Research Article

Nearly Radical Quadratic Functional Equations in p -2-Normed Spaces

M. Eshaghi Gordji,¹ H. Khodaei,² A. Ebadian,³ and G. H. Kim⁴

¹ Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

² Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran

³ Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran

⁴ Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com and A. Ebadian, ebadian.ali@gmail.com

Received 16 November 2011; Accepted 13 February 2012

Academic Editor: John Rassias

Copyright © 2012 M. Eshaghi Gordji et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We establish some stability results in 2-normed spaces for the radical quadratic functional equation $f(\sqrt{\sum_{i=1}^n (x_i + y_i)^2}) + f(\sqrt{\sum_{i=1}^n (x_i - y_i)^2}) = 2 \sum_{i=1}^n (f(x_i) + f(y_i))$ and then use subadditive functions to prove its stability in p -2-normed spaces.

1. Introduction and Preliminaries

The story of the stability of functional equations dates back to 1925 when a stability result appeared in the celebrated book by Pólya and Szeg [1]. In 1940, Ulam [2, 3] posed the famous Ulam stability problem which was partially solved by Hyers [4] in the framework of Banach spaces. Later Aoki [5] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [6] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. Găvruta [7] obtained the generalized result of T. M. Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function. On the other hand, Rassias, Găvruta, and several authors proved the Ulam-Gavruta-Rassias stability of several functional equations. For more details about the results concerning such problems, the reader is referred to [8–30].

Gajda and Ger [31] showed that one can get analogous stability results, for subadditive multifunctions. For further results see [32–42], among others.

The most famous functional equation is the Cauchy equation $f(x + y) = f(x) + f(y)$ any solution of which is called additive. It is easy to see that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = cx^2$ with c an arbitrary constant is a solution of the functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \quad (1.1)$$

So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known [43, 44] that a function $f : X \rightarrow Y$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B_1 : X \times X \rightarrow Y$ such that $f(x) = B_1(x, x)$ for all $x \in X$. The $B_1(x, y) = (1/4)(f(x + y) - f(x - y))$ for all $x, y \in X$.

We briefly recall some definitions and results used later on in the paper. For more details, the reader is referred to [45–49]. The theory of 2-normed spaces was first developed by Gähler [46] in the mid-1960s, while that of 2-Banach spaces was studied later by Gähler and White [47, 49].

Definition 1.1 (see [45]). Let $\mathcal{X}$ be a real linear space over $\mathbb{R}$ with $\dim \mathcal{X} > 1$ and $\|\cdot, \cdot\| : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ a function.

Then $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a linear 2-normed space if

$$(^2N_1) \quad \|x, y\| > 0 \text{ and } \|x, y\| = 0 \text{ if and only if } x \text{ and } y \text{ are linearly dependent,}$$

$$(^2N_2) \quad \|x, y\| = \|y, x\|,$$

$$(^2N_3) \quad \|\alpha x, y\| = |\alpha| \|x, y\|, \text{ for any } \alpha \in \mathbb{R},$$

$$(^2N_4) \quad \|x, y + z\| \leq \|x, y\| + \|x, z\|,$$

for all $x, y, z \in \mathcal{X}$. The function $\|\cdot, \cdot\|$ is called the 2-norm on $\mathcal{X}$.

Let $(\mathcal{X}, \|\cdot, \cdot\|)$ be a linear 2-normed space. If $x \in \mathcal{X}$ and $\|x, y\| = 0$, for all $y \in \mathcal{X}$, then $x = 0$. Moreover, for a linear 2-normed space $(\mathcal{X}, \|\cdot, \cdot\|)$, the functions $x \rightarrow \|x, y\|$ are continuous functions of $\mathcal{X}$ into $\mathbb{R}$ for each fixed $y \in \mathcal{X}$ (see [48]).

A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a Cauchy sequence if there are two points $y, z \in \mathcal{X}$ such that y and z are linearly independent, $\lim_{n,m \rightarrow \infty} \|x_n - x_m, y\| = 0$ and $\lim_{n,m \rightarrow \infty} \|x_n - x_m, z\| = 0$.

