_{P} |P|^{-\tau} \left\{ \int_{P} \left[ \sum_{Q \subset P} \left( |Q|^{-\alpha/q} \left| \langle f, \varphi_{Q} \rangle \right| \chi_{Q}(x) \right) \right]^{p/q} dx \right\}^{1/p} .
$$

It is clear that $\text{CMO}^{\alpha,0,q}_{r} = \mathcal{F}^{\alpha,r/q}_{q,q}$ for $0 < q < \infty$, and hence $\text{CMO}^{\alpha,0,q}_{r}$ “looks like” a special case of $\mathcal{F}^{\alpha,q}_{\tau}$.

The definition of $\text{CMO}^{\alpha,0,q}_{r}$ is independent of the choice of $\varphi \in \mathcal{S}$ satisfying (1.2). To show that, we need the following Plancherel-Pólya inequalities.

Theorem 1.4 (Plancherel-Pólya inequality for $0 < q < \infty$). Let $\varphi, \hat{\varphi} \in \mathcal{S}$ satisfy (1.2). For $\alpha, r \in \mathbb{R}$ and $0 < q < \infty$, if $f \in \mathcal{S}'/\mathcal{D}$ satisfies

$$
\sup_{P} \left\{ |P|^{-r} \sum_{k=-\log_{2}\mathcal{L}(P)}^{\infty} \sum_{\ell \in \mathbb{Z}} 2^{k\alpha} \sup_{u \in Q} |\hat{\varphi}_{k} * f(u)| \right\}^{q} |Q|^{1/q} < \infty,
$$

then

$$
\sup_{P} \left\{ |P|^{-r} \sum_{k=-\log_{2}\mathcal{L}(P)}^{\infty} \sum_{\ell \in \mathbb{Z}} 2^{k\alpha} \sup_{u \in Q} |\hat{\varphi}_{k} * f(u)| \right\}^{q} |Q|^{1/q} < \infty
$$

and

$$
\approx \sup_{P} \left\{ |P|^{-r} \sum_{k=-\log_{2}\mathcal{L}(P)}^{\infty} \sum_{\ell \in \mathbb{Z}} 2^{k\alpha} \inf_{u \in Q} |\hat{\varphi}_{k} * f(u)| \right\}^{q} |Q|^{1/q} .
$$
Theorem 1.5 (Plancherel-Pólya inequality for \( q = \infty \)). Let \( \varphi, \phi \in \mathcal{S} \) satisfy (1.2). For \( \alpha, r \in \mathbb{R} \), if \( f \in \mathcal{S}'/\mathcal{P} \) satisfies

\[
\sup_{Q} \left( |Q|^{-\alpha/n-r} \sup_{u \in Q} \bar{\varphi}_{k} * f(u) \right) < \infty, \tag{1.10}
\]

then

\[
\sup_{Q} \left( |Q|^{-\alpha/n-r} \sup_{u \in Q} \bar{\varphi}_{k} * f(u) \right) \approx \sup_{Q} \left( |Q|^{-\alpha/n-r} \inf_{u \in Q} |\bar{\varphi}_{k} * f(u)| \right). \tag{1.11}
\]

Remark 1.6. Let \( \varphi, \phi \in \mathcal{S} \) satisfy (1.2). Denote by \( \text{CMO}_{r}^{\alpha,q}(\varphi) \) the collection of all \( f \in \mathcal{S}'/\mathcal{P} \) satisfying \( \|f\|_{\text{CMO}_{r}^{\alpha,q}(\varphi)} < \infty \) defined in Definition 1.1 with respect to \( \varphi \). Then, by Theorem 1.4,

\[
\|f\|_{\text{CMO}_{r}^{\alpha,q}(\varphi)} \leq \sup_{P} \left\{ |P|^{-r} \sum_{k=\log_{q}(P)}^{\infty} \sum_{Q \subseteq P, \ell(Q)=2^{-k}} \left( 2^{k} \sup_{u \in Q} |\bar{\varphi}_{k} * f(u)| \right)^{q} |Q| \right\}^{1/q}
\]

\[
\leq C \sup_{P} \left\{ |P|^{-r} \sum_{k=\log_{q}(P)}^{\infty} \sum_{Q \subseteq P, \ell(Q)=2^{-k}} \left( 2^{k} \inf_{u \in Q} |\bar{\varphi}_{k} * f(u)| \right)^{q} |Q| \right\}^{1/q}
\]

\[
\leq C \|f\|_{\text{CMO}_{r}^{\alpha,q}(\varphi)} \quad \text{for } 0 < q < \infty.
\]

Similarly, \( \|f\|_{\text{CMO}_{r}^{\alpha,q}(\phi)} \leq \|f\|_{\text{CMO}_{r}^{\alpha,q}(\varphi)} \) by interchanging the roles of \( \varphi \) and \( \phi \). Hence, the definition of \( \text{CMO}_{r}^{\alpha,q}(\varphi) \) is independent of the choice of \( \varphi \) and, for short, denoted by \( \text{CMO}_{r}^{\alpha,q} \).

Also, Theorem 1.5 shows that \( \text{CMO}_{r}^{\alpha,\infty} \) is independent of the choice of \( \varphi \) satisfying (1.2) in the same argument.

Remark 1.7. The classical Plancherel-Pólya inequality [9] concludes that if \( \{x_{k}\} \) is an appropriate set of points in \( \mathbb{R}^{n} \), for example, lattice points, where the length of the mesh is sufficiently small, then

\[
\left( \sum_{k=1}^{\infty} |f(x_{k})|^{p} \right)^{1/p} \approx \|f\|_{p}
\]

for all \( 0 < p \leq \infty \) with a modification if \( p = \infty \).

Using the Calderón reproducing formula (either continuous or discrete version), several authors obtain the variant Plancherel-Pólya inequalities [10–13]. These inequalities give characterizations of the Besov spaces and the Triebel-Lizorkin spaces. Moreover, using these inequalities, one can show that the Littlewood-Paley \( g \)-function and Lusin area \( S \)-function are equivalent in \( L^{p} \)-norm.
Define a linear map $S_\varphi$ from $\mathcal{S}/\mathcal{D}$ into the family of complex sequences by
\[
S_\varphi(f) = \{\langle f, \varphi_Q \rangle \}_Q. \tag{1.14}
\]

Let $S_0$ denote the family of $f \in \mathcal{S}$ satisfying $\int x^k f(x) \, dx = 0$ for all $k \in (\mathbb{N} \cup \{0\})^n$. For $g \in \text{CMO}^{-a,q}_{-\alpha,q}$, define a linear functional $L_g$ by
\[
L_g(f) = \langle S_\varphi(g), S_\varphi(f) \rangle = \sum_Q \langle g, \varphi_Q \rangle \langle f, \varphi_Q \rangle \quad \text{for } f \in S_0. \tag{1.15}
\]

We now state our first main result as follows.

**Theorem 1.8** (duality for $F^a,q_p$). Suppose that $\alpha \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$.

(a) For $1 < q < \infty$, the dual of $F^a,q_p$ is $\text{CMO}^{-a,q}_{(q/p)-(q/q)}$ in the following sense.

(i) For $g \in \text{CMO}^{-a,q}_{(q/p)-(q/q)}$, the linear functional $L_g$ given by (1.15), defined initially on

\[
S_0, \quad \text{extends to a continuous linear functional on } F^a,q_p \quad \text{with } \|L_g\| \leq C\|g\|_{\text{CMO}^{-a,q}_{(q/p)-(q/q)}}.
\]

(ii) Conversely, every continuous linear functional $L$ on $F^a,q_p$ satisfies $L = L_g$ for some $g \in \text{CMO}^{-a,q}_{(q/p)-(q/q)}$ with $\|g\|_{\text{CMO}^{-a,q}_{(q/p)-(q/q)}} \leq C\|L\|.$

(b) For $0 < q \leq 1$, the dual of $F^a,q_p$ is $\text{CMO}^{-a+\alpha(n/p)-n,\infty}_r$ (any $r \in \mathbb{R}$) in the following sense.

(i) For $g \in \text{CMO}^{-a+\alpha(n/p)-n,\infty}_r$, the linear functional $L_g$ given by (1.15), defined initially on

\[
S_0, \quad \text{extends to a continuous linear functional on } F^a,q_p \quad \text{with } \|L_g\| \leq C\|g\|_{\text{CMO}^{-a+\alpha(n/p)-n,\infty}_r}.
\]

(ii) Conversely, every continuous linear functional $L$ on $F^a,q_p$ satisfies $L = L_g$ for some $g \in \text{CMO}^{-a+\alpha(n/p)-n,\infty}_r$ with $\|g\|_{\text{CMO}^{-a+\alpha(n/p)-n,\infty}_r} \leq C\|L\|.$

**Remark 1.9.** For $0 < p < 1$ and $0 < q \leq 1$, it follows immediately from [2, 3] (Verbitsky [4] corrected a gap of the proof) and definition that $(F^a,q_p)' = F^{-a+\alpha(n/p)-n,\infty}_\infty = \text{CMO}^{-a+\alpha(n/p)-n,\infty}_r$ (any $r \in \mathbb{R}$). Theorem 1.8 (b) shows a different approach to the duality and includes the case of $p = 1$.

For $p = 1 < q < \infty$, we have $\text{CMO}^{-a,q}_{(q/p)-(q/q)} = (F^a,q_1)' = F^{-a,q}_{\infty}$. For $0 < p < 1 < q < \infty$, $\text{CMO}^{-a,q}_{(q/p)-(q/q)} = (F^a,q_p)' = F^{-a,q}_{\infty}$, and hence $\text{CMO}^{-a,q}_{(q/p)-(q/q)} = \text{CMO}^{-a,q}_{\infty}$. That is, each $\text{CMO}^{-a,q}_{(q/p)-(q/q)}$ coincides with $\text{CMO}^{-a,q}_{\infty}$ for $\alpha, r \in \mathbb{R}$ and $0 < p < 1 < q < \infty$.

**Remark 1.10.** In Remark 1.2 we are aware that $\text{CMO}^{-a,q}$ generalize BMO by the viewpoint of spaces directly. Choosing $\alpha = 0$ and $q = 2$ in Theorem 1.8, we immediately have $(H^p)' = (F^0,2)' = \text{CMO}^{0,2}_{(2/p)-1}$ for $0 < p \leq 1$. In particular, $\text{BMO} = \text{CMO}_1^{0,2}$. Once again, we obtain that $\text{CMO}^{-a,q}$ generalize BMO by the viewpoint of duality. It was also proved in [14] that the dual of the multiparameter product Hardy space is the generalized multiparameter Carleson measure space (cf. [14] for more details).
Remark 1.11. For $\alpha, r \in \mathbb{R}$, in order to make each index works, we defined $\text{CMO}_r^{a,\infty}$ to be $\sup_{p} |P|^{-r} \sup_{Q \in P} |Q|^{-(a/n)-(1/2)} |\langle f, \psi_Q \rangle|$ in our earlier version and in [7]. In such a situation, for $0 < p, q \leq 1$, the dual of $F_p^{a,q}$ would be $\text{CMO}_r^{-a,\infty}$. In this paper, however, we follow the referee’s suggestion and adopt a more “natural” definition of $\text{CMO}_r^{a,\infty}$ in Definition 1.1, that is, the limit of $\text{CMO}_r^{a,q}$ as $q \to \infty$. The sequence space $c_r^{a,\infty}$ given in Definition 2.1 has a similar story as well.

