Research Article

Existence of Solutions for the $p(x)$ -Laplacian Problem with the Critical Sobolev-Hardy Exponent

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This paper deals with the $p(x)$ -Laplacian equation involving the critical Sobolev-Hardy exponent. Firstly, a principle of concentration compactness in $W_0^{1,p(x)}(\Omega)$ space is established, then by applying it we obtain the existence of solutions for the following $p(x)$ -Laplacian problem: $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = (h(x)|u|^{p_s^*(x)-2}u/|x|^{s(x)}) + f(x, u)$, $x \in \Omega$, $u = 0$, $x \in \partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 \in \Omega$, $1 < p^- \leq p(x) \leq p^+ < N$, and $f(x, u)$ satisfies $p(x)$ -growth conditions.

1. Introduction

In this paper we are concerned with the following $p(x)$ -Laplacian problem:

$$\begin{aligned} & -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u \\ & = \frac{h(x)|u|^{p_s^*(x)-2}u}{|x|^{s(x)}} + f(x, u), \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega, \end{aligned} \quad (1.1)$$

where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded domain, $p(x)$ is Lipschitz continuous, radially symmetric on $\overline{\Omega}$, and $1 < p^- \leq p(x) \leq p^+ < N$. $s(x)$ is Lipschitz continuous, radially symmetric on $\overline{\Omega}$ and $0 \leq s(x) \ll p(x)$. $p_s^*(x) = ((N - s(x))/(N - p(x)))p(x)$ is the critical Sobolev-Hardy exponent, and $p_0^*(x) = Np(x)/(N - p(x)) = p^*(x)$ is the critical Sobolev exponent. Throughout this paper we assume the following:

(F-1) $f(x, t)$ satisfies the Carathéodory condition.

(F-2) $|f(x, t)| \leq c_1 + c_2|t|^{q(x)-1}$, $q : \overline{\Omega} \rightarrow \mathbb{R}$ is measurable and satisfies $p(x) \ll q(x) \ll p_s^*(x)$ or $1 < q^- \leq q(x) \ll p(x)$, for any $x \in \overline{\Omega}$.

(F-3) $f(x, t) = f(|x|, t)$, for any $(x, t) \in \Omega \times \mathbb{R}$.

(F-4) $f(x, t) = -f(x, -t)$, for any $(x, t) \in \Omega \times \mathbb{R}$.

(F-5) $h(x) \in C(\overline{\Omega})$, $h(x) = h(|x|) > 0$ for any $0 \neq x \in \Omega$ and $h(0) = 0$.

In this paper, we mainly consider the singularity, that is, $\lim_{x \rightarrow 0} h(x) \cdot (1/|x|^{s(x)}) = \infty$. For example, let $h(x) = 1/|\ln|x||$ for $x \neq 0$; $h(x) = 0$ for $x = 0$; $s_0 = \inf_{x \in \overline{\Omega}} s(x) > 0$. It is easy to get $\lim_{x \rightarrow 0} (1/|\ln|x||) \cdot (1/|x|^{s(x)}) = \infty$.

Here we explain some notations employed in this paper: Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \rightarrow (1, \infty)$. For all $p(x) \in \mathbf{P}(\Omega)$, we denote $p^+ = \sup_{x \in \overline{\Omega}} p(x)$, $p^- = \inf_{x \in \overline{\Omega}} p(x)$, $p_s^{*+} = \sup_{x \in \overline{\Omega}} p_s^*(x)$, $p_s^{*-} = \inf_{x \in \overline{\Omega}} p_s^*(x)$ and denote by $p_1(x) \ll p_2(x)$ the fact that $\inf\{p_2(x) - p_1(x)\} > 0$. Denote by c_i , C , and k_i the generic positive constants. Denote by $|\Omega|$ the Lebesgue measure of Ω .

When $p(x) \equiv p$ is a constant function, the p -Laplacian problem related to Sobolev-Hardy inequality had been studied by many authors, either is the bounded domain or in the whole space $\mathbb{R}^N$, see, for example, [1–4]. In recent years, along with variable Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ being used, there are a lot of studies on $p(x)$ -Laplacian problems, see [5–8], and the theory on problems with $p(x)$ -growth conditions has important applications in nonlinear elastic mechanics and electrorheological fluids, see [9–12]. In [13], Fu discussed the existence of solutions for a class of $p(x)$ -Laplacian equation with critical growth by establishing a principle of concentration compactness. The method employed in this paper is a extension of the argument in [13, 14].

This paper is organized as follows: in Section 2 we deal with some preliminary materials and technical results; in Section 3 we give the proof of a principle of concentration compactness; in Section 4 we study the problem of $p(x)$ -Laplacian equation with the critical Sobolev-Hardy exponent.

2. Preliminaries

In this section we first recall some facts on variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open set, see [15–19] for the details.

Let $p(x) \in \mathbf{P}(\Omega)$ and

$$\|u\|_p = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \quad (2.1)$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of functions u such that $\int_{\Omega} |u(x)|^{p(x)} dx < \infty$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.1).

For a given $p(x) \in \mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as:

$$p' = \frac{p(x)}{p(x) - 1}. \quad (2.2)$$

Theorem 2.1. *Let $p(x) \in \mathbf{P}(\Omega)$. Then the inequality*

$$\int_{\Omega} |f(x) \cdot g(x)| dx \leq 2 \|f\|_p \|g\|_{p'} \quad (2.3)$$

holds for every $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$.

Theorem 2.2. Suppose that $p(x)$ satisfies

$$1 < p^- \leq p^+ < \infty. \quad (2.4)$$

Let $\text{meas } \Omega < \infty$, $p_1(x), p_2(x) \in \mathbf{P}(\Omega)$, then the necessary and sufficient condition for $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$ is that for almost all $x \in \Omega$ we have $p_1(x) \leq p_2(x)$, and in this case, the imbedding is continuous.

Theorem 2.3. Suppose that $p(x)$ satisfies (2.4). Let $\rho(u) = \int_{\Omega} |u(x)|^{p(x)} dx$. If $u, u_k \in L^{p(x)}(\Omega)$, then

- (1) $\|u\|_p < 1 (= 1; > 1)$ if and only if $\rho(u) < 1 (= 1; > 1)$.
- (2) If $\|u\|_p > 1$, then $\|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+}$.
- (3) If $\|u\|_p < 1$, then $\|u\|_p^{p^+} \leq \rho(u) \leq \|u\|_p^{p^-}$.
- (4) $\lim_{k \rightarrow \infty} \|u_k\|_p = 0$ if and only if $\lim_{k \rightarrow \infty} \rho(u_k) = 0$.
- (5) $\|u_k\|_p \rightarrow \infty$ if and only if $\rho(u_k) \rightarrow \infty$.