A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a convergent sequence if there is an $x \in \mathcal{X}$ such that $\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$, for all $y \in \mathcal{X}$. If $\{x_n\}$ converges to x , write $x_n \rightarrow x$ as $n \rightarrow \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n \rightarrow \infty} x_n = x$.

A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space. For a convergent sequence $\{x_n\}$ in a 2-normed space $\mathcal{X}$, $\lim_{n \rightarrow \infty} \|x_n, y\| = \|\lim_{n \rightarrow \infty} x_n, y\|$, for all $y \in \mathcal{X}$ (see [48]).

We fix a real number p with $0 < p \leq 1$, and let $\mathcal{Y}$ be a linear space. A p -2-norm is a function on $\mathcal{Y} \times \mathcal{Y}$ satisfying Definition 1.1; $(^2N_1)$, $(^2N_2)$, and $(^2N_4)$; the following: $\|\alpha x, y\| = |\alpha|^p \|x, y\|$, for all $x, y \in \mathcal{Y}$ and all $\alpha \in \mathbb{R}$. The pair $(\mathcal{Y}, \|\cdot, \cdot\|)$ is called a p -2-normed space if $\|\cdot, \cdot\|$ is a p -2-norm on $\mathcal{Y}$. A p -2-Banach space is a complete p -2-normed space.

We recall that a subadditive function is a function $\varphi_a : A \rightarrow B$, having a domain A and a codomain $(B, \leq)$ that are both closed under addition, with the following property:

$$\varphi_a(x + y) \leq \varphi_a(x) + \varphi_a(y), \quad (1.2)$$

for all $x, y \in A$. Let $\ell \in \{-1, 1\}$ be fixed. If there exists a constant L with $0 < L < 1$ such that a function $\varphi_a : A \rightarrow B$ satisfies

$$\ell\varphi_a(x+y) \leq \ell L^\ell(\varphi_a(x) + \varphi_a(y)), \quad (1.3)$$

for all $x, y \in A$, then we say that φ_a is contractively subadditive if $\ell = 1$, and φ_a is expansively superadditive if $\ell = -1$. It follows by the last inequality that φ_a satisfies the following properties:

$$\varphi_a(2^\ell x) \leq 2^\ell L \varphi_a(x), \quad \varphi_a(2^{\ell k} x) \leq (2^\ell L)^k \varphi_a(x), \quad (1.4)$$

for all $x \in A$ and integers $k \geq 1$.

Now, we consider the radical quadratic functional equation

$$f\left(\sqrt{\sum_{i=1}^n (x_i + y_i)^2}\right) + f\left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2}\right) = 2 \sum_{i=1}^n (f(x_i) + f(y_i)), \quad (1.5)$$

where $n \in \mathbb{N}$ is a fixed integer and prove generalized Ulam stability, in the spirit of Găvruta (see [7]), of this functional equation in 2-normed spaces. Moreover, in this paper, we investigate new results about the generalized Ulam stability by using subadditive functions in p -2-normed spaces for the radical quadratic functional equation (1.5).

2. Main Results

In this section, let X be a linear space, and let $\mathbb{R}$ and $\mathbb{R}^+$ denote the sets of real and positive real numbers, respectively. If a mapping $f : \mathbb{R} \rightarrow X$ satisfies the functional equation (1.5), by letting $x_i = y_i = 0$ ($1 \leq i \leq n$) in (1.5), we get $f(0) = 0$. Setting $x_i = y_i = x$ ($1 \leq i \leq n$) in (1.5) and using $f(0) = 0$, we get

$$f(\sqrt{4nx^2}) = 4nf(x), \quad (2.1)$$

for all $x \in \mathbb{R}$. Putting $x_i = 2x, y_i = 0$ ($1 \leq i \leq n$) in (1.5) and using $f(0) = 0$, we get

