Research Article

Some Results on an Infinite Family of Nonexpansive Mappings and an Inverse-Strongly Monotone Mapping in Hilbert Spaces

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We study the problem of approximating a common element in the common fixed point set of an infinite family of nonexpansive mappings and in the solution set of a variational inequality involving an inverse-strongly monotone mapping based on a viscosity approximation iterative method. Strong convergence theorems of common elements are established in the framework of Hilbert spaces.

1. Introduction and Preliminaries

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$, respectively. Let $C$ be a nonempty, closed, and convex subset of $H$. Let $A : C \to H$ be a mapping. Let $P_C$ be the metric projection from $H$ onto the subset $C$. The classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.1)$$

In this paper, we use $\text{VI}(C, A)$ to denote the solution set of the variational inequality. For a given point $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C, \quad (1.2)$$
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if and only if \( u = P_C z \). It is known that projection operator \( P_C \) is nonexpansive. It is also known that \( P_C \) satisfies

\[
\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.
\] (1.3)

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. The point \( u \in C \) is a solution of the variational inequality (1.1) if and only if \( u \in C \) satisfies the relation \( u = P_C (u - \lambda Au) \), where \( \lambda > 0 \) is a constant.

Recall the following definitions.

(a) \( A \) is said to be monotone if and only if

\[
\langle Ax - Ay, x - y \rangle \geq 0, \quad x, y \in C.
\] (1.4)

(b) \( A \) is said to be \( \alpha \)-strongly monotone if and only if there exists a positive real number \( \alpha \) such that

\[
\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad x, y \in C.
\] (1.5)

(c) \( A \) is said to be \( \alpha \)-inverse-strongly monotone if and only if there exists a positive real number \( \alpha \) such that

\[
\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.
\] (1.6)

(d) A mapping \( S : C \to C \) is said to be nonexpansive if and only if

\[
\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.
\] (1.7)

In this paper, we use \( F(S) \) to denote the fixed point set of \( S \).

(e) A mapping \( f : C \to C \) is said to be a \( \kappa \)-contraction if and only if there exists a positive real number \( \kappa \in (0,1) \) such that

\[
\|f(x) - f(y)\| \leq \kappa \|x - y\|, \quad \forall x, y \in C.
\] (1.8)

(f) A linear bounded operator \( B \) on \( H \) is strongly positive if and only if there exists a positive real number \( \gamma \) such that

\[
\langle Bx, x \rangle \geq \gamma \|x\|^2, \quad \forall x \in H.
\] (1.9)

(g) A set-valued mapping \( T : H \to 2^H \) is called monotone if and only if for all \( x, y \in H, \ f \in Tx, \) and \( g \in Ty \) imply \( \langle x - y, f - g \rangle \geq 0 \). A monotone mapping \( T : H \to 2^H \) is maximal if the graph of \( G(T) \) of \( T \) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping \( T \) is maximal if and only if for \( (x, f) \in H \times H, \ \langle x - y, f - g \rangle \geq 0 \) for every \( (y, g) \in G(T) \) implies \( f \in Tx \). Let \( A \) be a monotone
map of $C$ into $H$ and let $N_Cv$ be the normal cone to $C$ at $v \in C$, that is, $N_Cv = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$ and define

$$Tv = \begin{cases} \text{Av} + N_Cv, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$ \hfill (1.10)

Then $T$ is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [1] and the reference therein.

For finding a common element in the fixed point set of nonexpansive mappings and in the solution set of the variational inequality involving inverse-strongly mappings, Takahashi and Toyoda [2] introduced the following iterative process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0,$$ \hfill (1.11)

where $A$ is an $\alpha$-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a real number sequence in $(0, 1)$, and $\{\lambda_n\}$ is a real number sequence in $(0, 2\alpha)$. They showed that the sequence $\{x_n\}$ generated in (1.11) weakly converges to some point $z \in F(S) \cap VI(C, A)$ provided that $F(S) \cap VI(C, A)$ is nonempty.

In order to obtain a strong convergence theorem of common elements, Iiduka and Takahashi [3] considered the problem by the following iterative process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0,$$ \hfill (1.12)

where $x$ is a fixed element in $C$, $A$ is an $\alpha$-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a real number sequence in $(0, 1)$, and $\{\lambda_n\}$ is a real number sequence in $(0, 2\alpha)$. They showed that the sequence $\{x_n\}$ generated in (1.12) strongly converges to some point $z \in F(S) \cap VI(C, A)$ provided that $F(S) \cap VI(C, A)$ is nonempty.