We assume that k is a given positive integer.

Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_n$, we set $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, where $D_i = \partial/\partial x_i$ is the generalized derivative operator.

The generalized Sobolev space $W^{k,p(x)}(\Omega)$ is the class of functions f on Ω such that $D^\alpha f \in L^{p(x)}$ for every multi-index α with $|\alpha| \leq k$. $W^{k,p(x)}(\Omega)$ is endowed with the norm

$$\|f\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_p. \quad (2.5)$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.5).

In this paper we use the following equivalent norm of $W^{1,p(x)}(\Omega)$:

$$\|u\|_{1,p} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{\nabla u}{\lambda} \right|^{p(x)} + \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \quad (2.6)$$

Then we have the inequality $(1/2)(\|\nabla u\|_p + \|u\|_p) \leq \|u\|_{1,p} \leq 2(\|\nabla u\|_p + \|u\|_p)$.

Theorem 2.4. The spaces $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are separable reflexive Banach spaces if $p(x)$ satisfies (2.4).

Theorem 2.5. Suppose that $p(x)$ satisfies (2.4). Let $\varphi(u) = \int_{\Omega} |\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} dx$. If $u, u_k \in W^{1,p(x)}(\Omega)$, then

- (1) $\|u\|_{1,p} < 1 (= 1; > 1)$ if and only if $\varphi(u) < 1 (= 1; > 1)$.
- (2) If $\|u\|_{1,p} > 1$, then $\|u\|_{1,p}^{p^-} \leq \varphi(u) \leq \|u\|_{1,p}^{p^+}$.
- (3) If $\|u\|_{1,p} < 1$, then $\|u\|_{1,p}^{p^+} \leq \varphi(u) \leq \|u\|_{1,p}^{p^-}$.

- (4) $\lim_{k \rightarrow \infty} \|u_k\|_{1,p} = 0$ if and only if $\lim_{k \rightarrow \infty} \varphi(u_k) = 0$.
 (5) $\|u_k\|_{1,p} \rightarrow \infty$ if and only if $\varphi(u_k) \rightarrow \infty$.

Theorem 2.6. Let Ω be a bounded in $\mathbb{R}^N$, $p \in C(\overline{\Omega})$ and satisfies (2.4). Then for any measurable function $q(x)$ with $1 \leq q(x) \ll p^*(x)$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Theorem 2.7. If $p : \overline{\Omega} \rightarrow \mathbb{R}$ is Lipschitz continuous and satisfies (2.4), then for any measurable function $q(x)$ with $p(x) \leq q(x) \leq p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Next let us consider the weighted variable exponent Lebesgue space. Let $a(x) \in \mathbf{P}(\Omega)$ and $a(x) > 0$ for $x \in \Omega$. Define

$$L_{a(x)}^{p(x)}(\Omega) = \left\{ u \in P(\Omega) : \int_{\Omega} a(x) |u(x)|^{p(x)} dx < \infty \right\} \quad (2.7)$$

with the norm

$$\|u\|_{L_{a(x)}^{p(x)}(\Omega)} = \|u\|_{p,a} = \inf \left\{ \lambda > 0 : \int_{\Omega} a(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \quad (2.8)$$

then $L_{a(x)}^{p(x)}(\Omega)$ is a Banach space.

Theorem 2.8. Suppose that $p(x)$ satisfies (2.4). Let $\rho(u) = \int_{\Omega} a(x) |u(x)|^{p(x)} dx$. If $u, u_k \in L_{a(x)}^{p(x)}(\Omega)$, then

- (1) For $u \neq 0$, $\|u\|_{p,a} = \lambda$ if and only if $\rho(u/\lambda) = 1$.
 (2) $\|u\|_{p,a} < 1 (= 1; > 1)$ if and only if $\rho(u) < 1 (= 1; > 1)$.
 (3) If $\|u\|_{p,a} > 1$, then $\|u\|_{p,a}^{p^-} \leq \rho(u) \leq \|u\|_{p,a}^{p^+}$.
 (4) If $\|u\|_{p,a} < 1$, then $\|u\|_{p,a}^{p^+} \leq \rho(u) \leq \|u\|_{p,a}^{p^-}$.
 (5) $\lim_{k \rightarrow \infty} \|u_k\|_{p,a} = 0$ if and only if $\lim_{k \rightarrow \infty} \rho(u_k) = 0$.
 (6) $\|u_k\|_{p,a} \rightarrow \infty$ if and only if $\rho(u_k) \rightarrow \infty$.

Theorem 2.9. Let $\Omega \subset \mathbb{R}^n$ be a measurable subset. Suppose that $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caracheodory function and satisfies

$$|g(x, u)| \leq \alpha(x) + \beta |u|^{(p_1(x))/(p_2(x))} \quad \text{for any } x \in \Omega, \quad t \in \mathbb{R}, \quad (2.9)$$

where $p_i(x) \geq 1$, $i = 1, 2$, $\alpha(x) \in L^{p_2(x)}(\Omega)$, $\alpha(x) \geq 0$, $\beta \geq 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_g u)(x) = g(x, u(x))$ is a continuous and bounded operator.

Theorem 2.10. Assume that $0 \in \overline{\Omega}$ and the boundary of Ω possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega})$, $0 \leq s(x) < N$ for $x \in \overline{\Omega}$. If $q(x)$ satisfies $1 \leq q(x) < p_s^*(x)$ for $x \in \overline{\Omega}$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L_{|x|^{-s(x)}}^{q(x)}(\Omega)$.

Theorem 2.11. Assume that $0 \in \overline{\Omega}$ and the boundary of Ω possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega})$, $0 \leq s(x) \ll p(x)$ for $x \in \overline{\Omega}$. There is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L_{|x|^{-s(x)}}^{p_s^*(x)}(\Omega)$.