$$2f(\sqrt{4nx^2}) = 2nf(2x), \quad (2.2)$$

for all $x \in \mathbb{R}$. It follows from (2.1) and (2.2) that

$$f(2^m x) = 4^m f(x), \quad (2.3)$$

for all $x \in \mathbb{R}$ and integers $m \geq 1$. Setting $y_n = -y_n$ in (1.5) and then comparing it with (1.5), we obtain $f(-y_n) = f(y_n)$, for all $y_n \in \mathbb{R}$. Letting $x_i = y_i = 0$ ($2 \leq i \leq n$) in (1.5), we get

$$f\left(\sqrt{(x_1 + y_1)^2}\right) + f\left(\sqrt{(x_1 - y_1)^2}\right) = 2f(x_1) + 2f(y_1), \quad (2.4)$$

for all $x_1, y_1 \in \mathbb{R}$. It follows from (2.4) and the evenness of f that f satisfies (1.1). So we have the following lemma.

Lemma 2.1. *If a mapping $f : \mathbb{R} \rightarrow X$ satisfies the functional equation (1.5), then f is quadratic.*

Corollary 2.2. *If a mapping $f : \mathbb{R} \rightarrow X$ satisfies the functional equation (1.5), then there exists a symmetric biadditive mapping $B_1 : \mathbb{R} \times \mathbb{R} \rightarrow X$ such that $f(x) = B_1(x, x)$, for all $x \in \mathbb{R}$.*

Hereafter, we will assume that $\mathcal{X}$ is a 2-Banach space. First, using an idea of Găvruta [7], we prove the stability of (1.5) in the spirit of Ulam, Hyers, and Rassias.

Let ϕ be a function from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^+ \cup \{0\}$. A mapping $f : \mathbb{R} \rightarrow \mathcal{X}$ is called a ϕ -approximatively radical quadratic function if

$$\left\| f\left(\sqrt{\sum_{i=1}^n (x_i + y_i)^2}\right) + f\left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2}\right) - 2\sum_{i=1}^n (f(x_i) + f(y_i)), z \right\|_{\mathcal{X}} \leq \phi(x_1, \dots, x_n, y_1, \dots, y_n, z), \quad (2.5)$$

for all $x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R}$, where $n \in \mathbb{N}$ is a fixed integer.

Theorem 2.3. *Let $\ell \in \{-1, 1\}$ be fixed, and let $f : \mathbb{R} \rightarrow \mathcal{X}$ be a ϕ -approximatively radical quadratic function with $f(0) = 0$. If the function $\phi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^+ \cup \{0\}$ satisfies*

$$\Phi(x, z) := \sum_{j=(1-\ell)/2}^{\infty} \frac{1}{4^{\ell j}} \left(\phi\left(\overbrace{2^{\ell j}x, \dots, 2^{\ell j}x}^{2n}, z\right) + \frac{1}{2} \phi\left(\overbrace{2^{1+\ell j}x, \dots, 2^{1+\ell j}x}^n, \overbrace{0, \dots, 0}^n, z\right) \right) < \infty, \quad (2.6)$$

and $\lim_{m \rightarrow \infty} (1/4^{\ell m}) \phi(2^{\ell m}x_1, \dots, 2^{\ell m}x_n, 2^{\ell m}y_1, \dots, 2^{\ell m}y_n, z) = 0$, for all $x, x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R}$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{X}$, satisfies (1.5) and the inequality

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{X}} \leq \frac{1}{4n} \Phi(x, y), \quad (2.7)$$

for all $x, y \in \mathbb{R}$.