As applications, we first recall the Haar multipliers introduced in [15, 16]. Given a sequence $t = \{t_I\}_I$, where the $I$’s are dyadic intervals in $\mathbb{R}$, a Haar multiplier on $L^2(\mathbb{R})$ is a linear operator of the form

$$H_t f(x) := \sum_I t_I \langle f, h_I \rangle h_I(x), \quad f \in L^2(\mathbb{R}),$$

(1.16)

where $h_I$ are the Haar functions corresponding to $I$.

Using Meyer’s wavelets, we may generalize the above Haar multiplier to $\mathbb{R}^n$ and obtain a necessary and sufficient condition for the boundedness on Triebel-Lizorkin spaces. Let $\{\psi^i\}$ for $i \in E := \{1, 2, \ldots, 2^n - 1\}$ be Meyer’s wavelets (cf. [17], [18, pages 71–109]). Then, $\{\psi_Q\}$, where $i \in E$ and $Q$’s are dyadic cubes in $\mathbb{R}^n$, is a frame for $\dot{F}_p^{a,q}$ for $a \in \mathbb{R}$ and $0 < p, q \leq \infty$; that is, $\|f\|_{\dot{F}_p^{a,q}} \approx \sum_{i \in E} \|\langle f, \psi^i \rangle\|_{\dot{F}_p^{a,q}}$ for $f \in \dot{F}_p^{a,q}$. For $t = \{t_Q\}_Q$, define a wavelet multiplier $\tilde{T}_t$ on $\mathbb{R}^n$ by

$$\tilde{T}_t(f) = \sum_{i \in E} \sum_Q |Q|^{-1/2} t_Q \langle f, \psi_Q \rangle \psi_Q$$

(1.17)

for $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the above summation is well defined.

Theorem 1.12. Suppose that $\alpha, \beta \in \mathbb{R}$, $0 < p \leq 1$ and $0 < q < \infty$. Then,

(a) for $1 < q < \infty$, $\tilde{T}_t$ is bounded from $\dot{F}_p^{a,q}$ into $F_1^{a+\beta,1}$ if and only if $t \in c_r^{b,\alpha/q}$;

(b) for $0 < q \leq 1$ and $r \in \mathbb{R}$, $\tilde{T}_t$ is bounded from $\dot{F}_p^{a,q}$ into $\dot{F}_1^{a+\beta,1}$ if and only if $t \in c_r^{b+(n/p)-n,\infty}$, where $c_r^{b,\alpha/q}$ is given in Definition 2.1.

We consider another application. Let $\varphi$ and $\varrho$ in $\mathcal{S}$ satisfy (1.2) and (3.1). Choose a function $\Phi \in \mathcal{S}$ supported on $[0,1]^n$ and $\int \Phi = 1$. For $\alpha \in \mathbb{R}$ and $g \in \dot{F}_\infty^{a,\infty}$, define the paraproduct operator $\Pi_\alpha^g$ by

$$\Pi_\alpha^g(f) = \sum_Q \langle g, \psi_Q \rangle |Q|^{-1/2} \langle f, \phi_Q \rangle \psi_Q.$$

(1.18)

Thus, the adjoint operator $\Pi_\alpha^{*g}$ is

$$\Pi_\alpha^{*g}(f) = \sum_Q \langle g, \psi_Q \rangle |Q|^{-1/2} \langle f, \psi_Q \rangle \Phi_Q.$$

(1.19)

Then, $\Pi_\alpha 1 = g$ and $\Pi_\alpha^{*1} 1 = 0$ since $\langle 1, \phi_Q \rangle = |Q|^{1/2}$ and $\langle 1, \psi_Q \rangle = 0$. Also, if $g \in \dot{F}_\infty^{0,\infty}$, then
both \( \Pi_g \) and \( \Pi_g^* \) are singular integral operators satisfying the weak boundedness property. Moreover, \( \Pi_g \) is a Calderón-Zygmund operator (i.e., \( \Pi_g \) is bounded on \( L^2(\mathbb{R}^n) \)) if and only if \( g \in F^0_{\infty} \) by David-Journé’s T1 theorem [19] (also see [12, Theorems 5.4 and 5.8]). The authors showed a more general type of paraproduct operators in [12, page 688], which were derived from the discrete Calderón reproducing formula.

**Theorem 1.13.** Suppose that \( \beta \in \mathbb{R}, \ 0 < r \leq 1 \) and \( 0 < p \leq r < q < r/(1-r) \).

(i) For \( \alpha < 0 \), \( \Pi_g \) is bounded from \( F^\alpha_p \) into \( F^{\alpha+\beta,r}_r \) if and only if \( g \in \text{CMO}_{r(q-p)/p(q-r)}^{\beta,pr} \).

(ii) If \( \alpha \in \mathbb{R} \) with \( \alpha + \beta > 0 \) and \( g \in \text{CMO}_{r(q-p)/p(q-r)}^{\beta,pr} \), then \( \Pi_g^* \) is bounded from \( F^\alpha_p \) into \( F^{\alpha+\beta,r}_r \).

**Remark 1.14.** When \( r = 1, \ 0 < p \leq 1 < q < \infty \), and \( \beta \in \mathbb{R} \), Theorem 1.13 says that \( \Pi_g \) is bounded from \( F^\alpha_1 \) into \( F^{\alpha+\beta,1}_1 \) if and only if \( g \in \text{CMO}_{(q/p)-(q/q)}^{\beta,pr} \) for \( \alpha < 0 \), and \( \Pi_g^* \) is bounded from \( F^\alpha_1 \) into \( F^{\alpha+\beta,1}_1 \) for \( \alpha > -\beta \) provided \( g \in \text{CMO}_{(q/p)-(q/q)}^{\beta,pr} \). In 1995, Youssfi [20] showed that, for \( \beta \in \mathbb{R}, \ 1 < p < \infty, \ 1 \leq q \leq 2, \) and \( g \in \text{CMO}_{(q/p)-(q/q)}^{\beta,pr} \), \( \Pi_g^* \) is bounded from \( F^0_p \) into \( F^\beta_p \) if and only if \( g \in F^{\beta,pr}_\infty \). The special case of Theorem 1.13(i), \( p = r \), generalizes Youssfi’s result to \( 0 < p \leq 1 \).

More precisely, for \( \alpha < 0, \beta \in \mathbb{R}, \ 0 < p \leq 1, \) and \( p < q < p/(1-p) \), \( \Pi_g \) is bounded from \( F^\alpha_p \) to \( F^{\alpha+\beta,p}_r \) if and only if \( g \in \text{CMO}_{1}^{\beta,pa/(q-p)} = F^{\beta,pa/(q-p)}_\infty \).

The paper is organized as follows. In Section 2, we introduce the discrete version of the generalized Carleson measure spaces \( c^{\alpha,q}_r \) and show that the duals of sequence Triebel-Lizorkin spaces \( f^{\alpha,q}_p \) for \( 1 < q < \infty \) and \( 0 < q \leq 1 \) are \( c^{-(\alpha,q)}_r \) and \( c^{-(\alpha+1/n-p,n,\infty)}_r \) (for any \( r \in \mathbb{R} \), respectively). In Section 3, we prove the duals of homogeneous Triebel-Lizorkin spaces \( F^\alpha_p \) for \( 1 < q < \infty \) and \( 0 < q \leq 1 \) to be the generalized Carleson measure spaces \( \text{CMO}_{(q/p)-(q/q)}^{-(\alpha,q)}_r \) and \( \text{CMO}_{r(n/p)-n,\infty}^{-(\alpha+1/n-p,n,\infty)} \) (for any \( r \in \mathbb{R} \), respectively). In Section 4, we prove the Plancherel-Pólya inequalities that give us the independence of the choice of \( q \) for the definition of the generalized Carleson measure spaces. In the last section, we show the boundedness of wavelet multipliers and paraproduct operators. Throughout, we use \( C \) to denote a universal constant that does not depend on the main variables but may differ from line to line. Also, \( Q \) and \( P \) always mean the dyadic cubes in \( \mathbb{R}^n \), and, for \( r > 0 \), we denote by \( rQ \) the cube concentric with \( Q \) whose each edge is \( r \) times as long.

## 2. Sequence Spaces

In this section, we introduce sequence spaces \( c^{\alpha,q}_r \) and then characterize the duals of \( f^{\alpha,q}_p \) by means of \( c^{\alpha,q}_r \). Let us recall the definition of these sequence spaces \( f^{\alpha,q}_p \) defined in [2]. For \( \alpha \in \mathbb{R} \) and \( 0 < p, q \leq \infty \), the space \( f^{\alpha,q}_p \) consists all such sequences \( s = \{ s_Q \}_{Q \in \mathbb{R}} \) satisfying

\[
\left\| s \right\|_{f^{\alpha,q}_p} := \begin{cases} \left\| \sum_{Q} (|Q|^{-\alpha/n} - (1/2)) |s_Q| \chi_Q \right\|^{1/q}_{L^p} < \infty & \text{if } 0 < p < \infty, \\ \sup_{P} \left\{ |P|^{-1} \int_{P} \sum_{Q \subset P} (|Q|^{-\alpha/n} - (1/2)) |s_Q| \chi_Q(x) \right\}^{1/q} < \infty & \text{if } p = \infty. \end{cases}
\]  

(2.1)
For $\ell^q$, the previous $\ell^q$-norm is modified to the supremum norm for $0 < p < \infty$ and $q = \infty$. For $p = q = \infty$, we adopt the norm

$$\|s\|_{f_{\infty}^q} := \sup_Q |Q|^{-(\alpha/n) - (1/2)} |s_Q|.$$  \hspace{1cm} (2.2)

Note that $\|s\|_{f_{\infty}^q}$ is equivalent to the Carleson norm of the measure

$$\sum_Q \left( |Q|^{-(\alpha/n) - (1/2)} |s_Q| \right)^q |Q|^c_{(x,t)}.$$  \hspace{1cm} (2.3)

where $\delta_{(x,t)}$ is the point mass at $(x,t) \in \mathbb{R}^{n+1}$. See [2] for the details.