Iterative methods for nonexpansive mappings have been applied to solve convex minimization problems; see, for example, [4–8] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping $S$ on a real Hilbert space $H$:

$$\min_{x \in F(S)} \frac{1}{2} \langle Bx, x \rangle - \langle x, b \rangle,$$ \hfill (1.13)

where $B$ is a linear bounded self-adjoint operator, and $b$ is a given point in $H$. In [4], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n b, \quad n \geq 0,$$ \hfill (1.14)

strongly converges to the unique solution of the minimization problem (1.13) provided that the sequence $\{\alpha_n\}$ satisfies certain conditions.
Recently, Marino and Xu [5] considered the problem by viscosity approximation method. They study the following iterative process:

\[ x_0 \in C, \quad x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n f(x_n), \quad n \geq 0, \]

(1.15)

where \( f \) is a contraction. They proved that the sequence \( \{x_n\} \) generated by the above iterative scheme strongly converges to the unique solution of the variational inequality

\[ \langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C, \]

(1.16)

which is the optimality condition for the minimization problem \( \min_{x \in F(S)} (Bx, x) - h(x) \), where \( h \) is a potential function for \( \delta f \) (i.e., \( h'(x) = \delta f(x) \) for \( x \in H \)).

Concerning a family of nonlinear mappings has been considered by many authors; see, for example, [9–21] and the references therein. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see, for example, [16, 17].

Recently, Qin et al. [18] considered a general iterative algorithm for an infinite family of nonexpansive mapping in the framework of Hilbert spaces. To be more precise, they introduced the following general iterative algorithm:

\[ x_0 \in C, \quad x_{n+1} = \lambda_n f(x_n) + \beta_n x_n + \left( (1 - \beta_n)I - \lambda_n A \right) W_n x_n, \quad n \geq 0, \]

(1.17)

where \( f \) is a contraction on \( H \), \( A \) is a strongly positive bounded linear operator, \( W_n \) are nonexpansive mappings which are generated by a finite family of nonexpansive mapping \( T_1, T_2, \ldots \) as follows:

\[
\begin{align*}
U_{n,n+1} &= I, \\
U_{n,n} &= \gamma_n T_n U_{n,n+1} + (1 - \gamma_n)I, \\
U_{n,n-1} &= \gamma_{n-1} T_{n-1} U_{n,n} + (1 - \gamma_{n-1})I, \\

&\vdots \\
U_{n,k} &= \gamma_k T_k U_{n,k+1} + (1 - \gamma_k)I, \\
u_{n,k-1} &= \gamma_{k-1} T_{k-1} U_{n,k} + (1 - \gamma_{k-1})I, \\

&\vdots \\
U_{n,2} &= \gamma_2 T_2 U_{n,3} + (1 - \gamma_2)I, \\
W_n &= U_{n,1} = \gamma_1 T_1 U_{n,2} + (1 - \gamma_1)I, 
\end{align*}
\]

(1.18)

where \( \{\gamma_1\}, \{\gamma_2\}, \ldots \) are real numbers such that \( 0 \leq \gamma \leq 1 \), \( T_1, T_2, \ldots \) become an infinite family of mappings of \( C \) into itself. Nonexpansivity of each \( T_i \) ensures the nonexpansivity of \( W_n \).
Concerning $W_n$ we have the following lemmas which are important to prove our main results.

**Lemma 1.1** (see [19]). Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $T_1, T_2, \ldots$ be nonexpansive mappings of $C$ into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \leq \eta < 1$ for any $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n \to \infty} U_{nk} x$ exists.

Using Lemma 1.1, one can define the mapping $W$ of $C$ into itself as follows. $W x = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x$, for every $x \in C$. Such a $W$ is called the $W$-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$. Throughout this paper, we will assume that $0 < \gamma_n \leq \eta < 1$ for all $n \geq 1$.

**Lemma 1.2** (see [19]). Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $T_1, T_2, \ldots$ be nonexpansive mappings of $C$ into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \leq \eta < 1$ for any $n \geq 1$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.

Motivated by the above results, in this paper, we study the problem of approximating a common element in the common fixed point set of an infinite family of nonexpansive mappings, and in the solution set of a variational inequality involving an inverse-strongly monotone mapping based on a viscosity approximation iterative method. Strong convergence theorems of common elements are established in the framework of Hilbert spaces.