Proof. Let $u \in W^{1,p(x)}(\Omega)$. Note that

$$\begin{aligned} \int_{\Omega} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx &= \int_{\Omega} \frac{|u|^{s(x)} |u|^{p_s^*(x)-s(x)}}{|x|^{s(x)}} dx \\ &\leq C_1 \left\| \left| \frac{u}{x} \right|^{s(x)} \right\|_{p/s} \left\| |u|^{N(p(x)-s(x))/(N-p(x))} \right\|_{p/(p-s)}. \end{aligned} \quad (2.10)$$

By Theorems 2.7 and 2.10, we have $\|u\|_{p,|x|^{-p}} \leq C_2 \|u\|_{1,p} < \infty$ and $\|u\|_{p^*} \leq C_3 \|u\|_{1,p} < \infty$. So we get

$$\begin{aligned} \int_{\Omega} \left(\left| \frac{u}{x} \right|^{s(x)} \right)^{p(x)/s(x)} dx &= \int_{\Omega} \left| \frac{u}{x} \right|^{p(x)} dx < \infty, \\ \int_{\Omega} |u|^{(N(p(x)-s(x))/(N-p(x))) \cdot (p(x)/(p(x)-s(x)))} dx &= \int_{\Omega} |u|^{p^*(x)} dx < \infty. \end{aligned} \quad (2.11)$$

Furthermore, we obtain $\int_{\Omega} |u|^{p_s^*(x)} / |x|^{s(x)} dx < \infty$. This shows $W^{1,p(x)}(\Omega) \subset L_{|x|^{-s(x)}}^{p_s^*(x)}(\Omega)$, then by the closed graph theorem in Banach space, we get the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L_{|x|^{-s(x)}}^{p_s^*(x)}(\Omega)$. $\square$

3. The Principle of Concentration Compactness

In this section, we will establish the principle of concentration compactness in $W_0^{1,p(x)}(\Omega)$.

We denote by $\mathcal{M}(\overline{\Omega})$ the space of finite nonnegative Borel measures on $\overline{\Omega}$. A sequence $\mu_n \rightarrow \mu$ weakly-* in $\mathcal{M}(\overline{\Omega})$ is defined by $(\mu_n, u) \rightarrow (\mu, u)$, for any $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$.

We first give two lemmas. From [13] we can obtain the proof of the following lemmas. Assume that $p(x)$ is Lipschitz continuous satisfying (2.4) and $s(x)$ is continuous on $\overline{\Omega}$.

Lemma 3.1. Let $\{u_n\} \subset L_{|x|^{-s(x)}}^{p(x)}(\Omega)$ be bounded, and $u_n \rightarrow u \in L_{|x|^{-s(x)}}^{p(x)}(\Omega)$ a.e. on Ω , then

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{|u_n|^{p(x)}}{|x|^{s(x)}} - \frac{|u_n - u|^{p(x)}}{|x|^{s(x)}} dx = \int_{\Omega} \frac{|u|^{p(x)}}{|x|^{s(x)}} dx. \quad (3.1)$$

Lemma 3.2. Let $\delta > 0$, $0 < r < R < 1$, and $r/R \leq k(\delta) = \min\{\exp(-(\delta/(2\tilde{C}))^{n/p^-(1-n)}), e^{-|s^{n-1}|^{1/(n-1)}}\}$, where $\tilde{C} = ((1/((1 + (\delta/2))^{1/(p^+-1)} - 1)) + 1)^{p^+-1} \max\{2C^{p^+}, 2C^{p^-}\} |s^{n-1}|^{p^-/n}, |s^{n-1}|$

denotes the surface area of the unit sphere in $\mathbb{R}^n$ and C satisfies the inequality $\|u\|_{p^*(x)} \leq C\|\nabla u\|_{p(x)}$. Then for every $u \in W_0^{1,p(x)}(\Omega)$,

$$\int_{B_r(x_0)} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx \leq C^* \max \left\{ \left(\int_{B_R(x_0)} |\nabla u|^{p(x)} + |u|^{p(x)} dx + \delta \max \{ \|u\|_{1,p}^{p^+}, \|u\|_{1,p}^{p^-} \} \right)^{p_s^{*-}/p^+}, \right. \\ \left. \left(\int_{B_R(x_0)} |\nabla u|^{p(x)} + |u|^{p(x)} dx + \delta \max \{ \|u\|_{1,p}^{p^+}, \|u\|_{1,p}^{p^-} \} \right)^{p_s^{*+}/p^-} \right\}, \quad (3.2)$$

where $C^* = \sup \{ \int_{\Omega} |u|^{p_s^*(x)} / |x|^{s(x)} dx : \|u\|_{1,p} \leq 1, u \in W_0^{1,p(x)}(\Omega) \}$.

Theorem 3.3. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ with $\|u_n\|_{1,p} \leq 1$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega), \\ |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \longrightarrow \mu \quad \text{weakly-}^* \text{ in } \mathcal{M}(\overline{\Omega}), \\ \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \longrightarrow \nu \quad \text{weakly-}^* \text{ in } \mathcal{M}(\overline{\Omega}), \quad (3.3)$$

as $n \rightarrow \infty$. Then the limit measures are of the form

$$\mu = |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0 + \tilde{\mu}, \quad \mu(\overline{\Omega}) \leq 1, \\ \nu = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0, \quad \nu(\overline{\Omega}) \leq C^*, \quad (3.4)$$

where J is a countable set, $\{\mu_j\} \subset [0, \infty)$, $\{\nu_j\} \subset [0, \infty)$, $\mu_0 \geq 0$, $\nu_0 \geq 0$, $\{x_j\} \in \overline{\Omega}$, $\tilde{\mu} \in \mathcal{M}(\overline{\Omega})$ is a nonatomic positive measure. δ_{x_j} and δ_0 are atomic measures which concentrate on x_j and 0, respectively. C^* is as defined in Lemma 3.2. The atoms and the regular part satisfy the generalized Sobolev inequalities

$$\nu(\overline{\Omega}) \leq C^* \max \left\{ \mu(\overline{\Omega})^{p_s^{*+}/p^-}, \mu(\overline{\Omega})^{p_s^{*-}/p^+} \right\}, \\ \nu_j \leq C^* \max \left\{ \mu_j^{p_s^{*+}/p^-}, \mu_j^{p_s^{*-}/p^+} \right\}, \\ \nu_0 \leq C^* \max \left\{ \mu_0^{p_s^{*+}/p^-}, \mu_0^{p_s^{*-}/p^+} \right\}. \quad (3.5)$$