To study the duals of $f_{p}^{\alpha,q}$, we introduce a discrete version of the generalized Carleson measure spaces $c^{\alpha,q}_r$.

**Definition 2.1.** For $\alpha, r \in \mathbb{R}$ and $0 < q \leq \infty$, the space $c^{\alpha,q}_r$ is the collection of all sequences $t = \{t_Q\}_Q$ satisfying $\|t\|_{c^{\alpha,q}_r} < \infty$, where

$$\|t\|_{c^{\alpha,q}_r} := \begin{cases} \sup_P |P|^{-r} \sum_{Q \subset P} \left( |Q|^{-(\alpha/n) - (1/2)} |t_Q| |\chi_Q(x)| \right)^q dx & \text{for } 0 < q < \infty, \\ \sup_P \sup_{Q \subset P} |Q|^{-(\alpha/n) - (1/2)} |t_Q| = \sup_Q |Q|^{-(\alpha/n) - (1/2)} |t_Q| & \text{for } q = \infty. \end{cases}$$  \hspace{1cm} (2.4)

It is obvious that

$$\|t\|_{c^{\alpha,q}_r} = \|t\|_{f_{p}^{\alpha,q}}$$

and $\|t\|_{c^{\alpha,q}_r} = \|t\|_{c^{\alpha,q}_r}$ for $\alpha, r \in \mathbb{R}$. Using embedding theorem, Frazier and Jawerth [2, equation (5.14) and Theorem 5.9] obtained that, for $\alpha \in \mathbb{R}$ and $0 < q < \infty$, the dual of $f_{p}^{\alpha,q}$ is $f_{p}^{\alpha+(n/p)\cdot q, \infty}$ when $0 < p < 1$, and the dual of $f_{p}^{\alpha,q}$ is $f_{p}^{\alpha,q}$. Note that $c^{\alpha,q}_r = \{0\}$ for $r < 0$ and $0 < q < \infty$. Here we give the dual relationship between sequence spaces $f_{p}^{\alpha,q}$ and $c^{\alpha,q}_r$.

**Theorem 2.2** (duality for $f_{p}^{\alpha,q}$). Suppose that $\alpha \in \mathbb{R}$, $0 < p \leq 1$, and $0 < q < \infty$.

(a) For $1 < q < \infty$, the dual of $f_{p}^{\alpha,q}$ is $c_{(q/p)\cdot q, (-q)/q}^{\alpha}$ in the following sense.

(i) For $t = \{t_Q\}_Q \in c_{(q/p)\cdot q, (-q)/q}^{\alpha}$, the linear functional $\ell_t$ on $f_{p}^{\alpha,q}$ given by $\ell_t(s) = \sum_Q s_Q t_Q$ is continuous with $\|\ell_t\| \leq C\|t\|_{c_{(q/p)\cdot q, (-q)/q}^{\alpha}}$ for $s = \{s_Q\}_Q \in f_{p}^{\alpha,q}$.

(ii) Conversely, every continuous linear functional $\ell$ on $f_{p}^{\alpha,q}$ satisfies $\ell = \ell_t$ for some $t \in c_{(q/p)\cdot q, (-q)/q}^{\alpha}$ with $\|t\|_{c_{(q/p)\cdot q, (-q)/q}^{\alpha}} \leq C\|\ell\|$.

(b) For $0 < q \leq 1$, the dual of $f_{p}^{\alpha,q}$ is $c_{r}^{\alpha+(n/p)\cdot q, \infty}$ (any $r \in \mathbb{R}$) in the following sense.
For simpler proof of Frazier-Jawerth’s result for the duality of $\dot{f}^{a,d}_{p}$, without loss of generality, we may assume that $s\in c_{r}^{-a+(n/p)-n,\infty}$.

(ii) Conversely, every continuous linear functional $\ell$ on $\dot{f}^{a,d}_{p}$ satisfies $\ell = \ell_{t}$ for some $t \in c_{r}^{-a+(n/p)-n,\infty}$, with $\|t\|_{c_{r}^{-a+(n/p)-n,\infty}} \leq C\|\ell\|$.

Remark 2.3. For $a \in \mathbb{R}$ and $0 < q < \infty$, sequence spaces $c_{r}^{a,q} = f^{a,q}_{\infty}$ and $c_{r}^{a,\infty} = f^{a,\infty}_{\infty}$ (for any $r \in \mathbb{R}$) by definitions. Theorem 2.2 shows that $(f^{a,q}_{1})' = f^{a,q}_{\infty}$, which gives a different but simpler proof of Frazier-Jawerth’s result for the duality of $f^{a,q}_{1}$ (cf. [2, Theorem 5.9]).

Proof of Theorem 2.2. For $s = \{s_{Q}\}_{Q} \in f^{a,q}_{p}$ and $t = \{t_{Q}\}_{Q} \in c_{r}^{-a,d}_{\infty}$, set $\tilde{s} = \{\tilde{s}_{Q}\}_{Q}$ and $\tilde{t} = \{\tilde{t}_{Q}\}_{Q}$ to be

$$
\tilde{s}_{Q} = |Q|^{-a/n}s_{Q}, \quad \tilde{t}_{Q} = |Q|^{a/n}t_{Q}.
$$

(2.6)

Then, $\ell_{t}(\tilde{s}) = \ell_{t}(s)$.

Also,

$$
\|\tilde{s}\|_{f^{a,q}_{p}} = \|s\|_{f^{a,q}_{p}}, \quad \|\tilde{t}\|_{c_{r}^{-a,d}_{\infty}} = \|t\|_{c_{r}^{-a,d}_{\infty}}.
$$

(2.7)

Without loss of generality, we may assume that $a = 0$.

We first consider the case $1 < q < \infty$. Let $t \in c_{(q/p)-(q/a)}^{0,d}$ and define a linear functional $\ell_{t}$ on $f^{0,q}_{p}$ by

$$
\ell_{t}(s) = \sum_{Q} s_{Q}t_{Q} \quad \text{for } s \in f^{0,q}_{p}.
$$

(2.8)

For $s = \{s_{Q}\}_{Q} \in f^{0,q}_{p}$, let

$$
V_{q}(x) := \left(\sum_{Q} (|Q|^{-1/2}|s_{Q}x\Omega_{Q}|)^{q} \right)^{1/q}.
$$

(2.9)

For $k \in \mathbb{Z}$, let

$$
\Omega_{k} := \left\{ x \in \mathbb{R}^{n} : 2^{k} < V_{q}(x) \leq 2^{k+1} \right\},
$$

$$
\tilde{\Omega}_{k} := \left\{ x \in \mathbb{R}^{n} : M\chi_{\Omega_{k}}(x) > \frac{1}{2} \right\},
$$

$$
B_{k} := \left\{ \text{dyadic } Q : |Q\cap \Omega_{k}| > \frac{|Q|}{2}, \ |Q\cap \Omega_{j+1}| \leq \frac{|Q|}{2} \text{ for some } j \geq k \right\},
$$

(2.10)

where $M$ is the Hardy-Littlewood maximal function. Then, for each dyadic cube $Q$, there exists exactly a $k \in \mathbb{Z}$ such that $Q \in B_{k}$. For every $Q \in B_{k}$, let $\tilde{Q}$ denote the maximal
dyadic cube in $B_k$ containing $Q$. Then all of such $\tilde{Q}$’s are pairwise disjoint. Thus, by Hölder’s inequality for $q$ and the inequality $(a + b)^p \leq a^p + b^p$ for $0 < p \leq 1$,

$$
\left| \sum_{Q} s_Q t_Q \right| \leq \sum_{k \in Z} \sum_{Q \in B_k} \sum_{Q \cap \tilde{Q} \neq \emptyset} \left( |Q|^{-\left(1/2\right) + \left(1/q\right)} |s_Q| \right) \left( |Q|^{-\left(1/2\right) - \left(1/q\right)} |t_Q| \right)
$$

$$
\leq \left\{ \sum_{k \in Z} \sum_{Q \in B_k} \left( \sum_{Q \cap \tilde{Q} \neq \emptyset} \left( |Q|^{-\left(1/2\right) + \left(1/q\right)} |s_Q| \right)^q \right)^{p/q} \left( \sum_{Q \cap \tilde{Q} \neq \emptyset} \left( |Q|^{-\left(1/2\right) - \left(1/q\right)} |t_Q| \right)^q \right)^{p/q} \right\}^{1/p}
$$

$$
\leq \|t\|_{c^d_{\left(\frac{q}{q-1}\right)-\left(\frac{d}{q}\right)}} \left\{ \sum_{k \in Z} \left| \tilde{\Omega}_k \right|^{1-(p/q)} \left( \sum_{Q \in B_k} \left( |Q|^{-\left(1/2\right) + \left(1/q\right)} |s_Q| \right)^q \right)^{p/q} \right\}^{1/p}.
$$

(2.11)

Since $\tilde{Q} \in B_k$ implies $\tilde{Q} \subseteq \tilde{\Omega}_k$, the disjointness of $\tilde{Q}$’s and Hölder’s inequality yield

$$
\left| \sum_{Q} s_Q t_Q \right| \leq \|t\|_{c^d_{\left(\frac{q}{q-1}\right)-\left(\frac{d}{q}\right)}} \left\{ \sum_{k \in Z} \left| \tilde{\Omega}_k \right|^{1-(p/q)} \left( \sum_{Q \in B_k} \left( |Q|^{-\left(1/2\right) + \left(1/q\right)} |s_Q| \right)^q \right)^{p/q} \right\}^{1/p}.
$$

(2.12)

We claim that $\sum_{Q \in B_k} \left( |Q|^{-\left(1/2\right) + \left(1/q\right)} |s_Q| \right)^q \leq C 2^{k q} |\tilde{\Omega}_k|$ for $k \in Z$ and $0 < q < \infty$. Assume the claim for the moment. The weak $(1, 1)$ boundedness of $M$ gives $|\tilde{\Omega}_k| \leq C |\Omega_k|$, and hence

$$
\left| \sum_{Q} s_Q t_Q \right| \leq C \|t\|_{c^d_{\left(\frac{q}{q-1}\right)-\left(\frac{d}{q}\right)}} \left( \sum_{k \in Z} \left| \tilde{\Omega}_k \right|^{1-(p/q)} \left( 2^{k q} |\tilde{\Omega}_k| \right)^{p/q} \right)^{1/p}
$$

$$
\leq C \|t\|_{c^d_{\left(\frac{q}{q-1}\right)-\left(\frac{d}{q}\right)}} \left( \sum_{k \in Z} 2^{k p} |\Omega_k| \right)^{1/p}
$$

$$
\leq C \|t\|_{c^d_{\left(\frac{q}{q-1}\right)-\left(\frac{d}{q}\right)}} \|V_t\|_{L^p}
$$

$$
= C \|t\|_{c^d_{\left(\frac{q}{q-1}\right)-\left(\frac{d}{q}\right)}} \|s\|_{f^p}.
$$