In order to prove our main results, we need the following lemmas.

**Lemma 1.3** (see [5]). Assume $B$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\gamma > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho \gamma$.

**Lemma 1.4** (see [22]). Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + \delta_n, \quad (1.19)$$

where $\gamma_n$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n \to \infty} \delta_n / \gamma_n < 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

**Lemma 1.5** (see [23]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (1.20)$$

Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$. 

Lemma 1.6 (see [14, 15]). Let $K$ be a nonempty closed convex subset of a Hilbert space $H$, $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i)$, $\{\gamma_n\}$ be a real sequence such that $0 < \gamma_n \leq b < 1$ for each $n \geq 1$. If $C$ is any bounded subset of $K$, then $\lim_{n \to \infty} \sup_{x \in C} \|Wx - W_n x\| = 0$.

Lemma 1.7 (see [5]). Let $H$ be a Hilbert space. Let $B$ be a strongly positive linear bounded self-adjoint operator with the constant $\bar{\gamma} > 0$ and $f$ a contraction with the constant $\kappa$. Assume that $0 < \gamma < \bar{\gamma}/\kappa$. Let $T$ be a nonexpansive mapping with a fixed point $x_0 \in H$ of the contraction $x \mapsto \gamma f(x) + (I - tB)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of $T$, which solves the variational inequality

$$\langle (A - \gamma f)\bar{x}, z - \bar{x} \rangle \leq 0, \quad \forall z \in F(T).$$

(1.21)

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\bar{x} = \bar{x}$.

2. Main Results

Theorem 2.1. Let $H$ be a real Hilbert space and $C$ a nonempty closed convex subset of $H$. Let $A : C \to H$ be an $\alpha$-inverse-strongly monotone mapping and $f : C \to C$ a $\kappa$-contraction. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings from $C$ into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$. Let $B$ be a strongly positive linear bounded self-adjoint operator of $C$ into itself with the constant $\bar{\gamma} > 0$. Let $\{x_n\}$ be a sequence generated in

$$x_1 \in C,$$

$$y_n = \beta_n \gamma f(x_n) + (I - \beta_n B)W_n P_C(I - r_n A)x_n,$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P_C y_n, \quad n \geq 1,$$

where $W_n$ is generated in (1.18), $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0, 1)$. Assume that the control sequence $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following restrictions:

(i) $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$;

(iii) $\lim_{n \to \infty} |r_{n+1} - r_n| = 0$;

(iv) $\{r_n\} \subset [a, b]$ for some $a, b$ with $0 < a < b < 2\alpha$.

Assume that $0 < \gamma < \bar{\gamma}/\kappa$. Then $\{x_n\}$ strongly converges to some point $q$, where $q \in F$, where $q = P_F(\gamma f + (I - B))(q)$, which solves the variation inequality

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad \forall p \in F.'$$

(2.2)
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Proof. First, we show that the mapping \( I - r_n A \) is nonexpansive. Notice that

\[
\| (I - r_n A)x - (I - r_n A)y \|^2 = \| x - y - r_n (Ax - Ay) \|^2 \\
= \| x - y \|^2 - 2 r_n (x - y, Ax - Ay) + r_n^2 \| Ax - Ay \|^2 \\
\leq \| x - y \|^2 + r_n (r_n - 2 \alpha) \| Ax - Ay \|^2 \\
\leq \| x - y \|^2, \quad \forall x, y \in C,
\]

which implies that the mapping \( I - r_n A \) is nonexpansive. Since the condition (i), we may assume, with no loss of generality, that \( \beta_n < \| B \|^{-1} \) for all \( n \). From Lemma 1.3, we know that if \( 0 < \rho \leq \| B \|^{-1} \), then \( \| I - \rho B \| \leq 1 - \rho \gamma \). Letting \( p \in F \), we have

\[
\| y_n - p \| = \| \beta_n (\gamma f(x_n) - Bp) + (I - \beta_n B) (W_n p (I - r_n A)x_n - p) \| \\
\leq \beta_n \| \gamma f(x_n) - Bp \| + (1 - \beta_n \gamma) \| W_n p (I - r_n A)x_n - p \| \\
\leq \beta_n \| \gamma f(x_n) - f(p) \| + \beta_n \| \gamma f(p) - Bp \| + (1 - \beta_n \gamma) \| x_n - p \| \\
= [1 - \beta_n (\gamma \kappa)] \| x_n - p \| + \beta_n \| \gamma f(p) - Bp \|.
\]