Proof. By Lemma 3.2, for every $\delta > 0$, there exists $k(\delta) > 0$ such that for $0 < r < R$ with $r/R \leq k(\delta)$,

$$\begin{aligned} & \int_{B_r(0)} \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} dx \\ & \leq C^* \max \left\{ \left(\int_{B_R(0)} |\nabla u_n|^{p(x)} + |u_n|^{p(x)} dx + \delta \max \left\{ \|u_n\|_{1,p}^{p^+}, \|u_n\|_{1,p}^{p^-} \right\} \right)^{p_s^{*-}/p^+} \right. \\ & \quad \left. \left(\int_{B_R(0)} |\nabla u_n|^{p(x)} + |u_n|^{p(x)} dx + \delta \max \left\{ \|u_n\|_{1,p}^{p^+}, \|u_n\|_{1,p}^{p^-} \right\} \right)^{p_s^{**+}/p^-} \right\}. \end{aligned} \quad (3.6)$$

Let $\eta_1 \in C_0^\infty(B_r(0))$ and $\eta_2 \in C_0^\infty(B_{2R}(0))$ such that $0 \leq \eta_1, \eta_2 \leq 1$, $\eta_1 \equiv 1$ in $B_{r/2}(0)$ and $\eta_2 \equiv 1$ in $B_R(0)$. Then we have

$$\begin{aligned} & \int_{B_r(0)} \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \eta_1 dx \longrightarrow \int_{B_r(0)} \eta_1 d\nu, \\ & \int_{B_{2R}(0)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) \eta_2 dx \longrightarrow \int_{B_{2R}(0)} \eta_2 d\mu. \end{aligned} \quad (3.7)$$

Thus,

$$\int_{B_r(0)} \eta_1 d\nu \leq C^* \max \left\{ \left(\int_{B_{2R}(0)} \eta_2 d\mu + \delta \right)^{p_s^{*-}/p^+}, \left(\int_{B_{2R}(0)} \eta_2 d\mu + \delta \right)^{p_s^{**+}/p^-} \right\}. \quad (3.8)$$

Furthermore,

$$\nu(\{0\}) \leq \nu(B_{r/2}(0)) \leq C^* \max \left\{ (\mu(B_{2R}(0)) + \delta)^{p_s^{*-}/p^+}, (\mu(B_{2R}(0)) + \delta)^{p_s^{**+}/p^-} \right\}. \quad (3.9)$$

Let $\delta \rightarrow 0$ and $R \rightarrow 0$, then we get

$$\nu(\{0\}) \leq C^* \max \left\{ \mu(\{0\})^{p_s^{*-}/p^+}, \mu(\{0\})^{p_s^{**+}/p^-} \right\}, \quad (3.10)$$

that is,

$$\nu_0 \leq C^* \max \left\{ \mu_0^{p_s^{*-}/p^+}, \mu_0^{p_s^{**+}/p^-} \right\}. \quad (3.11)$$

By Theorem 2.11 and the definition of C^* , we have

$$\int_{\Omega} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx \leq C^* \max \left\{ \left(\int_{\Omega} |\nabla u|^{p(x)} + |u|^{p(x)} dx \right)^{p_s^{*-}/p^+}, \left(\int_{\Omega} |\nabla u|^{p(x)} + |u|^{p(x)} dx \right)^{p_s^{**+}/p^-} \right\}. \quad (3.12)$$

Similar to the proof of Theorem 3.1 in [13], we get

$$v = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j \in J} v_j \delta_{x_j} + v_0 \delta_0 \quad (3.13)$$

and the other results. $\square$

4. Existence of Solutions

Let $O(N)$ be the group of orthogonal linear transformations in $\mathbb{R}^N$, and G is a subgroup of $O(N)$. For $x \neq 0$, we denote the cardinality of $G_x = \{gx : g \in G\}$ by $|G_x|$ and set $|G| = \inf_{x \in \mathbb{R}^N, x \neq 0} |G_x|$. An open subset $\Omega \subset \mathbb{R}^N$ is G -invariant if $g\Omega = \Omega$ for any $g \in G$.

Definition 4.1. Let Ω be a G -invariant open subset of $\mathbb{R}^N$. The action of G on $W_0^{1,p(x)}(\Omega)$ is defined by $gu(x) = u(g^{-1}x)$ for any $u \in W_0^{1,p(x)}(\Omega)$. The subspace of invariant functions is defined by

$$W_{0,G}^{1,p(x)}(\Omega) = \left\{ u \in W_0^{1,p(x)}(\Omega) : gu = u, \forall g \in G \right\}. \quad (4.1)$$

A functional $I : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}^N$ is G -invariant if $I \circ g = I$ for any $g \in G$.
Set

$$I(u) = \int_{\Omega} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) - \frac{h(x)}{p_s^*(x)} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} - F(x, u) dx, \quad (4.2)$$

$$F(x, t) = \int_0^t f(x, s) ds.$$

The critical points of $I(u)$, that is,

$$0 = I'(u)\varphi = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi + |u|^{p(x)-2} u \varphi - h(x) \frac{|u|^{p_s^*(x)-2} u}{|x|^{s(x)}} \varphi - f(x, u) \varphi dx \quad (4.3)$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, are weak solutions of the problem (1.1). So next we need only to consider the existence of nontrivial critical points of $I(u)$.

In this paper, assume that $G = O(N)$ and Ω is $O(N)$ -invariant. By (F-3) and (F-5), we get that I is $O(N)$ -invariant. By the principle of symmetric criticality of Krawcewicz and Marzantowicz [20], u is a critical point of I if and only if u is a critical point of $\tilde{I} = I|_{W_{0,O(N)}^{1,p(x)}(\Omega)}$.

So we only need to prove the existence of critical points of $\tilde{I}$ on $W_{0,O(N)}^{1,p(x)}(\Omega)$.

Lemma 4.2. Any $(PS)_c$ sequence $\{u_n\} \subset W_{0,O(N)}^{1,p(x)}(\Omega)$ possesses a convergent subsequence.