(2.13)

To prove the claim, we note that, for $k \in Z$ and $0 < q < \infty$,

$$
2^{q(k+1)} |\tilde{\Omega}_k| \geq \int_{\tilde{\Omega}_k \setminus \cup_{j=k+1}^{\infty} \Omega_j} (V_q(x))^q dx
$$

$$
= \int_{\tilde{\Omega}_k \setminus \cup_{j=k+1}^{\infty} \Omega_j} \sum_{Q} \left( |Q|^{-1/2} |s_Q| \chi_Q(x) \right)^q dx
$$

$$
\geq \sum_{Q \in B_k} \left( |Q|^{-1/2} |s_Q| \right)^q \left( |\tilde{\Omega}_k \setminus \Omega_j| \cap Q \right) \quad \text{for some } j \geq k + 1.
$$

(2.14)
which implies
\[ 2^{q(k+1)}|\tilde{\Omega}_k| \geq \frac{1}{2} \sum_{Q_k \in B_k} \left( |Q|^{-1/(2+1/q)} |s_Q| \right)^q. \] (2.15)

For \(0 < q \leq 1\), with a modification, we have
\[
\left| \sum_Q s_Q t_Q \right| \leq \sum_{k \in Z} \sum_{Q_k \in B_k} \left( \sum_{Q \subseteq Q_k \in Q_{\delta k}} |Q|^{1/2} |s_Q| \right) \left( |Q|^{-1/(p+1/2)} |t_Q| \right) \left( \frac{|Q|}{|Q|} \right)^{(1/p)-1} |Q|^{(1/p)-1}
\]
\[
\leq \|t\|_{L_p^{(\omega/p)-\omega}} \left\{ \sum_{k \in Z} \sum_{Q_k \in B_k} \left( \sum_{Q \subseteq Q_k \in Q_{\delta k}} |Q|^{1/2} |s_Q| \right) |\tilde{Q}|^{1-p} \right\}^{1/p}
\]
\[
\leq C \|t\|_{L_p^{(\omega/p)-\omega}} \|V_p\|_{L_p}
\]
\[
\leq C \|t\|_{L_p^{(\omega/p)-\omega}} \|s\|_{L_p^{p/q}}.
\] (2.16)

On the other hand, suppose that \(\ell\) is a continuous linear functional on \(f_p^{0,q}\). For each dyadic cube \(P\), write \(e^p = \{(e^p)_Q\}_Q\) to be the sequence defined by
\[
(e^p)_Q = \begin{cases} 1 & \text{if } Q = P, \\ 0 & \text{if } Q \neq P. \end{cases}
\] (2.17)

Let \(t_P = \ell(e^p)\) and \(t = \{t_P\}_P\). Then, for \(s = \{s_Q\}_Q \in f_p^{0,q}\),
\[
\ell(s) = \sum_Q s_Q t_Q = \ell_t(s).
\] (2.18)

Fix a dyadic cube \(P\). For \(1 < q < \infty\), let \(X\) be the sequence space consisting of \(s = \{s_Q\}_Q \subseteq L_p^{p/q}\), and define a counting measure on dyadic cubes \(Q \subseteq P\) by \(d\sigma(Q) = |Q|/|P|^{(q/p)-(q/q)}\). Then,
\[
\left( \frac{1}{|P|^{(q/p)-(q/q)}} \sum_{Q \subseteq P} \left( |Q|^{-1/(2+1/q)} |s_Q| |t_Q| \right)^q \right)^{1/q}
\]
\[
= \sup_{\|s\|_{L_p^{p/q}} \leq 1} \left| \frac{1}{|P|^{(q/p)-(q/q)}} \sum_{Q \subseteq P} |Q|^{-1/2} |s_Q| |s_Q|^{1/2} |t_Q| \right|
\]
\[
\leq \|\ell\| \sup_{\|s\|_{L_p^{p/q}} \leq 1} \left\{ \frac{|s_Q|^{1/2}}{|P|^{(q/p)-(q/q)}} \right\}_{Q \subseteq P}.
\] (2.19)
Note that

\[
\left\| \left\{ \frac{s_Q|Q^{1/2}}{|P|^{(q/p)-(q'/q)}} \right\}_{Q \in \mathcal{P}} \right\| \leq \frac{1}{|P|^{(q/p)-(q'/q)}} \left\{ \left( \sum_{Q \in \mathcal{P}} |Q|s_Q^q \right)^{p/q} \cdot |P|^{-1-(q'/q)} \right\}^{1/p} 
\leq C \|s\|_{\ell^p(X,d\sigma)}.
\]

Thus,

\[
\left( \frac{1}{|P|^{(q/p)-(q'/q)}} \sum_{Q \in \mathcal{P}} \left( |Q|^{-(1/2)+(1/q')} |t_Q| \right)^{q'} \right)^{1/q'} \leq C \|\ell\|,
\]

and hence \( t \in c_{(q/p)-(q'/q)}^0 \). For \( 0 < q \leq 1 \), consider \( e^p \) defined before. Then, \( \|e^p\|_{\ell^p_{\alpha,q}} = |P|^{-(1/2)+(1/p)} \) and

\[
\left( \frac{1}{|P|^{(1/2)-(1/p)} |t_P|} \right) \|e^p\|_{\ell^p_{\alpha,q}} = |t_P| = \|\ell(e^p)\| \leq \|\ell\| \|e^p\|_{\ell^p_{\alpha,q}}.
\]

Hence, \( \|t\|_{c_{(q/p)-n,n}} = \sup_P |P|^{(1/2)-(1/p)} |t_P| \leq \|\ell\| \). This completes the proof. \( \square \)

### 3. Proof of the Main Theorem

Let us recall the \( \varphi \)-transform identity given by Frazier and Jawerth [2]. Choose a function \( \varphi \in S \) satisfying (1.2). Then there exists a function \( \varphi' \in S \) satisfying the same conditions as \( \varphi \) such that \( \sum_{\xi \in \mathbb{Z}} \hat{\varphi}(2^{-\xi})\hat{\varphi'}(2^{-\xi}) = 1 \) for \( \xi \neq 0 \). The \( \varphi\)-transform identity is given by

\[
f = \sum_Q \langle f, \varphi_Q \rangle \varphi_Q,
\]

where the identity holds in the sense of \( S'/\mathcal{P} \), \( S_0 \), and \( \mathcal{F}^\alpha_{p,q} \)-norm.

Define a linear map \( S_\varphi \) from \( S'/\mathcal{P} \) into the family of complex sequences by

\[
S_\varphi(f) = \{ \langle f, \varphi_Q \rangle \}_{Q \in \mathcal{P}}
\]

and another linear map \( T_\varphi \) from the family of complex sequences into \( S'/\mathcal{P} \) by

\[
T_\varphi(\{s_Q \}_{Q}) = \sum_Q s_Q \varphi_Q.
\]

Then, \( T_\varphi \circ S_\varphi|_{\mathcal{F}^\alpha_{p,q}} \) is the identity on \( \mathcal{F}^\alpha_{p,q} \) by [2, Theorem 2.2].

**Proposition 3.1.** Suppose that \( \alpha \in \mathbb{R} \) and, \( 0 < p, q < +\infty \), and \( \varphi, \varphi' \in S \) satisfy (1.2) and (3.1). The linear operators \( S_\varphi : \mathcal{F}^\alpha_{p,q} \mapsto \mathcal{F}^\alpha_{p,q} \) and \( T_\varphi : \mathcal{F}^\alpha_{p,q} \mapsto \mathcal{F}^\alpha_{p,q} \) defined by (3.2) and (3.3), respectively, are
bounded. Furthermore, $T_v \circ S_v$ is the identity on $f_{p,q}^{\alpha,q}$. In particular, $\|f\|_{f_{p,q}^{\alpha,q}} \approx \|S_v(f)\|_{f_{p,q}^{\alpha,q}}$ and $f_{p,q}^{\alpha,q}$ can be identified with a complemented subspace of $f_{p,q}^{\alpha,q}$.

Figures 1 and 2 illustrate the relationship among $f_{p,q}^{\alpha,q}$, $f_{p,q}^{\alpha,q}$, $CMO_{\alpha,q}^{\alpha,q}$, $a_{\alpha,q}^{\alpha,q}$, and $c_{\alpha,q}^{\alpha,q}$.

One recalls the almost diagonality given by Frazier and Jawerth [2]. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$, let $J = n / (\min\{1, p, q\})$. One says that a matrix $A = \{a_{QP}\}_{Q,P}$ is $(\alpha, p, q)$-almost diagonal if there exists $\varepsilon > 0$ such that

$$\sup_{Q,P} \frac{|a_{QP}|}{w_{QP}(\varepsilon)} < +\infty,$$

where

$$w_{QP}(\varepsilon) = \left( \frac{\ell(Q)}{\ell(P)} \right)^\alpha \left( 1 + \frac{|x_Q - x_P|}{\max(\ell(P), \ell(Q))} \right)^{-J-\varepsilon} \cdot \min \left\{ \left( \frac{\ell(Q)}{\ell(P)} \right)^{(\varepsilon+n)/2}, \left( \frac{\ell(P)}{\ell(Q)} \right)^{(\varepsilon+n)/2} \right\}.$$

**Lemma 3.2.** For $\alpha, r \in \mathbb{R}$ and $0 < q < \infty$, an $(\alpha + nr, q, q)$-almost diagonal matrix is bounded on $c_{\alpha,q}^{\alpha,q}$. Furthermore, when $r \geq 0$, an $(\alpha + nr, \infty, \infty)$-almost diagonal matrix is bounded on $c_{\alpha,q}^{\alpha,\infty}$.

We postpone the proof of Lemma 3.2 until the end of Section 4.
Let \( \alpha, r \in \mathbb{R} \). For \( q = \infty \), we have \( \ell_{c_r,\infty}^\alpha = f_{c_r,\infty}^\alpha \) and \( \text{CMO}_{c_r,\infty}^\alpha = F_{c_r,\infty}^\alpha \). Thus, \( S_\varphi : \text{CMO}_{c_r,\infty}^\alpha \rightarrow c_r,\infty \) and \( T_\varphi : c_r,\infty \rightarrow \text{CMO}_{c_r,\infty}^\alpha \) are bounded by Proposition 3.1. For \( 0 < q < \infty \) and \( f \in \text{CMO}_{c_r,q}^\alpha \), let \( s = \{ s_Q \}_Q = S_\varphi (f) \). Then, the \( \varphi \)-transform identity (3.1) shows that \( f = \sum_Q s_Q \varphi_Q \) and \( \| f \|_{\text{CMO}_{c_r,q}^\alpha} = \| S_\varphi (f) \|_{c_r,q} = \| s \|_{c_r,q} \). In particular, \( \| f \|_{\text{CMO}_{c_r,q}^\alpha} = \| S_\varphi (f) \|_{c_r,q} = \| S_\varphi (f) \|_{F_{c_r,q}^\alpha} \approx \| f \|_{F_{c_r,q}^\alpha} \).