On the other hand, we have

\[
\| x_{n+1} - p \| = \| \alpha_n (x_n - p) + (1 - \alpha_n) (P_C y_n - p) \| \\
\leq \alpha_n \| x_n - p \| + (1 - \alpha_n) \| y_n - p \| \\
\leq \alpha_n \| x_n - p \| + (1 - \alpha_n) \| (1 - \beta_n (\gamma \kappa)) \| x_n - p \| + \beta_n \| \gamma f(p) - Bp \|. 
\]

By simple induction, we have

\[
\| x_n - p \| \leq \max \left\{ \| x_0 - p \|, \frac{\| Bp - \gamma f(p) \|}{\gamma \kappa} \right\}, 
\]

which gives that the sequence \( \{ x_n \} \) is bounded, so is \( \{ y_n \} \).

Next, we prove \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \). Put \( \rho_n = P_C (I - r_n A)x_n \). Next, we compute

\[
\| \rho_n - \rho_{n+1} \| = \| P_C (I - r_n A)x_n - P_C (I - r_{n+1} A)x_{n+1} \| \\
\leq \| (I - r_n A)x_n - (I - r_{n+1} A)x_{n+1} \| \\
= \| (x_n - r_n Ax_n) - (x_{n+1} - r_n Ax_{n+1}) + (r_{n+1} - r_n) Ax_{n+1} \| \\
\leq \| x_n - x_{n+1} \| + |r_{n+1} - r_n| M_1, 
\]

where \( M_1 \) is a constant.
where $M_1$ is an appropriate constant such that $M_1 \geq \sup_{n \geq 1} \{ \|Ax_n\| \}$. It follows that

$$
\begin{align*}
\|y_n - y_{n+1}\| &= \|(I - \beta_{n+1}B)(W_{n+1}\rho_{n+1} - W_n\rho_n) - (\beta_{n+1} - \beta_n)BW_n\rho_n \\
&\quad + \gamma [\beta_{n+1}(f(x_{n+1}) - f(x_n)) + f(x_n)(\beta_{n+1} - \beta_n)] \|
\end{align*}
$$

(2.8)

where $M_2$ is an appropriate constant such that

$$
M_2 \geq \max\left\{ \sup_{n \geq 1} \{ \|BW_n\rho_n\| \}, \gamma \sup_{n \geq 1} \{ \|f(x_n)\| \} \right\}.
$$

(2.9)

Since $T_i$ and $U_{n,i}$ are nonexpansive, we have from (1.18) that

$$
\begin{align*}
\|W_{n+1}\rho_n - W_n\rho_n\| &= \|\gamma_1 T_i U_{n+1,2}\rho_n - \gamma_1 T_i U_{n,2}\rho_n\| \\
&\leq \gamma_1 \|U_{n+1,2}\rho - U_{n,2}\rho_n\| \\
&= \gamma_1 \|\gamma_2 T_2 U_{n+1,3}\rho_n - \gamma_2 T_2 U_{n,3}\rho_n\| \\
&\leq \gamma_1 \|U_{n+1,3}\rho_n - U_{n,3}\rho_n\| \\
&\leq \cdots \\
&\leq \gamma_1 \|\gamma_2 \cdots \gamma_n \| U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \\
&\leq M_3 \prod_{i=1}^n \gamma_i, \\
\end{align*}
$$

where $M_3 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \leq M_3$, for all $n \geq 0$. Substitute (2.7) and (2.10) into (2.8) yields that

$$
\begin{align*}
\|y_n - y_{n+1}\| &\leq \left[1 - \beta_{n+1}(\gamma - \kappa\gamma)\right]\|x_{n+1} - x_n\| \\
&\quad + M_4 \left(|r_{n+1} - r_n| + |\beta_{n+1} - \beta_n| + \prod_{i=1}^n \gamma_i, \\
\end{align*}
$$

(2.11)

where $M_4$ is an appropriate appropriate constant such that $M_4 \geq \max\{M_1, M_2, M_3\}$. From the conditions (i) and (iii), we have

$$
\limsup_{n \to \infty} \left( \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \right) \leq 0.
$$

(2.12)

By virtue of Lemma 1.5, we obtain that

$$
\lim_{n \to \infty} \|y_n - x_n\| = 0.
$$

(2.13)
On the other hand, we have
\[ \|x_{n+1} - x_n\| = (1 - \alpha_n)\|x_n - P_C y_n\| \leq \|x_n - y_n\|. \tag{2.14} \]
This implies from (2.13) that
\[ \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{2.15} \]