Proof. Suppose that $\tilde{I}(u_n) \rightarrow c$, $c \in \mathbb{R}$, and $\tilde{I}'(u_n) \rightarrow 0$ in $(W_{0,O(N)}^{1,p(x)}(\Omega))^*$. Let $l(x) = (p(x) + p_s^*(x))/2$ and $|\nabla(1/l(x))| \leq C$. Denote $a = \inf_{x \in \bar{\Omega}}((1/p(x)) - (1/l(x))) > 0$ and $b = \inf_{x \in \bar{\Omega}}((1/l(x)) - (1/p_s^*(x))) > 0$. Then we have

$$\begin{aligned} & \tilde{I}(u_n) - \left\langle \tilde{I}'(u_n), \frac{u_n}{l(x)} \right\rangle \\ &= \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{l(x)} \right) (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) + h(x) \left(\frac{1}{l(x)} - \frac{1}{p_s^*(x)} \right) \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \\ & \quad + \frac{1}{l(x)} f(x, u_n) u_n - F(x, u_n) dx - \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \left(\frac{1}{l(x)} \right) u_n dx \\ &\geq \int_{\Omega} a (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) + b h(x) \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} + \frac{1}{l(x)} f(x, u_n) u_n - F(x, u_n) dx \\ & \quad - \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \left(\frac{1}{l(x)} \right) u_n dx. \end{aligned} \quad (4.4)$$

By Young's inequality, for $\varepsilon_1 \in (0, 1)$, we get

$$\left| |\nabla u_n|^{p(x)-2} \nabla u_n \cdot u_n \right| \leq \varepsilon_1 |\nabla u_n|^{p(x)} + \varepsilon_1 |u_n|^{p_s^*(x)} + C(\varepsilon_1). \quad (4.5)$$

By (F-2), $|(1/l(x)) f(x, u_n) u_n - F(x, u_n)| \leq C(|u_n| + |u_n|^{q(x)})$, then we have for $\varepsilon_2 \in (0, 1)$

$$|u_n| + |u_n|^{q(x)} \leq \varepsilon_2 |u_n|^{p_s^*(x)} + C(\varepsilon_2). \quad (4.6)$$

From $h(x)/|x|^{s(x)} \rightarrow \infty$ as $x \rightarrow 0$, we get that there exists $\bar{H} > 0$ such that $h(x)/|x|^{s(x)} > \bar{H}$ for any $x \in \Omega$, so we have

$$\begin{aligned} & \tilde{I}(u_n) - \left\langle \tilde{I}'(u_n), \frac{u_n}{l(x)} \right\rangle \\ &\geq \int_{\Omega} a (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx + \int_{\Omega} b \bar{H} |u_n|^{p_s^*(x)} dx - C \varepsilon_1 \int_{\Omega} |\nabla u_n|^{p(x)} dx \\ & \quad - C(\varepsilon_1 + \varepsilon_2) \int_{\Omega} |u_n|^{p_s^*(x)} dx - C(\varepsilon_1) - C(\varepsilon_2). \end{aligned} \quad (4.7)$$

Take ε_1 and ε_2 sufficiently small such that $C \varepsilon_1 < a/2$ and $C(\varepsilon_1 + \varepsilon_2) \leq b \bar{H}$, thus,

$$c + 1 > I(u_n) \geq \int_{\Omega} \frac{a}{2} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) dx - C, \quad (4.8)$$

if n is sufficiently large. Furthermore, we obtain $\|u_n\|_{1,p} < \infty$.

Note that

$$\begin{aligned}
& \int_{\Omega} \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) (\nabla u_n - \nabla u) dx \\
& \leq \left| \left\langle \tilde{I}'(u_n), u_n - u \right\rangle \right| + \int_{\Omega} \left| \left(|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right) (u_n - u) \right| dx \\
& \quad + \left| \left\langle \tilde{I}'(u), u_n - u \right\rangle \right| + \int_{\Omega} \left| h(x) \left(\frac{|u_n|^{p_s^*(x)-2} u_n}{|x|^{s(x)}} - \frac{|u|^{p_s^*(x)-2} u}{|x|^{s(x)}} \right) (u_n - u) \right| dx \\
& \quad + \int_{\Omega} |(f(x, u_n) - f(x, u))(u_n - u)| dx \\
& \triangleq \sum_{i=1}^5 I_i.
\end{aligned} \tag{4.9}$$

Because $\{u_n\}$ is bounded in $W_{0,O(N)}^{1,p(x)}(\Omega)$, there exists a subsequence (still denoted by u_n) such that $u_n \rightharpoonup u$ weakly in $W_{0,O(N)}^{1,p(x)}(\Omega)$. Then we have $u_n \rightarrow u$ in $L^{q(x)}(\Omega)$. It is easy to get $I_1 \rightarrow 0$, $I_2 \rightarrow 0$, and $I_3 \rightarrow 0$. By (F-2)

$$\begin{aligned}
& \int_{\Omega} |f(x, u_n)|^{q'(x)} dx \\
& \leq \int_{\Omega} \left(c_1 + c_2 |u_n|^{q(x)-1} \right)^{q'(x)} dx \\
& \leq C \int_{\Omega} (1 + |u_n|)^{(q(x)-1)q'(x)} dx \\
& \leq C \left(|\Omega| + \int_{\Omega} |u_n|^{q(x)} dx \right).
\end{aligned} \tag{4.10}$$

Then we have that $\|f(x, u_n)\|_{q'}$ is bounded. By

$$I_5 \leq 2\|f(x, u_n)\|_{q'} \|u_n - u\|_q + 2\|f(x, u)\|_{q'} \|u_n - u\|_{q'}, \tag{4.11}$$

we get $I_5 \rightarrow 0$.

Next we show that $I_4 \rightarrow 0$. Note that

$$\begin{aligned}
I_4 & \leq h^0 \left(\int_{\Omega} \frac{|u_n|^{p_s^*(x)-1}}{|x|^{s(x)}} |u_n - u| dx + \int_{\Omega} \frac{|u|^{p_s^*(x)-1}}{|x|^{s(x)}} |u_n - u| dx \right) \\
& \leq 2h^0 \left(\left\| \frac{|u_n|^{p_s^*(x)-1}}{|x|^{s(x)/p_s^*(x)}} \right\|_{p_s^*} \left\| \frac{u_n - u}{|x|^{s(x)/p_s^*(x)}} \right\|_{p_s^*} + \left\| \frac{|u|^{p_s^*(x)-1}}{|x|^{s(x)/p_s^*(x)}} \right\|_{p_s^*} \left\| \frac{u_n - u}{|x|^{s(x)/p_s^*(x)}} \right\|_{p_s^*} \right),
\end{aligned} \tag{4.12}$$

where $h^0 = \max_{x \in \bar{\Omega}} h(x)$. By Theorem 2.11, $\| |u_n|^{p_s^*(x)-1} / |x|^{s(x)/p_s^{**}(x)} \|_{p_s^{**}}$ is bounded. If we show that there exists a subsequence (still denoted by $\{u_n\}$) such that $\int_{\Omega} |u_n - u|^{p_s^*(x)} / |x|^{s(x)} dx \rightarrow 0$ as $n \rightarrow \infty$, then $I_4 \rightarrow 0$.