Furthermore, for \( s \in c_r,\alpha \),

\[
\| T_\varphi (s) \|_{\text{CMO}_{c_r,q}^\alpha} = \left\| \sum_P s_p \varphi_P \right\|_{\text{CMO}_{c_r,q}^\alpha} = \left\| \left\{ \sum_P s_p \varphi_P, \varphi_Q \right\} \right\|_Q = \| A \|_{c_r,q},
\]

(3.6)

where \( A := \{ \langle \varphi_P, \varphi_Q \rangle \} \) is \((\alpha + nr, q, q)\)-almost diagonal (cf. [2, Lemma 3.6]) and hence \( A \) is bounded on \( c_r,\alpha \) by Lemma 3.2. Therefore, \( S_\varphi \) is bounded from \( \text{CMO}_{c_r,q}^\alpha \) to \( c_r,\alpha \) and \( T_\varphi \) is bounded from \( c_r,\alpha \) to \( \text{CMO}_{c_r,q}^\alpha \).

We summarize that \( T_\varphi \circ S_\varphi |_{\text{CMO}_{c_r,q}^\alpha} \) is also the identity on \( \text{CMO}_{c_r,q}^\alpha \).

**Proposition 3.3.** For \( (\alpha, r, q) \in \mathbb{R} \times \mathbb{R} \times (0, \infty) \) or \( (\alpha, r, q) \in \mathbb{R} \times \mathbb{R} \times \{ \infty \} \), the linear operators \( S_\varphi : \text{CMO}_{c_r,q}^\alpha \rightarrow c_r,\alpha \) and \( T_\varphi : c_r,\alpha \rightarrow \text{CMO}_{c_r,q}^\alpha \) are bounded. Furthermore, \( T_\varphi \circ S_\varphi \) is the identity on \( \text{CMO}_{c_r,q}^\alpha \) and \( \| f \|_{\text{CMO}_{c_r,q}^\alpha} = \| S_\varphi (f) \|_{c_r,\alpha} \).

In particular, \( \| f \|_{\text{CMO}_{c_r,q}^\alpha} = \| S_\varphi (f) \|_{c_r,\alpha} = \| S_\varphi (f) \|_{F_{c_r,q}^\alpha} \approx \| f \|_{F_{c_r,q}^\alpha} \) for \( \alpha \in \mathbb{R} \) and \( 0 < q < \infty \), and \( \| f \|_{\text{CMO}_{c_r,q}^\alpha} = \| S_\varphi (f) \|_{c_r,\alpha} = \| S_\varphi (f) \|_{F_{c_r,q}^\alpha} \approx \| f \|_{F_{c_r,q}^\alpha} \) for \( \alpha, r \in \mathbb{R} \).

Theorem 1.8 can be proved as a consequence of Propositions 3.1–3.3 and a duality result between two sequence spaces.

**Proof of Theorem 1.8.** First let us consider the case for \( 1 < q < \infty \). Let \( g \in \text{CMO}_{c_r,q}^{-\alpha} \). Then, by Proposition 3.3, \( \| g \|_{\text{CMO}_{c_r,q}^{-\alpha}} = \| S_\varphi (g) \|_{c_r,\alpha} \). It follows from Theorem 2.2 that \( \ell_{S_\varphi (g)} \) is a continuous linear functional on \( F_{c_r,q}^{\alpha} \) and \( \| \ell_{S_\varphi (g)} \| \approx \| S_\varphi (g) \|_{c_r,\alpha} \). Hence, for \( f \in S_\varphi \),

\[
| L_{\varphi} (f) | \leq C \| S_\varphi (g) \|_{c_r,\alpha} \| S_\varphi (f) \|_{F_{c_r,q}^\alpha} \leq C \| g \|_{\text{CMO}_{c_r,q}^{-\alpha}} \| f \|_{F_{c_r,q}^\alpha}.
\]

(3.7)

Since \( S_\varphi \) is dense in \( F_{c_r,q}^{\alpha} \), the functional \( L_{\varphi} \) can be extended to a continuous linear functional on \( F_{c_r,q}^{\alpha} \) satisfying \( \| L_{\varphi} \| \leq C \| g \|_{\text{CMO}_{c_r,q}^{-\alpha}} \).

Conversely, let \( L \in (F_{c_r,q}^{\alpha}), \) and set \( \ell = L \circ T_\varphi \) on \( F_{c_r,q}^{\alpha} \). By Proposition 3.1, \( \ell \in (F_{c_r,q}^{\alpha})' \). Thus, by Theorem 2.2, there exists \( t = \{ t_Q \}_Q \in c_r,\alpha \) such that

\[
\ell'(s_Q) = \sum_Q s_Q t_Q \quad \text{for} \quad s_Q \in F_{c_r,q}^{\alpha},
\]

and \( \| t \|_{c_r,\alpha} \approx \| \ell \| \leq C \| L \| \). For \( f \in F_{c_r,q}^{\alpha} \), we have

\[
\ell \circ S_\varphi (f) = L \circ T_\varphi \circ S_\varphi (f) = L(f).
\]

(3.9)
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So, for $f \in S_0$ and letting $g = T_\varphi(t) = \sum_Q t_Q \varphi_Q$,

$$L(f) = \ell \circ S_\varphi(f) = \sum_Q \langle f, \varphi_Q \rangle t_Q = \langle t, S_\varphi(f) \rangle.$$  \hspace{1cm} (3.10)

It follows from [2, equations (2.7)-(2.8)] that $\langle g, f \rangle = \langle S_\varphi(g), S_\varphi(f) \rangle$ and $\langle t, S_\varphi(f) \rangle = \langle T_\varphi(t), f \rangle$ for $f \in S_0$ and $g \in S'/\mathcal{D}$. This shows that $L(f) = \langle T_\varphi(t), f \rangle = L_g(f)$ for $f \in S_0$.

Proposition 3.3 and Theorem 2.2 give

$$\|g\|_{C_{MO}^{a,q}_{(p/q);(q)}(\psi)} \leq C \|t\|_{C_{MO}^{a,q}_{(q/p);(p)}(\varphi)} \leq C \|L\|.$$  \hspace{1cm} (3.11)

A similar argument gives the desired result for $0 < q \leq 1$ with a slight modification, and hence the proof is finished. \hfill $\square$

Remark 3.4. As pointed out by one of the referees, Yang and Yuan [8, Theorem 1] show that if $\tau > 1/p$ and $0 < p, q < \infty$, then $F_{p,q}^{a,q} = F_{\infty,\infty}^{a+\nu(r/n,p),\infty}$, where the definition of $F_{p,q}^{a,q}$ is given in Remark 1.3. Thus, for $0 < p < 1$ and $1 < q < \infty$,

$$(F_{p,q}^{a,q})' = F_{\infty,\infty}^{a-(n/p)-(n,\infty)} = F_{q,q}^{a-(1/p)-(1/q)} = C_{MO}^{a,q}_{(q/p)-(q/q)}'$$  \hspace{1cm} (3.12)

which demonstrates a different approach to the duality.

4. Proofs of the Plancherel-Pólya Inequalities

In this section we demonstrate the Plancherel-Pólya inequalities.

Proof of Theorem 1.4. Without loss of generality, we may assume that $\alpha = 0$. By (3.1), we rewrite $\tilde{\varphi}_j * f(u)$ as

$$\tilde{\varphi}_j * f(u) = \sum_Q \langle f, \varphi_Q \rangle \int \tilde{\varphi}_j(u - x) \varphi_Q(x) dx$$

$$= \sum_{k \in \mathbb{Z}} \sum_Q |Q| \langle f, \varphi_k(\cdot - x_Q) \rangle \int \tilde{\varphi}_j(u - x) \varphi_k(x - x_Q) dx.$$  \hspace{1cm} (4.1)

Using the inequality [2, page 151, equation (B.5)]

$$\left| \int \tilde{\varphi}_j(u - x) \varphi_k(x - x_Q) dx \right| \leq C 2^{-k|j-k|} \frac{2^{-j/k}}{(2^{-j/k} + |u - x_Q|)^{n+1}},$$  \hspace{1cm} (4.2)
where \( j \wedge k = \min\{j, k\} \) and \( K > 1 + nr \), we obtain

\[
\left| \tilde{\phi}_j * f(u) \right| \leq C \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{(j-k)^2}} 2^{-K[j-k]} |Q| \frac{2^{-j/k}}{2^{-j/k} + |x_Q - x_Q'|}^{n+1} |\tilde{\phi}_k * f(x_Q)|. \tag{4.3}
\]

Thus, for \( \ell'(Q') = 2^{-j} \),

\[
\left( \sup_{u \in Q'} \left| \tilde{\phi}_j * f(u) \right| \right)^q \leq C \left( \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{(j-k)^2}} 2^{-K[j-k]} |Q| \frac{2^{-j/k}}{2^{-j/k} + |x_Q - x_Q'|}^{n+1} |\tilde{\phi}_k * f(x_Q)| \right)^q \tag{4.4}
\]

where the last inequality is followed by Hölder’s inequality and

\[
\sum_{k \in \mathbb{Z}} 2^{-K[j-k]} |Q| \frac{2^{-j/k}}{2^{-j/k} + |x_Q - x_Q'|}^{n+1} \leq C. \tag{4.5}
\]

Denote \( T_Q \) by

\[
T_Q := \inf_{u \in Q} \left| \tilde{\phi}_k * f(u) \right|^q. \tag{4.6}
\]

Since \( x_Q \) can be replaced by any point in \( Q \) in the last inequality,

\[
\left( \sup_{u \in Q'} \left| \tilde{\phi}_j * f(u) \right| \right)^q \leq C \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_{(j-k)^2}} 2^{-K[j-k]} |Q| \frac{2^{-j/k}}{2^{-j/k} + |x_Q - x_Q'|}^{n+1} T_Q. \tag{4.7}
\]