Next, we show \( \lim_{n \to \infty} \|W \rho_n - \rho_n\| = 0 \). Observing that
\[ y_n - W_n \rho_n = \beta_n (\gamma f(x_n) - B W_n \rho_n) \tag{2.16} \]
and the condition (i), we can easily get
\[ \lim_{n \to \infty} \|W_n \rho_n - y_n\| = 0. \tag{2.17} \]

Notice that
\[
\|\rho_n - p\|^2 = \|P_C (I - r_n A) x_n - P_C (I - r_n A) p\|^2 \\
\leq \|(x_n - p) - r_n (A x_n - A p)\|^2 \\
= \|x_n - p\|^2 - 2r_n \langle x_n - p, A x_n - A p\rangle + r_n^2 \|A x_n - A p\|^2 \\
\leq \|x_n - p\|^2 - 2r_n \alpha \|A x_n - A p\|^2 + r_n^2 \|A x_n - A p\|^2 \\
= \|x_n - p\|^2 - r_n (2 \alpha - r_n) \|A x_n - A p\|^2. \tag{2.18} \]

On the other hand, we have
\[
\|y_n - p\|^2 = \|\beta_n (\gamma f(x_n) - B p) + (I - \beta_n B) (W_n \rho_n - p)\|^2 \\
\leq (\beta_n \|\gamma f(x_n) - B p\| + (1 - \beta_n \bar{\gamma}) \|\rho_n - p\|^2)^2 \\
\leq \beta_n \|\gamma f(x_n) - B p\|^2 + \|\rho_n - p\|^2 + 2 \bar{\beta}_n \|\gamma f(x_n) - B p\| \|\rho_n - p\|, \tag{2.19} \]
from which it follows that
\[
\|x_{n+1} - p\|^2 = \|\alpha_n (x_n - p) + (1 - \alpha_n) (P_C y_n - p)\|^2 \\
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|y_n - p\|^2 \\
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \\
\times \left[ \beta_n \|\gamma f(x_n) - B p\|^2 + \|\rho_n - p\|^2 + 2 \beta_n \|\gamma f(x_n) - B p\| \|\rho_n - p\| \right]. \tag{2.20} \]
Substituting (2.18) into (2.20), we arrive at
\[ \|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \beta_n \|\gamma f(x_n) - Bp\|^2 \\
- (1 - \alpha_n)r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 \\
+ 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|. \]  
(2.21)

It follows that
\begin{align*}
(1 - \alpha_n)r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 \\
& \leq \beta_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\| \\
& \leq \beta_n\|\gamma f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|. \quad \text{(2.22)}
\end{align*}

In view of the restrictions (i), and (iv), we find from (2.15) that
\[ \lim_{n \to \infty} \|Ax_n - Ap\| = 0. \]  
(2.23)

Observe that
\begin{align*}
\|\rho_n - p\|^2 &= \|P_C(I - r_n A)x_n - P_C(I - r_n A)p\|^2 \\
& \leq \langle (I - r_n A)x_n - (I - r_n A)p, \rho_n - p \rangle \\
& = \frac{1}{2}\left\{ \|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n\|^2 \right\} \\
& \leq \frac{1}{2}\left\{ \|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n\|^2 - 2r_n\|Ax_n - Ap\|^2 \right\} \\
& = \frac{1}{2}\left\{ \|x_n - p\|^2 + \|\rho_n - p\|^2 - \|x_n - \rho_n\|^2 - r_n^2\|Ax_n - Ap\|^2 \\
& \quad + 2r_n\langle x_n - \rho_n, Ax_n - Ap \rangle \right\},
\end{align*}
(2.24)

which yields that
\[ \|\rho_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - \rho_n\|^2 + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\|. \]  
(2.25)

Substituting (2.25) into (2.20), we have
\begin{align*}
\|x_{n+1} - p\|^2 & \leq \|x_n - p\|^2 + \beta_n\|\gamma f(x_n) - Bp\|^2 + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| \\
& \quad - (1 - \alpha_n)\|\rho_n - x_n\|^2 + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|. \quad \text{(2.26)}
\end{align*}
This implies that
\[
(1 - \alpha_n) \| \rho_n - x_n \|^2 \\
\leq \| x_n - p \|^2 - |x_{n+1} - p|^2 + \beta_n \| yf(x_n) - Bp \|^2 + 2r_n \| \rho_n - x_n \| \| Ax_n - Ap \| \\
+ 2\rho_n \| yf(x_n) - Bp \| \| \rho_n - p \|
\]
(2.27)