As $u_n \rightharpoonup u$ weakly in $W_{0,O(N)}^{1,p(x)}(\Omega)$, passing to a subsequence, still denoted by $\{u_n\}$, by Theorem 3.3 we assume that there exist $\mu, \nu \in \mathcal{M}(\bar{\Omega})$ and $\{x_j\}_{j \in J}$ in $\bar{\Omega}$ such that $|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \rightarrow \mu$ weakly-* in $\mathcal{M}(\bar{\Omega})$ and $|u_n|^{p_s^*(x)} / |x|^{s(x)} \rightarrow \nu$ weakly-* in $\mathcal{M}(\bar{\Omega})$, where

$$\begin{aligned} \mu &= |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0 + \tilde{\mu}, \\ \nu &= \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0. \end{aligned} \quad (4.13)$$

J is a countable set, $\{\mu_j\} \subset [0, \infty)$, $\{\nu_j\} \subset [0, \infty)$, $\mu_0 \geq 0$, $\nu_0 \geq 0$, $\tilde{\mu} \in \mathcal{M}(\bar{\Omega})$ is a nonatomic positive measure. Take $\eta \equiv 1$, then

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \eta dx = \int_{\Omega} \eta d\nu = \int_{\Omega} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx + \sum_{j \in J} \nu_j + \nu_0. \quad (4.14)$$

We claim $\nu_0 = 0$ and $\nu_j = 0$ for any $j \in J$. First we consider ν_0 .

For any $\varepsilon > 0$, choose $\varphi_0 \in C_0^\infty(B_{2\varepsilon}(0))$ such that $0 \leq \varphi_0 \leq 1$, $\varphi_0 = 1$ on $B_\varepsilon(0)$ and $|\nabla \varphi_0| \leq 2/\varepsilon$. Then

$$\begin{aligned} \langle \tilde{I}'(u_n), u_n \varphi_0 \rangle &= \int_{\Omega} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) \varphi_0 dx - \int_{\Omega} h(x) \frac{|u_n|^{p_s^*(x)} \varphi_0}{|x|^{s(x)}} dx \\ &\quad - \int_{\Omega} f(x, u_n) u_n \varphi_0 dx + \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi_0 u_n dx. \end{aligned} \quad (4.15)$$

We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{B_{2\varepsilon}(0)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) \varphi_0 dx &= \int_{B_{2\varepsilon}(0)} \varphi_0 d\mu, \\ \lim_{n \rightarrow \infty} \int_{B_{2\varepsilon}(0)} \frac{|u_n|^{p_s^*(x)} \varphi_0}{|x|^{s(x)}} dx &= \int_{B_{2\varepsilon}(0)} \varphi_0 d\nu. \end{aligned} \quad (4.16)$$

By Theorem 2.1,

$$\begin{aligned} &\int_{B_{2\varepsilon}(0)} \left| |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi_0 u_n \right| dx \\ &\leq 2 \|u_n \nabla \varphi_0\|_{p, B_{2\varepsilon}(0)} \left\| |\nabla u_n|^{p(x)-1} \right\|_{p', B_{2\varepsilon}(0)} \\ &\leq C \|u_n \nabla \varphi_0\|_{p, B_{2\varepsilon}(0)}. \end{aligned} \quad (4.17)$$

By Theorem 2.6, we have $u_n \rightarrow u$ in $L^{p(x)}(\Omega)$, then

$$\lim_{n \rightarrow \infty} \int_{B_{2\varepsilon}(0)} |u_n \nabla \varphi_0|^{p(x)} dx = \int_{B_{2\varepsilon}(0)} |u \nabla \varphi_0|^{p(x)} dx. \quad (4.18)$$

Furthermore,

$$\begin{aligned} \int_{B_{2\varepsilon}(0)} |u \nabla \varphi_0|^{p(x)} dx &\leq 2 \left\| |\nabla \varphi_0|^{p(x)} \right\|_{N/p, B_{2\varepsilon}(0)} \left\| |u|^{p(x)} \right\|_{N/(N-p), B_{2\varepsilon}(0)}, \\ &\int_{B_{2\varepsilon}(0)} |\nabla \varphi_0|^N dx \leq 4^N \omega_N, \end{aligned} \quad (4.19)$$

where ω_N is the volume of the unit ball. By $\lim_{\varepsilon \rightarrow 0} \int_{B_{2\varepsilon}(0)} |u|^{p^*(x)} dx = 0$, then we have

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi_0 u_n dx = 0. \quad (4.20)$$

Since $\|f(x, u_n)\|_{q'}$ is bounded and by Theorem 2.9 we have

$$\lim_{n \rightarrow \infty} \int_{B_{2\varepsilon}(0)} |f(x, u_n) - f(x, u)|^{q'(x)} dx = 0. \quad (4.21)$$

From

$$\begin{aligned} &\int_{B_{2\varepsilon}(0)} |f(x, u_n) u_n - f(x, u) u| dx \\ &\leq 2 \|f(x, u_n)\|_{q'} \|u_n - u\|_q + 2 \|f(x, u_n) - f(x, u)\|_{q'} \|u\|_q, \end{aligned} \quad (4.22)$$

we have

$$\lim_{n \rightarrow \infty} \int_{B_{2\varepsilon}(0)} f(x, u_n) u_n \varphi_0 dx = \int_{B_{2\varepsilon}(0)} f(x, u) u \varphi_0 dx. \quad (4.23)$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{B_{2\varepsilon}(0)} f(x, u_n) u_n \varphi_0 dx = \lim_{\varepsilon \rightarrow 0} \int_{B_{2\varepsilon}(0)} f(x, u) u \varphi_0 dx = 0. \quad (4.24)$$

Thus, we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \langle \tilde{I}'(u_n), u_n \varphi_0 \rangle = \int_{B_{2\varepsilon}(0)} \varphi_0 d\mu - \int_{B_{2\varepsilon}(0)} h(x) \varphi_0 dv - \int_{B_{2\varepsilon}(0)} f(x, u) u \varphi_0 dx \\ &\quad + \lim_{n \rightarrow \infty} \int_{B_{2\varepsilon}(0)} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi_0 \cdot u_n dx. \end{aligned} \quad (4.25)$$

Furthermore, we obtain

$$0 = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \langle \tilde{I}'(u_n), u_n \varphi_0 \rangle = \mu_0 - h(0) \nu_0. \quad (4.26)$$

As $h(0) = 0$, $\mu_0 = 0$, thus, $\nu_0 = 0$.