Given a dyadic cube \( P \) with \( \ell(P) = 2^{-k_0} \), the above estimates yield

\[
\sum_{j=k_0}^\infty \sum_{Q \subset P} \left( \sup_{u \in Q'} \left| \tilde{\phi}_j * f(u) \right| \right)^q |Q'| \leq C \sum_{j=k_0}^\infty \sum_{Q \subset P} \sum_{k \in \mathbb{Z}} \sum_{Q' \subset Q} 2^{-K[j-k]} |Q'| \frac{2^{-j/k}}{2^{-j/k} + |x_Q - x_Q'|}^{n+1} T_Q |Q| \tag{4.8}
\]

\[
:= CA_1 + CA_2,
\]
where

\[ A_1 = \sum_{j=k_0}^{\infty} \sum_{Q \in P} \sum_{k<k_0} \sum_{Q \in \mathcal{Q}} 2^{-K^{|j-k|}} |Q'| \frac{2^{-|j-k|}}{(2^{-|j-k|} + |x_Q - x_Q|)^{\omega+1}} T_Q |Q|, \]

\[ A_2 = \sum_{j=k_0}^{\infty} \sum_{Q \in P} \sum_{k<k_0} \sum_{Q \in \mathcal{Q}} 2^{-K^{|j-k|}} |Q'| \frac{2^{-|j-k|}}{(2^{-|j-k|} + |x_Q - x_Q|)^{\omega+1}} T_Q |Q|. \]  

(4.9)

Then, \( A_1 \) can be further decomposed as

\[ A_1 = \sum_{j=k_0}^{\infty} \sum_{Q \in P} \sum_{k<k_0} \sum_{Q \in \mathcal{Q}} 2^{-K^{|j-k|}} |Q'| \frac{2^{-|j-k|}}{(2^{-|j-k|} + |x_Q - x_Q|)^{\omega+1}} T_Q |Q| \]

\[ + \sum_{j=k_0}^{\infty} \sum_{Q \in P} \sum_{k<k_0} \sum_{Q \in \mathcal{Q}, Q' \neq \emptyset} 2^{-K^{|j-k|}} |Q'| \frac{2^{-|j-k|}}{(2^{-|j-k|} + |x_Q - x_Q|)^{\omega+1}} T_Q |Q| \]

\[ := A_{11} + A_{12}. \]  

(4.10)

There are \( 3^n \) dyadic cubes in \( 3P \) with the same side length as \( P \), so

\[ \sum_{Q \in \mathcal{Q}} T_Q |Q| \leq 3^n \sup_{P' \in \mathcal{P}} \sum_{Q \in \mathcal{Q}, Q \subseteq P'} T_Q |Q|. \]  

(4.11)

Thus,

\[ |P|^r A_{11} \leq C|P|^r \sum_{j=k_0}^{\infty} \sum_{Q \in \mathcal{Q}} \sum_{k<k_0} \sum_{Q \in \mathcal{Q}} 2^{-K^{|j-k|}} |Q'| \frac{2^{-|j-k|}}{(2^{-|j-k|} + |x_Q - x_Q|)^{\omega+1}} T_Q |Q| \]

\[ \leq C \sup_{P'} |P'|^r \sum_{k=-\log_3 \ell(P')}^{\infty} \sum_{k<k_0} \inf_{u \in Q} |\tilde{f}_k * f(u)|^q |Q|. \]  

(4.12)

Next we decompose the set of dyadic cubes \( \{Q : Q \cap 3P = \emptyset, \ell(Q) = \ell(P)\} \) into \( \{B_i\}_{i \in \mathbb{N}} \) according to the distance between each \( Q \) and \( P \). Namely, for each \( i \in \mathbb{N} \),

\[ B_i := \left\{ P' : P' \cap 3P = \emptyset, \ell(P') = \ell(P), 2^{i-k_0} \leq |y_{P'} - y_P| < 2^{i-k_0+1} \right\}, \]  

(4.13)
where \( y_Q \) denotes the center of \( Q \). Then, we obtain

\[
|P|^{-\tau} A_{12} \leq C \sum_{i=1}^{\infty} \sum_{P \in B_i} |P|^{-\tau} \sum_{j=k_0}^{\infty} \sum_{Q \subseteq P \atop \ell(Q) \geq 2^{-i}} \sum_{k \geq k_0} \sum_{Q \cap P \atop \ell(Q) \geq 2^{-i}} 2^{-K[j-k]} |Q| \tag{4.14}
\]

\[
\times \frac{2^{-j\bar{k}k}}{(2^{-j\bar{k}k} + |x_P - x_P|)^{n+1}} T_Q|Q|.
\]

Since \( \sum_{Q \subseteq P \atop \ell(Q) \geq 2^{-i}} |Q| = |P| \) for each \( j \geq k_0 \) and \( |x_P - x_P| \approx 2^{j-k_0} \) for \( P' \in B_i \), the right-hand side of (4.14) is dominated by

\[
C \sum_{i=1}^{\infty} \sum_{P \in B_i} |P| \frac{2^{-k_0}}{2^{(i-k_0)(n+1)}} \left( \sum_{j=k_0}^{\infty} 2^{k_0-(j\bar{k}k)+|k-j|} \right) \left( |P|^{-\tau} \sum_{Q \subseteq P \atop \ell(Q) \geq 2^{-i}} T_Q|Q| \right). \tag{4.15}
\]

There are at most \( 2^{(i+2)n} \) cubes in \( B_i \), and hence

\[
|P|^{-\tau} A_{12} \leq C \left\{ \sup_{P'} |P'|^{-\tau} \sum_{k \geq k_0} \sum_{Q \subseteq P'} \sum_{Q \subseteq P \atop \ell(Q) \geq 2^{-i}} T_Q|Q| \right\} \sum_{i=1}^{\infty} |P| \frac{2^{-k_0}}{2^{(i-k_0)(n+1)}} 2^{2in} \tag{4.16}
\]

\[
= C \sup_{P'} |P'|^{-\tau} \sum_{k=-\log_2 \ell(P')}^{\infty} \sum_{Q \subseteq P'} \inf_{u \in Q} \left| \tilde{\varphi}_k * f(u) \right| |Q|.
\]

To estimate \( A_2 \), for \( i \in \mathbb{N} \) and \( k < k_0 \), set

\[
E_{i,k} := \left\{ Q : \ell(Q) = 2^{-k}, \quad x_Q \in 2^{i}P \setminus 2^{i-1}P \right\}. \tag{4.17}
\]

Then, \( |x_Q - x_P| \approx 2^{j-k_0} \) for \( Q \in E_{i,k} \) and

\[
A_2 = \sum_{j=k_0}^{\infty} \sum_{k < k_0} \sum_{i=1}^{\infty} \sum_{Q \in E_{i,k}} 2^{-K[j-k]} |P| \frac{2^{-j\bar{k}k}}{|Q|^{-\tau}} \frac{2^{-j\bar{k}k}}{(2^{-j\bar{k}k} + |x_Q - x_P|)^{n+1}} T_Q|Q| |Q|^{\tau} T_Q|Q|. \tag{4.18}
\]

Since, for \( Q \in E_{i,k} \),

\[
|Q|^{\tau} T_Q|Q| \leq \sup_{P'} |P'|^{-\tau} \sum_{m=-\log_2 \ell(P')}^{\infty} \sum_{Q \subseteq P'} \sum_{Q \subseteq P \atop \ell(Q) \geq 2^{-m}} T_Q|Q'| \tag{4.19}
\]
and the number of dyadic cubes contained in $E_{i,k}$ is at most $2^{(i+k-k_0)n}$,

$$|P|^r A_2 \leq C \left\{ \sup_{P'} |P'|^{-r} \sum_{m=-\log_\ell \ell'(|P|)}^{\infty} \sum_{Q \in \Omega \atop \ell'(Q) = \ell} T_Q |Q'| \right\}$$

$$\times \sum_{j=k_0}^{\infty} \sum_{k=k_0}^{\infty} 2^{(k-j)n} 2^{K(k-j)} 2^{-k(n+1)} \frac{2^{k(n+1)}}{2^{(i-k_0)(n+1)}} 2^{(i+k-k_0)n}$$

(4.20)

where the condition $K > 1 + nr$ is used in the last equality. Combining the estimates of $A_1$ and $A_2$, we prove Theorem 1.4.

By modifying the proof above, we may easily show Theorem 1.5. Detailed verifications are left to the reader.

We now return to show Lemma 3.2.

Proof of Lemma 3.2. For $r < 0$, $c_{r^d} = \{0\}$, and hence the result holds. For $r = 0$, $c_0 = f_q^{a_d}$, and so the matrix is bounded by $[2, \text{Theorem 3.3}].$ To complete the proof, it suffices to show the boundedness of $(\alpha + nr, q, q')$-almost diagonal matrices for the case $r > 0$.

We may assume that $\alpha = 0$ since the case implies the general case. The proof is similar to the proof of Theorem 1.4. Here, we only outline the proof. First let us consider the case for $q > 1$. Let $A = \{a_{Q,P}\}_{Q,P}$ be an $(nr, q, q')$-almost diagonal matrix. Then, for $\ell'(Q) = 2^{-k}$,

$$\left| (A_{n})_Q \right| \leq C \sum_{j \in \mathbb{Z}} \sum_{Q \in \Omega \atop \ell(Q) = 2^{-j}} 2^{(j-k)(nr+q(1+q)/2)} \left( 1 + 2^j |x_Q - x_P| \right)^{-n} |Q|^{q} \left| P \right|^{q}$$

$$\left( |Q|^{-1/2} \left| (A_{n})_Q \right| \right)^{q} \leq C \sum_{j \in \mathbb{Z}} \sum_{Q \in \Omega \atop \ell(Q) = 2^{-j}} 2^{(j-k)(nr+q(1+q)/2)} \left( 1 + 2^j |x_Q - x_P| \right)^{-n} \left( |Q|^{-1/2} |P|^{q} \right)^{q}$$

(4.21)

due to Hölder’s inequality. Given a dyadic cube $R$ with $\ell(R) = 2^{-\delta}$,

$$\sum_{k \geq \delta} \sum_{Q \subseteq R \atop \ell(Q) = 2^{-k}} \left( |Q|^{-1/2} \left| (A_{n})_Q \right| \right)^{q} |Q| \leq CI + CII,$$