In view of the restrictions (i) and (ii), we find from (2.15) and (2.23) that
\[
\lim_{n \to \infty} \| \rho_n - x_n \| = 0.
\]
(2.28)

On the other hand, we have
\[
\| \rho_n - W_n \rho_n \| \leq \| x_n - \rho_n \| + \| x_n - y_n \| + \| y_n - W_n \rho_n \|.
\]
(2.29)

It follows from (2.13), (2.17) and (2.28) that \( \lim_{n \to \infty} \| W_n \rho_n - \rho_n \| = 0 \). From Lemma 1.6, we find that \( \| W_n \rho_n - W \rho_n \| \to 0 \) as \( n \to \infty \). Notice that
\[
\| W_n \rho_n - \rho_n \| \leq \| W_n \rho_n - \rho_n \| + \| W \rho_n - W \rho_n \|,
\]
(2.30)

from which it follows that
\[
\lim_{n \to \infty} \| W_n \rho_n - \rho_n \| = 0.
\]
(2.31)

Next, we show \( \limsup_{n \to \infty} \langle yf(q) - Bq, x_n - q \rangle \leq 0 \), where \( q = P_F(\gamma f + (I - B))(q) \). To show it, we choose a subsequence \( \{ x_{n_i} \} \) of \( \{ x_n \} \) such that
\[
\limsup_{i \to \infty} \langle yf(q) - Bq, x_{n_i} - q \rangle = \lim_{i \to \infty} \langle yf(q) - Bq, x_{n_i} - q \rangle.
\]
(2.32)

As \( \{ x_{n_i} \} \) is bounded, we have that there is a subsequence \( \{ x_{n_{i_j}} \} \) of \( \{ x_{n_i} \} \) converges weakly to \( p \). We may assume, without loss of generality, that \( x_{n_i} \to p \). Hence we have \( p \in F \). Indeed, let us first show that \( p \in VI(C, A) \). Put
\[
Tw_1 = \begin{cases} 
Aw_1 + N_C w_1, & w_1 \in C, \\
\emptyset, & w_1 \notin C.
\end{cases}
\]
(2.33)

Since \( A \) is inverse-strongly monotone, we see that \( T \) is maximal monotone. Let \( (w_1, w_2) \in G(T) \). Since \( w_2 - Aw_1 \in N_C w_1 \) and \( \rho_n \in C \), we have
\[
\langle w_1 - \rho_n, w_2 - Aw_1 \rangle \geq 0.
\]
(2.34)
On the other hand, from $\rho_n = P_{C}(I - r_n A)x_n$, we have

$$\langle w_1 - \rho_n, \rho_n - (I - r_n A)x_n \rangle \geq 0$$

(2.35)

and hence

$$\langle w_1 - \rho_n, \rho_n - \frac{x_n - r_n A \rho_n}{r_n} \rangle \geq 0.$$  

(2.36)

It follows that

$$\langle w_1 - \rho_n, w_2 \rangle \geq \langle w_1 - \rho_n, Aw_1 \rangle$$

$$\geq \langle w_1 - \rho_n, Aw_1 \rangle - \frac{\langle w_1 - \rho_n, \rho_n - x_n \rangle + Ax_n \rangle}{r_n}$$

$$\geq \langle w_1 - \rho_n, Aw_1 - \frac{\rho_n - x_n}{r_n} - Ax_n \rangle$$

$$= \langle w_1 - \rho_n, Aw_1 - A\rho_n \rangle + \langle w_1 - \rho_n, A\rho_n - Ax_n \rangle$$

$$- \frac{\langle w_1 - \rho_n, \rho_n - x_n \rangle}{r_n}$$

$$\geq \langle w_1 - \rho_n, A\rho_n - Ax_n \rangle - \frac{\langle w_1 - \rho_n, \rho_n - x_n \rangle}{r_n}$$,