Next we consider ν_j for any $j \in J$. Suppose $\exists j_0 \in J$ such that $\nu_{j_0} > 0$. Note that $u_n \in W_{0,O(N)}^{1,p(x)}(\Omega)$, then for any $g \in O(N)$, $\nu(gx_{j_0}) = \nu(x_{j_0}) > 0$. By $|O(N)| = \infty$, we get $\nu(\{gx_{j_0} : g \in O(N)\}) = \infty$. As the measure ν is finite, that is a contradiction. So we obtain that $\nu_0 = 0$ and $\nu_j = 0$ for any $j \in J$. Thus,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} dx = \int_{\Omega} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx. \quad (4.27)$$

By Lemma 3.1, we obtain $\lim_{n \rightarrow \infty} \int_{\Omega} |u_n - u|^{p_s^*(x)} / |x|^{s(x)} dx = 0$, that is, $u_n \rightarrow u$ strongly in $L_{|x|^{-s(x)}}^{p_s^*(x)}(\Omega)$.

We obtain that $\{u_n\}$ possesses a subsequence (still denoted by $\{u_n\}$), such that $I_i \rightarrow 0$, $i = 1, \dots, 5$, as $n \rightarrow \infty$. Thus, $\int_{\Omega} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u) (\nabla u_n - \nabla u) dx \rightarrow 0$, as $n \rightarrow \infty$. As in the proof of Theorem 3.1 in [5], we divide Ω into two parts:

$$\Omega_1 = \{x \in \Omega : p(x) \geq 2\}, \quad \Omega_2 = \{x \in \Omega : p(x) < 2\}. \quad (4.28)$$

We have

$$\int_{\Omega_1} |\nabla u_n - \nabla u|^{p(x)} dx + \int_{\Omega_2} |\nabla u_n - \nabla u|^{p(x)} dx \rightarrow 0, \quad (4.29)$$

that is, $\int_{\Omega} |\nabla u_n - \nabla u|^{p(x)} dx \rightarrow 0$. Then $u_n \rightarrow u$ in $W_{0,O(N)}^{1,p(x)}(\Omega)$. $\square$

Since $W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space, $W_{0,O(N)}^{1,p(x)}(\Omega)$ is also a separable and reflexive Banach space. So there exist $\{e_n\}_{n=1}^{\infty} \subset W_{0,O(N)}^{1,p(x)}(\Omega)$ and $\{e_n^*\}_{n=1}^{\infty} \subset (W_{0,O(N)}^{1,p(x)}(\Omega))^*$ such that

$$\begin{aligned} \langle e_j^*, e_i \rangle &= \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \\ W_{0,O(N)}^{1,p(x)}(\Omega) &= \overline{\text{span}\{e_n : n = 1, 2, \dots\}}, \\ (W_{0,O(N)}^{1,p(x)}(\Omega))^* &= \overline{\text{span}\{e_n^* : n = 1, 2, \dots\}}. \end{aligned} \quad (4.30)$$

For $k = 1, 2, \dots$, denote $X_k = \text{span}\{e_k\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Theorem 4.3. *Under assumptions (F-1)–(F-5), the problem (1.1) admits a sequence of solutions $\{u_n\} \subset W_{0,O(N)}^{1,p(x)}(\Omega)$ such that $I(u_n) \rightarrow \infty$.*

Proof. Set $\varphi(u) = \int_{\Omega} F(x, u) dx$. We first show that $\varphi(u)$ is weakly strongly continuous. Let $u_n \rightharpoonup u$ weakly in $W_{0,O(N)}^{1,p(x)}(\Omega)$. So we have $u_n \rightarrow u$ in $L^{q(x)}(\Omega)$. Note that

$$|F(x, u)| \leq C(|u| + |u|^{q(x)}) \leq C(1 + |u|^{q(x)}), \quad (4.31)$$

then by Theorem 2.9 we obtain $F(x, u_n) \rightarrow F(x, u)$ in $L^1(\Omega)$. By Proposition 3.5 in [18],

$$\beta_k = \beta_k(r) = \sup_{u \in Z_k, \|u\|_{1,p} \leq r} \int_{\Omega} |F(x, u)| dx \rightarrow 0, \quad (4.32)$$

as $k \rightarrow \infty$ for $r > 0$.

Set

$$\theta_k = \theta_k(r) = \sup_{u \in Z_k, \|u\|_{1,p} \leq r} \int_{\Omega} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx. \quad (4.33)$$

Next we show $\theta_k \rightarrow \sum_{j \in J} \nu_j + \nu_0$ as $k \rightarrow \infty$. Note that $0 \leq \theta_{k+1} \leq \theta_k$, then $\theta_k \rightarrow \theta \geq 0$, as $k \rightarrow \infty$. There exists $u_k \in Z_k$ with $\|u_k\|_{1,p} \leq r$ such that $0 \leq \theta_k - \int_{\Omega} (|u_k|^{p_s^*(x)} / |x|^{s(x)}) dx < 1/k$, for each $k = 1, 2, \dots$. As $W_{0,O(N)}^{1,p(x)}(\Omega)$ is reflexive, passing to a subsequence, still denoted by $\{u_k\}$, we assume $u_k \rightharpoonup u$ weakly in $W_{0,O(N)}^{1,p(x)}(\Omega)$. We claim $u = 0$. In fact, for any e_m^* , we have $e_m^*(u_k) = 0$, when $k > m$, then $e_m^*(u_k) \rightarrow 0$ as $k \rightarrow \infty$. It is immediate to get $e_m^*(u) = 0$ for any $m \in \mathbb{N}$. Then we have $u = 0$. By Theorem 3.3, there exist a finite measure ν and a sequence $\{x_j\} \subset \overline{\Omega}$ such that

$$\frac{|u_k|^{p_s^*(x)}}{|x|^{s(x)}} \rightharpoonup \nu = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0, \quad (4.34)$$

where J is countable. Set $\eta \equiv 1$, we obtain $\int_{\Omega} (|u_k|^{p_s^*(x)} / |x|^{s(x)}) \eta dx \rightarrow \sum_{j \in J} \nu_j + \nu_0$. So we have $\lim_{k \rightarrow \infty} \theta_k = \sum_{j \in J} \nu_j + \nu_0 \leq \nu(\overline{\Omega}) < \infty$.