(4.22)

where

$$I = \sum_{k \geq \delta} \sum_{Q \subseteq R \atop \ell(Q) = 2^{-k}} \sum_{j \geq \delta} \sum_{P_j \subseteq \Omega \atop \ell(P_j) = 2^{-j}} 2^{(j-k)(nr+q(1+q)/2)} \left( 1 + 2^j |x_Q - x_P| \right)^{-n} \left( |P|^{-1/2} |S_j| \right)^{q} |P|,$$

$$II = \sum_{k \geq \delta} \sum_{Q \subseteq R \atop \ell(Q) = 2^{-k}} \sum_{j \geq \delta} \sum_{P_j \subseteq \Omega \atop \ell(P_j) = 2^{-j}} 2^{(j-k)(nr+q(1+q)/2)} \left( 1 + 2^j |x_Q - x_P| \right)^{-n} \left( |P|^{-1/2} |S_j| \right)^{q} |P|.$$
Then, $I$ can be further decomposed as

$$I = \sum_{k \geq 0} \sum_{Q \subseteq R} \sum_{j \geq 2^k} \sum_{P \subseteq 3R} 2^{(j-k)(nr+n(c/2))} \left(1 + 2^j |x_Q - x_P|\right)^{-n-c} \left(|P|^{-1/2}|s_p|\right)^q |P|$$

$$+ \sum_{k \geq 0} \sum_{Q \subseteq R} \sum_{j \geq 2^k} \sum_{P \supseteq 3R} 2^{(j-k)(nr+n(c/2))} \left(1 + 2^j |x_Q - x_P|\right)^{-n-c} \left(|P|^{-1/2}|s_p|\right)^q |P|$$

$:= I_{11} + I_{12}.$

The same argument showed in the proof of Theorem 1.4 for the term $A_1$ gives us

$$|R|^{-r} I \leq C \|s\|_{c^{r,q}_{\epsilon,\lambda}}^q.$$  \hfill (4.25)

To estimate $II$, let $i \in \mathbb{N}$ and $j < \delta$, let

$$E_{i,j} := \left\{ Q : \ell(Q) = 2^{-j}, \ x_Q \in 2^i R \setminus 2^{i-1} R \right\}.$$  \hfill (4.26)

Then, using the same argument as Theorem 1.4 for $A_2$, we have

$$|R|^{-r} II \leq C \|s\|_{c^{r,q}_{\epsilon,\lambda}}^q.$$  \hfill (4.27)

Both estimates for $I$ and $II$ show the desired result for $q > 1$. When $q \leq 1$, we modify the previous proof by replacing Hölder’s inequality with $q$-triangle inequality to get the result.

When $q = \infty$ and $r \geq 0$, the space $c^{a,\infty} = f^{a,\infty}$, and hence an $(a + nr, \infty, \infty)$-almost diagonal matrix is bounded on $c^{a,\infty}$ by Proposition 5.3.

**Remark 4.1.** Note that $c^{a,q}_{\epsilon,\lambda} = f^{a,q}_{\epsilon,\lambda}$. By a duality argument and [2, Theorem 3.3 and page 81], one can show that the $(\alpha + n, q, q)$-almost diagonal matrix is bounded on $f^{a,q}_{\epsilon,\lambda}$. When $q > 1$ and $r > 1$, we can prove Lemma 3.2 by duality in Theorem 2.2. Let $A = \{a_{QP}\}_{Q,P}$ be an $(nr, q, q)$-almost diagonal matrix. Also define the transpose of $A$ by $A' = \{a_{QP}\}_{Q,P}$. For $q > 1$ and $r > 1$, let $p = (q + q')/(q' + q)$. Then, $p < 1$. Since $A$ is $(nr, q, q)$-almost diagonal, $A'$ is $(0, p, q')$-almost diagonal by a calculation for a different value of $\epsilon$. Thus, by Theorem 2.2 (a) and Proposition 5.3, $A'$ is bounded on $c^{0,q}_{\epsilon,\lambda}$.

**5. Applications**

We define another wavelet multiplier on $\mathbb{R}^n$ by using $\psi$-transform identity as follows. Let $\varphi$ and $\psi$ in $S$ satisfy (1.2) and (3.1). For a sequence $t = \{t_Q\}_Q$ where the $Q$’s are dyadic cubes in $\mathbb{R}^n$, define the wavelet multiplier $T_t$ by

$$T_t(f) = \sum_Q |Q|^{-1/2} t_Q(f, \varphi_Q) \psi_Q$$  \hfill (5.1)
for \( f \in \mathcal{S}'/\mathcal{D} \) such that the above summation is well defined. Thus, we have the following characterization.

**Theorem 5.1.** Suppose that \( \alpha, \beta \in \mathbb{R}, 0 < p \leq 1 \), and \( 0 < q < \infty \). Then,

(a) for \( 1 < q < \infty \), \( T_i \) is bounded from \( f_p^{\alpha,q} \) into \( f_1^{\alpha+\beta,1} \) if \( t \in c_{(q'/p)-(q'/q)}^{\beta,q} \)

(b) for \( 0 < q \leq 1 \) and \( r \in \mathbb{R} \), \( T_i \) is bounded from \( f_p^{\alpha,q} \) into \( f_1^{\alpha+\beta,1} \) if \( t \in c_r^{\beta+(n/p)-n,\infty} \).

**Proof.** We show the case \( \alpha = 0 \) only, which implies the general case by (2.7). For \( \beta \in \mathbb{R}, 0 < p \leq 1 \), and \( 1 < q < \infty \), let \( f \in f_p^{\alpha,q} \) and \( t \in c_{(q'/p)-(q'/q)}^{\beta,q} \). It follows from Theorem 2.2 and Proposition 3.1 that

\[
\| T_i(f) \|_{f_1^{\alpha,1}} \leq C \left\| \{ |Q|^{-1/2} t_Q \langle f, \varphi_Q \rangle \} \right\|_{f_1^{\beta,1}} \\
= C \sum_Q \left\| \{ |Q|^{-\beta/n^*} \langle f, \varphi_Q \rangle \} \|_{f_1^{\beta,1}} \right\|_{c_{(q'/p)-(q'/q)}^{\beta,q}} \\
\leq C \| \{ \langle f, \varphi_Q \rangle \} \|_{f_p^{\alpha,q}} \left\| \{ |Q|^{-\beta/n^*} t_Q \} \right\|_{c_{(q'/p)-(q'/q)}^{\beta,q}} \\
\leq C \| f \|_{f_p^{\alpha,q}} \| t \|_{c_{(q'/p)-(q'/q)}^{\beta,q}}.
\]

This shows that \( T_i \) is bounded from \( f_p^{\alpha,q} \) into \( f_1^{\beta,1} \) and \( \| T_i \| \leq C \| t \|_{c_{(q'/p)-(q'/q)}^{\beta,q}} \). A similar argument yields the boundedness of \( T_i \) for the case \( 0 < q \leq 1 \).

In order to prove Theorem 1.12, we demonstrate a similar result in sequence spaces first. For a sequence \( t = \{ t_Q \}_Q \), define \( D_t \) by

\[
D_t(s) = \left\{ |Q|^{-1/2} t_Q s_Q \right\}_Q \quad \text{for} \quad s = \{ s_Q \}_Q \text{ with finitely many nonzero terms.}
\]

**Theorem 5.2.** Suppose that \( \alpha, \beta \in \mathbb{R}, 0 < p \leq 1 \), and \( 0 < q < \infty \). Then,

(a) for \( 1 < q < \infty \), \( D_t \) is extendible to be bounded from \( f_p^{\alpha,q} \) into \( f_1^{\alpha+\beta,1} \) if and only if \( t \in c_{(q'/p)-(q'/q)}^{\beta,q} \)

(b) for \( 0 < q \leq 1 \) and \( r \in \mathbb{R} \), \( D_t \) is extendible to be bounded from \( f_p^{\alpha,q} \) into \( f_1^{\alpha+\beta,1} \) if and only if \( t \in c_r^{\beta+(n/p)-n,\infty} \).
Proof. We still assume that $\alpha = 0$. For $\beta \in \mathbb{R}$, $0 < p \leq 1$, and $1 < q < \infty$, let $s = \{s_{Q}\}_{Q} \in f^{\beta,q}_{p}$ and $t = \{t_{Q}\}_{Q} \in c^{(q/p)-(q/q)}$. It follows from Theorem 2.2 that

$$
\|D_t(s)\|_{f^{\beta,1}_1} = \sum_{Q} \left| |Q|^{-\beta/n} |t_{Q}| \right| s_{Q} \leq C \|s\|_{f^{\beta,q}_p} \left\| \left\{ |Q|^{-\beta/n} t_{Q} \right\}_{Q} \right\|_{c^{(q/p)-(q/q)}},
$$

(5.4)

Conversely, suppose that $D_t$ maps from $f^{\alpha,q}_p$ into $f^{\beta,1}_1$ boundedly. For $t = \{t_{Q}\}_{Q}$, let $\tilde{t} = \{ |Q|^{-\beta/n} t_{Q} \}_{Q}$. Define a linear functional $\ell_t^{*}$ by

$$
\ell_t^{*}(s) = \sum_{Q} s_{Q} \tilde{t}_{Q} \text{ for } s = \{s_{Q}\}_{Q} \text{ with finitely many nonzero terms.}
$$

(5.5)

Then,

$$
|\ell_t^{*}(s)| \leq \sum_{Q} \left( |Q|^{-\beta/n} |t_{Q}| \right) |s_{Q}| = \|D_t(s)\|_{f^{\beta,1}_1}.
$$

(5.6)

The assumption shows that $\ell_t^{*}$ is a continuous linear functional on $f^{\alpha,q}_p$. Using Theorem 2.2, we have $\tilde{t} \in c^{(q/p)-(q/q)}$ and hence $t \in c^{(q/p)-(q/q)}$.

For $0 < q \leq 1$, a similar argument gives the desired result of (b). \qed

Proof of Theorem 1.12. The “if” part follows from Theorem 5.1. To show the “only if” part, define $T^{\dagger}_{q}$ by

$$
T^{\dagger}_{q}(f) = \sum_{Q} |Q|^{-1/2} t_{Q} \left\langle f, \varphi_{q}^{j} \right\rangle \varphi_{q}^{j}.
$$

(5.7)

The boundedness of $T_{q}$ says that $T^{\dagger}_{q}$ is bounded from $F^{\alpha,q}_{p}$ into $F^{\beta,1}_{1}$. Clearly,

$$
S_{q^{*}} \circ T^{\dagger}_{q} \circ T_{q}(s) = D_{t}(s) \text{ for } s \in f^{\alpha,q}_{p}.
$$

(5.8)

It follows from Proposition 3.1 that $D_{t}$ is bounded from $f^{\alpha,q}_{p}$ into $f^{\beta,1}_{1}$, and hence $t \in c^{(q/p)-(q/q)}$ for $1 < q < \infty$ and $t \in c^{(q/(q/p))-(q/q)}$ for $0 < q \leq 1$ and $r \in \mathbb{R}$ by Theorem 5.2. \qed

In order to study the boundedness of the paraproduct operators acting on Triebel-Lizorkin spaces, we need more results described as follows.