(2.37)

which implies from (2.28) that $\langle w_1 - p, w_2 \rangle \geq 0$. We have $p \in T^{-1}0$ and hence $p \in VI(C,A)$. Next, let us show $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Since Hilbert spaces are Opial’s spaces, from (2.31), we have

$$\liminf_{i \to \infty} \|\rho_n - p\| < \liminf_{i \to \infty} \|\rho_n - Wp\|$$

$$= \liminf_{i \to \infty} \|\rho_n - W\rho_n + W_n\rho_n - Wp\|$$

$$\leq \liminf_{i \to \infty} \|W\rho_n - Wp\|$$

$$\leq \liminf_{i \to \infty} \|\rho_n - p\|,$$

(2.38)

which derives a contradiction. Thus, we have from Lemma 1.2 that $p \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. On the other hand, we have

$$\limsup_{n \to \infty} \langle yf(q) - Bq, x_n - q \rangle = \lim_{n \to \infty} \langle yf(q) - Bq, x_n - q \rangle$$

$$= \langle yf(q) - Bq, p - q \rangle$$

(2.39)

$$\leq 0.$$
Finally, we show $x_n \rightharpoonup q$ strongly as $n \to \infty$. Notice that

$$
\|y_n - q\|^2 = \|\beta_n (\gamma f(x_n) - Bq) + (I - \beta_n B)(W_n p_n - q)\|^2
\leq (1 - \beta_n \bar{\gamma})^2 \|W_n p_n - q\|^2 + 2\beta_n \langle \gamma f(x_n) - Bq, y_n - q \rangle
\leq (1 - \beta_n \bar{\gamma})^2 \|x_n - q\|^2 + \kappa \gamma \beta_n \left( \|x_n - q\|^2 + \|y_n - q\|^2 \right)
+ 2\beta_n \langle \gamma f(q) - Bq, y_n - q \rangle.
$$

(2.40)

Therefore, we have

$$
\|y_n - q\|^2 \leq \frac{(1 - \beta_n \bar{\gamma})^2 + \beta_n \gamma \kappa}{1 - \beta_n \gamma \kappa} \|x_n - q\|^2 + \frac{2\beta_n}{1 - \alpha_n \gamma \kappa} \langle \gamma f(q) - Bq, y_n - q \rangle
\leq \left[ \frac{1 - 2\beta_n (\bar{\gamma} - \gamma \kappa)}{1 - \beta_n \gamma \kappa} \right] \|x_n - q\|^2
+ \frac{2\beta_n (\bar{\gamma} - \gamma \kappa)}{1 - \beta_n \gamma \kappa} \left[ \frac{1}{\bar{\gamma} - \gamma \kappa} \langle \gamma f(q) - Bq, y_n - q \rangle + \frac{\alpha_n \gamma^2}{2(\bar{\gamma} - \gamma \kappa)} M_5 \right],
$$

(2.41)

where $M_5$ is an appropriate constant. On the other hand, we have

$$
\|x_{n+1} - p\|^2 = \|\alpha_n(x_n - p) + (1 - \alpha_n)(P_{C_y} y_n - p)\|^2
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|P_{C_y} y_n - p\|^2
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|y_n - p\|^2.
$$

(2.42)

Substitute (2.41) into (2.42) yields that

$$
\|x_{n+1} - p\|^2 \leq \left[ 1 - \alpha_n \frac{2\beta_n (\bar{\gamma} - \gamma \kappa)}{1 - \beta_n \gamma \alpha} \right] \|x_n - q\|^2
+ (1 - \alpha_n) \frac{2\beta_n (\bar{\gamma} - \gamma \kappa)}{1 - \beta_n \gamma \alpha} \left[ \frac{1}{\bar{\gamma} - \gamma \kappa} \langle \gamma f(q) - Bq, y_n - q \rangle + \frac{\beta_n \gamma^2}{2(\bar{\gamma} - \gamma \kappa)} M_5 \right].
$$

(2.43)
Put \( l_n = (1 - \alpha_n)(2\beta_n(\gamma - \alpha_n\gamma)/(1 - \beta_n\alpha\gamma)) \) and
\[
t_n = \frac{1}{\gamma - \alpha\gamma} \langle \gamma f(q) - Bq, y_n - q \rangle + \frac{\beta_n\gamma^2}{2(\gamma - \alpha\gamma)} M_5. \tag{2.44}
\]
That is,
\[
\|x_{n+1} - q\|^2 \leq (1 - l_n)\|x_n - q\|^2 + l_n t_n. \tag{2.45}
\]
Notice that
\[
\langle \gamma f(q) - Bq, y_n - q \rangle = \langle \gamma f(q) - Bq, y_n - x_n \rangle + \langle \gamma f(q) - Bq, x_n - q \rangle
\leq \|\gamma f(q) - Bq\|\|y_n - x_n\| + \langle \gamma f(q) - Bq, x_n - q \rangle. \tag{2.46}
\]
From (2.13) and (2.39) that
\[
\limsup_{n \to \infty} \langle \gamma f(q) - Aq, y_n - q \rangle \leq 0. \tag{2.47}
\]
It follows from the condition (i) and (2.47) that
\[
\lim_{n \to \infty} l_n = 0, \quad \sum_{n=1}^{\infty} l_n = \infty, \quad \limsup_{n \to \infty} t_n \leq 0. \tag{2.48}
\]
Apply Lemma 1.4 to (2.45) to conclude \( x_n \to q \) as \( n \to \infty \). This completes the proof. \( \Box \)