For any $n \in \mathbb{N}$, there exists a positive integer k_n such that $\beta_k(n) \leq 1$ and $\theta_k(n) \leq \sum_{j \in J} \nu_j + \nu_0 + 1$ for all $k \geq k_n$. Assume that $k_n < k_{n+1}$ for each n . Define $\{r_k : k = 1, 2, \dots\}$ in the following way:

$$r_k = \begin{cases} n, & k_n \leq k < k_{n+1}, \\ 1, & 1 \leq k < k_1. \end{cases} \quad (4.35)$$

Then we get $r_k \rightarrow \infty$ as $k \rightarrow \infty$. Hence, for $u \in Z_k$ with $\|u\|_{1,p} = r_k$, we get

$$\begin{aligned} \tilde{I}(u) &\geq \frac{1}{p^+} \|u\|_{1,p}^{p^-} - \frac{h^0}{p_s^{*-}} \theta_k(r_k) - \beta_k(r_k) \\ &\geq \frac{1}{p^+} \|u\|_{1,p}^{p^-} - \frac{h^0}{p_s^{*-}} \left(\sum_{j \in J} v_j + v_0 + 1 \right) - 1, \end{aligned} \quad (4.36)$$

where h^0 is as defined in Lemma 4.2. So

$$\inf_{u \in Z_k, \|u\|_{1,p} = r_k} \tilde{I}(u) \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (4.37)$$

Note that for $\varepsilon \in (0, 1)$, $|F(x, u)| \leq C\varepsilon |u|^{p_s^*(x)} + C(\varepsilon)$, then

$$\int_{\Omega} F(x, u) dx \leq C\varepsilon \int_{\Omega} |u|^{p_s^*(x)} dx + C(\varepsilon) |\Omega|. \quad (4.38)$$

We have

$$\tilde{I}(u) \leq \int_{\Omega} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \int_{\Omega} \frac{\overline{H} |u|^{p_s^*(x)}}{p_s^{*+}} dx + C\varepsilon \int_{\Omega} |u|^{p_s^*(x)} dx + C(\varepsilon) |\Omega|. \quad (4.39)$$

Take ε sufficiently small so that $C\varepsilon \leq \overline{H}/2p_s^{*+}$, then

$$\tilde{I}(u) \leq \int_{\Omega} |\nabla u|^{p(x)} + |u|^{p(x)} dx - m \int_{\Omega} |u|^{p_s^*(x)} dx + C, \quad (4.40)$$

where $m = \overline{H}/2p_s^{*+}$. Since the dimension of Y_k is finite, any two norms on Y_k are equivalent, then $k_1 \|u\|_{1,p} \leq \|u\|_{p_s^*} \leq k_2 \|u\|_{1,p}$, $k_1, k_2 > 0$. As in the proof of Theorem 4.2 in [13], we can find hypercubes $\{Q_i\}_{i=1}^Q$ which mutually have no common points such that $\overline{\Omega} \subseteq \bigcup_{i=1}^Q \overline{Q_i}$ and $p_i^+ = \sup_{y \in \Omega_i} p(y) < \inf_{y \in \Omega_i} p_s^*(y) = p_{si}^{*-}$, where $\Omega_i = Q_i \cap \Omega$. Then we have

$$\begin{aligned} \tilde{I}(u) &\leq \sum_{\|u\|_{1,p,\Omega_i} > 1} \left(\|u\|_{1,p,\Omega_i}^{p_i^+} - mk_2^{p_{si}^{*-}} \|u\|_{1,p,\Omega_i}^{p_{si}^{*-}} \right) \\ &\quad + \sum_{\|u\|_{1,p,\Omega_i} \leq 1} \left(\|u\|_{1,p,\Omega_i}^{p_i^-} - mk_2^{p_{si}^{*+}} \|u\|_{1,p,\Omega_i}^{p_{si}^{*+}} \right) + C \\ &\leq \sum_{\|u\|_{1,p,\Omega_i} > 1} \left(\|u\|_{1,p,\Omega_i}^{p_i^+} - mk_2^{p_{si}^{*-}} \|u\|_{1,p,\Omega_i}^{p_{si}^{*-}} \right) + Q + C. \end{aligned} \quad (4.41)$$

Let $f_i(t) = t^{p_i^+} - mk_2^{p_{si}^{*-}} t^{p_{si}^{*-}}$, for $i = 1, \dots, Q$. Take $s_i > 0$ such that $f_i(s_i) = \max_{t \geq 0} f_i(t) \geq f_i(0) = 0$. Denote $g_i(t) = t^{p_i^+} - mk_2^{p_{si}^{*-}} t^{p_{si}^{*-}} + \sum_{j=1}^Q f_j(s_j) + Q + C$, for $i = 1, \dots, Q$. By $\lim_{t \rightarrow \infty} g_i(t) = -\infty$, there

exists $t_0 > 0$ such that $g_i(t) \leq 0$ for $t \in [t_0, +\infty)$, for all $i = 1, \dots, Q$. For any $k = 1, 2, \dots$, take $\|u\|_{1,p} = \rho_k = \max\{Qt_0, r_k + 1\}$. Note that $\exists i_0$ such that

$$\|u\|_{1,p,\Omega_{i_0}} \geq \frac{1}{Q} \sum_{i=1}^Q \|u\|_{1,p,\Omega_i} \geq \frac{\rho_k}{Q} \geq t_0. \quad (4.42)$$

Then we have $g_{i_0}(\|u\|_{1,p,\Omega_{i_0}}) \leq 0$. Thus,

$$\tilde{I}(u) \leq g_{i_0}(\|u\|_{1,p,\Omega_{i_0}}) = \sum_{i=1}^Q f_i(s_i) + f_{i_0}(\|u\|_{1,p,\Omega_{i_0}}) + Q + C \leq 0. \quad (4.43)$$

Therefore, $\tilde{I}(u) \leq 0$ for $u \in Y_K \cap S_{\rho_k}$, where $S_{\rho_k} = \{u : \|u\|_{1,p} = \rho_k\}$. From Lemma 4.2 we have that $\tilde{I}(u)$ satisfies $(PS)_c$ condition. In view of (F-4), by Fountain Theorem [21], we conclude the result. $\square$

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