**Proposition 5.3 ([2, pages 54 and 81]).** For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$, an $(\alpha, p, q)$-almost diagonal matrix is bounded on $f^{\alpha,q}_{p}$.
Lemma 5.4. Define a matrix by \( G = \{ (\psi_p, \Phi_Q) \} \). Then, for \( \alpha < 0 \) and \( 0 < p, q \leq +\infty \), \( G \) is \((\alpha, p, q)\)-almost diagonal and hence is bounded on \( f^{a_{p,q}}\).

Proof. For \( \ell(P) \leq \ell(Q) \), since \( \int x^r \psi_p(x) dx = 0 \) for all \( r \), by \cite[page 150, Lemma B.1]{2}, we have

\[
|\langle \psi_p, \Phi_Q \rangle| \leq C \left( \ell(Q) / \ell(P) \right)^\alpha \left( 1 + \frac{|x_Q - x_P|}{\ell(P)} \right)^{-1 - \epsilon / q} \left( \ell(P) / \ell(Q) \right)^{(n+\epsilon)/2 + J - n}
\]

for \( \epsilon > 0 \) and \( \alpha < J - n + (\epsilon/2) \), where \( J = n / \min\{1, p, q\} \) and \( C \) is independent of \( P \) and \( Q \).

For \( \ell(Q) < \ell(P) \), by \cite[page 152, Lemma B.2]{2}, we obtain

\[
|\langle \psi_p, \Phi_Q \rangle| \leq C \left( \ell(Q) / \ell(P) \right)^\alpha \left( 1 + \frac{|x_Q - x_P|}{\ell(P)} \right)^{-1 - \epsilon / q} \left( \ell(P) / \ell(Q) \right)^{(n-2\alpha)/2}
\]

(5.10)

Choosing \( \epsilon = -2\alpha \), we obtain the result. \( \square \)

We now can prove Theorem 1.13.

Proof of Theorem 1.13. To simplify notations, let \( q_0 = qr / (q - r) \) and \( (1/p_0) = (1/p) - (1/q) + (1/q_0) \). The requirement \( p \leq r < q < r / (1 - r) \) guarantees that \( p_0 \leq 1 \leq q_0 \). Now assume that \( g \in CMO_{(q_0/p_0)-(q_0/q_0)}^{\beta_{d_0}} \) and \( f \in F^{a_{p,q}}_p \). To prove part (i), by (3.1) we rewrite \( \Pi_s(f) \) as

\[
\Pi_s(f) = \sum_Q \langle g, \varphi_Q \rangle |Q|^{-1/2} \left( \sum_P \langle f, \varphi_p \rangle \xi_p, \Phi_Q \right) \varphi_Q
\]

(5.11)

where \( s = \{ \langle f, \varphi_p \rangle \}_p \). Proposition 3.1 and Theorem 2.2 give

\[
\left\| \Pi_s(f) \right\|_{F^{a_{p,q}}_p} \leq C \left\| \left\{ |Q|^{-1/2} \langle g, \varphi_Q \rangle (G_s)_Q \right\}_Q \right\|_{F^{a_{p,q}}_p}
\]

\[
= C \sum_Q \left( \langle g, \varphi_Q \rangle \right)^{r'} \left( |Q|^{-\frac{(\beta/n) - (1/2)}{2} + (1/2r)} \right)^{r'} \left( |Q|^{-\frac{(\alpha/n) - (1/2)}{2} + (1/2r)} (G_s)_Q \right)^{r'}
\]

\[
\leq C \left\| \left\{ \langle g, \varphi_Q \rangle \right\}_Q \right\|_{F^{a_{p,q}}_p} \left\| \left\{ \langle G_s, \varphi_Q \rangle \right\}_Q \right\|_{F^{a_{p,q}}_p}
\]

(5.12)
It is clear that

\[ \left\| \left\{ |Q|^{-\left(\beta/n\right) - (1/2) + (1/2r)} \left( \langle g, \varphi Q \rangle \right) \right\} \right\|_{L^{r/\left(q/r\right)}} \]

\[ = \sup_{p^*} \left\{ |P|^{-r(q/r')}((1/p) - (1/q)) \int_{Q \subseteq P} \left| \langle g, \varphi Q \rangle \right|_{X_Q(x)}^{r(q/r')} \, dx \right\}^{1/(q/r')} \]

\[ = \left\| \left\{ \langle g, \varphi Q \rangle \right\} \right\|_{L^{\left(q/r\right)\left(q/r\right)}}, \tag{5.13} \]

and

\[ \left\| \left\{ |Q|^{-\left(\alpha/n\right) - (1/2) + (1/2r)} (G_s)_{Q} \right\} \right\|_{L^{r/\left(q/r\right)}} \]

\[ = \left\| \left( \sum_{Q} \left( |Q|^{-\left(\alpha/n\right) - (1/2)} (G_s)_{Q} \right) \right)^{r/q} \, \right\|_{L^{r/\left(q/r\right)}} \]

\[ = \left\| (G_s) \right\|_{L^{r/\left(q/r\right)}}, \tag{5.14} \]

Hence, by Propositions 3.1 and 3.3, and Lemma 5.4,

\[ \left\| \Pi_f (f) \right\|_{L^{r+\beta/r}} \leq C \left\| \left\{ \langle g, \varphi Q \rangle \right\} \right\|_{L^{\left(q/r\right)\left(q/r\right)}} \left\| (G_s) \right\|_{L^{r+\beta/r}} \]

\[ \leq C \left\| g \right\|_{CMC_{\left(q/r\right)\left(q/r\right)}} \left\| s \right\|_{L^{p/q}} \]

\[ \leq C \left\| g \right\|_{CMC_{\left(q/r\right)\left(q/r\right)}} \left\| f \right\|_{L^{p/q}}. \tag{5.15} \]

Next suppose that \( \Pi_f \) is bounded from \( f_{p,q}^{\alpha} \) into \( f_{r}^{\alpha+\beta} \). Without loss of generality, we may assume that \( \alpha = 0 \). A computation yields

\[ \left( |P|^{-q_0((1/p) + (1/q)) - 1} \int_{Q \subseteq P} \left( |Q|^{-\left(\beta/n\right) - (1/2)} \left( \langle g, \varphi Q \rangle \right) \right)^{q_0} \, dx \right)^{1/q_0} \]

\[ = |P|^{-1/(p + 1)} \left( \int_{Q \subseteq P} \left( |Q|^{-\left(\beta/n\right) - (1/2)} \left( \langle g, \varphi Q \rangle \right) \right)^{q/(q-r)} \, dx \right)^{(q-r)/q} \]

\[ \leq C |P|^{-1/p} \left( \int_{Q \subseteq P} \left( |Q|^{-\left(\beta/n\right) - (1/2)} \left( \langle g, \varphi Q \rangle \right) \right) \, dx \right)^{1/r}. \tag{5.16} \]
Fix an integer $N > (n/p) - n$. Choose a function $\theta \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\theta(x) = 1$ on $[0,1]^n$, $\theta(x) = 0$ if $x \notin [0,1]^n$ and $\int x^\gamma \theta(x) dx = 0$ for all multi-indices $\gamma$ with $|\gamma| \leq N$. By the molecular theory [2, page 56], it follows that $\theta \in \mathcal{F}_{\rho A}$. For each dyadic cube $P$, define $\theta^P$ by

$$
\theta^P(x) = \frac{x - x_P}{\ell(P)}.
$$

Then, $\langle \theta^P, \Phi_Q \rangle = \int \Phi_Q(x) dx = |Q|^{1/2}$ for all dyadic cubes $Q \subseteq P$ and $\| \theta^P \|_{\mathcal{F}_{\rho A}} = C|P|^{1/p}$ by the translation invariance and the dilation properties of $\mathcal{F}_{\rho A}$. By Proposition 3.1,

$$
\left\| \Pi_g(\theta^P) \right\|_{\mathcal{F}_{\rho A}} \approx \left\| \left\{ \langle g, \varphi_Q \rangle |Q|^{-1/2} \langle \theta^P, \Phi_Q \rangle \right\}_Q \right\|_{\mathcal{F}_{\rho A}} \geq \left( \int_P \sum_{Q \subseteq P} \left( |Q|^{-(\beta/n) - (1/2)} |\langle g, \varphi_Q \rangle | \right)^{\beta} dx \right)^{1/\beta},
$$

and hence, by the boundedness of $\Pi_g$,

$$
\left( |P|^{-\rho_B((1/p_0) + (1/q_0) - 1)} \int_P \sum_{Q \subseteq P} \left( |Q|^{-(\beta/n) - (1/2)} |\langle g, \varphi_Q \rangle | \right)^{\beta} dx \right)^{1/\beta} \leq C.
$$

Taking the supremum on $P$, we show that $g \in \text{CMO}_{(\rho_B, \rho_B)}^{(\beta, \beta)}$.

To prove part (ii), assume that $g \in \text{CMO}_{(\rho_B, \rho_B)}^{(\beta, \beta)}$ and $f \in \mathcal{F}_{\rho A}$. Let $t = \{ \langle g, \varphi_Q \rangle \}_Q$ and $s = \{ \langle f, \varphi_Q \rangle \}_Q$. By Proposition 3.1,

$$
\left\| \Pi^*_g(f) \right\|_{\mathcal{F}_{\rho A}} \approx \left\| \left\{ \sum_P |P|^{-1/2} \langle g, \varphi_P \rangle \langle \Phi_P, \varphi_Q \rangle \langle f, \varphi_P \rangle \right\}_Q \right\|_{\mathcal{F}_{\rho A}} \leq C |Dts|_{\mathcal{F}_{\rho A}^*},
$$

where $\tilde{G} := \{ \langle \Phi_P, \varphi_Q \rangle \}_{Q,P}$ is the transpose of $\{ \langle \varphi_P, \Phi_Q \rangle \}_{Q,P}$. Since $\alpha + \beta > 0$, by Lemma 5.4, $\tilde{G}$ is $(\alpha + \beta, r, r)$-almost diagonal and hence is bounded on $\mathcal{F}_{\rho A}^{\alpha + \beta, r}$. Following the same argument as the proof of part (i), we get

$$
\left\| \Pi^*_g(f) \right\|_{\mathcal{F}_{\rho A}^*} \leq C |Dts|_{\mathcal{F}_{\rho A}^{\alpha + \beta, r}} = C \sum_Q \left( |Q|^{-(\beta/n) - (1/2)} \left| \langle g, \varphi_Q \rangle \right| \right)^{\beta} \cdot \left( |Q|^{-(\alpha/n) - (1/2r)} \left| \langle f, \varphi_Q \rangle \right| \right)^{\alpha} \leq C \left\| g \right\|_{\text{CMO}_{(\alpha/n)}^{(\beta, \beta)}} \left\| f \right\|_{\mathcal{F}_{\rho A}^{\alpha + \beta, r}},
$$

which completes the proof. \qed
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