For a single nonexpansive mapping, we have from Theorem 2.1 the following.

**Corollary 2.2.** Let \( H \) be a real Hilbert space and \( C \) a nonempty closed convex subset of \( H \). Let \( A : C \to H \) be an \( \alpha \)-inverse-strongly monotone mapping and \( f : C \to C \) a \( \kappa \)-contraction. Let \( T \) be a nonexpansive mapping from \( C \) into itself such that \( F := F(T) \cap \text{VI}(C, A) \neq \emptyset \). Let \( B \) be a strongly positive linear bounded self-adjoint operator of \( C \) into itself with the constant \( \gamma > 0 \). Let \( \{x_n\} \) be a sequence generated in
\[
x_1 \in C, \quad y_n = \beta_n \gamma f(x_n) + (I - \beta_n B)TP_C(I - r_n A)x_n, \tag{2.49}
\]
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)PCy_n, \quad n \geq 1,
\]
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are real number sequences in \((0, 1)\). Assume that the control sequence \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{r_n\} \) satisfy the following restrictions:

(i) \( \lim_{n \to \infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty \);

(ii) \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1 \);
Let \( C \) be a nonempty closed convex subset of \( H \). Let \( f : C \rightarrow C \) be a \( \kappa \)-contraction. Let \( T \) be a nonexpansive mapping from \( C \) into itself such that \( F(T) \neq \emptyset \). Let \( B \) be a strongly positive linear bounded self-adjoint operator of \( C \) into itself with the constant \( \overline{\gamma} > 0 \).

Let \( \{x_n\} \) be a sequence generated in
\[
x_1 \in C,
\]
\[
y_n = \beta_n \gamma f(x_n) + (1 - \beta_n)Bx_n,
\]
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)P_C y_n, \quad n \geq 1,
\]
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are real number sequences in \((0,1)\). Assume that the control sequence \( \{\alpha_n\} \), and \( \{\beta_n\} \) satisfy the following restrictions:
(i) \( \lim_{n \to \infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty; \)
(ii) \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1. \)

Assume that \( 0 < \gamma < \overline{\gamma}/\kappa \). Then \( \{x_n\} \) strongly converges to some point \( q \), where \( q \in F(T) \), where \( q = P_T(\gamma f + (I - B))(q) \), which solves the variation inequality
\[
\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad \forall p \in F(T).
\] (2.52)

If \( B \) is the identity mapping, then Theorem 2.1 is reduced to the following.

**Corollary 2.4.** Let \( H \) be a real Hilbert space and \( C \) a nonempty closed convex subset of \( H \). Let \( A : C \rightarrow H \) be an \( \alpha \)-inverse-strongly monotone mapping and \( f : C \rightarrow C \) a \( \kappa \)-contraction. Let \( \{T_i\}_{i=1}^{\infty} \) be an infinite family of nonexpansive mappings from \( C \) into itself such that \( F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated in
\[
x_1 \in C,
\]
\[
y_n = \beta_n f(x_n) + (1 - \beta_n)W_n P_C(I - r_n A)x_n,
\]
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n, \quad n \geq 1,
\]
where \( W_n \) is generated in (1.18), \( \{\alpha_n\} \) and \( \{\beta_n\} \) are real number sequences in \((0,1)\). Assume that the control sequence \( \{\alpha_n\} \), \( \{\beta_n\} \), and \( \{r_n\} \) satisfy the following restrictions:
(i) \( \lim_{n \to \infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty; \)
(ii) \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1; \)
Then \( \{ x_n \} \) strongly converges to some point \( q \), where \( q \in F \), where \( q = P_F f(q) \), which solves the variation inequality

\[
\langle f(q) - q, p - q \rangle \leq 0, \quad \forall p \in F.
\]  

(2.54)

References