Proof. Letting $x_i = x + y, y_i = x - y$ ($1 \leq i \leq n$) in (2.5), we get

$$\begin{aligned} & \left\| f(\sqrt{4nx^2}) + f(\sqrt{4ny^2}) - 2nf(x+y) - 2nf(x-y), z \right\|_{\mathcal{X}} \\ & \leq \phi \left(\overbrace{x+y, \dots, x+y}^n, \overbrace{x-y, \dots, x-y}^n, z \right), \end{aligned} \quad (2.8)$$

for all $x, y, z \in \mathbb{R}$. Setting $x_i = y_i = x$ ($1 \leq i \leq n$) in (2.5), we get

$$\left\| f(\sqrt{4nx^2}) - 4nf(x), z \right\|_{\mathcal{X}} \leq \phi \left(\overbrace{x, \dots, x}^{2n}, z \right), \quad (2.9)$$

for all $x, z \in \mathbb{R}$. Replacing y by x in (2.8), we obtain

$$\left\| f(\sqrt{4nx^2}) - nf(2x), z \right\|_{\mathcal{X}} \leq \frac{1}{2} \phi \left(\overbrace{2x, \dots, 2x}^n, \overbrace{0, \dots, 0}^n, z \right), \quad (2.10)$$

for all $x, z \in \mathbb{R}$. It follows from (2.9) and (2.10) that

$$\left\| 4f(x) - f(2x), y \right\|_{\mathcal{X}} \leq \frac{1}{n} \phi \left(\overbrace{x, \dots, x}^{2n}, y \right) + \frac{1}{2n} \phi \left(\overbrace{2x, \dots, 2x}^n, \overbrace{0, \dots, 0}^n, y \right), \quad (2.11)$$

for all $x, y \in \mathbb{R}$. Thus,

$$\begin{aligned} \left\| f(x) - \frac{1}{4}f(2x), y \right\|_{\mathcal{X}} & \leq \frac{1}{4n} \phi \left(\overbrace{x, \dots, x}^{2n}, y \right) + \frac{1}{8n} \phi \left(\overbrace{2x, \dots, 2x}^n, \overbrace{0, \dots, 0}^n, y \right), \\ \left\| f(x) - 4f\left(\frac{x}{2}\right), y \right\|_{\mathcal{X}} & \leq \frac{1}{n} \phi \left(\overbrace{\frac{x}{2}, \dots, \frac{x}{2}}^{2n}, y \right) + \frac{1}{2n} \phi \left(\overbrace{x, \dots, x}^n, \overbrace{0, \dots, 0}^n, y \right), \end{aligned} \quad (2.12)$$

for all $x, y \in \mathbb{R}$. Hence,

$$\begin{aligned} & \left\| \frac{1}{4^{\ell k}} f(2^{\ell k} x) - \frac{1}{4^{\ell r}} f(2^{\ell r} x), y \right\|_{\mathcal{X}} \\ & \leq \frac{1}{4n} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} \frac{1}{4^{\ell j}} \left(\phi \left(\overbrace{2^{\ell j} x, \dots, 2^{\ell j} x}^{2n}, y \right) + \frac{1}{2} \phi \left(\overbrace{2^{1+\ell j} x, \dots, 2^{1+\ell j} x}^n, \overbrace{0, \dots, 0}^n, y \right) \right) \end{aligned} \quad (2.13)$$

for all $x, y \in \mathbb{R}$ and integers $r > k \geq 0$. Thus, $\{(1/4^{\ell m})f(2^{\ell m}x)\}$ is a Cauchy sequence in the 2-Banach space $\mathcal{X}$. Hence, we can define a mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{X}$ by $\mathcal{F}(x) := \lim_{m \rightarrow \infty} (1/4^{\ell m})f(2^{\ell m}x)$, for all $x \in \mathbb{R}$. That is,

$$\lim_{m \rightarrow \infty} \left\| \frac{1}{4^{\ell m}} f(2^{\ell m}x) - \mathcal{F}(x), y \right\|_{\mathcal{X}} = 0, \quad (2.14)$$

for all $x, y \in \mathbb{R}$. In addition, it is clear from (2.5) that the following inequality:

$$\begin{aligned} & \left\| \mathcal{F} \left(\sqrt{\sum_{i=1}^n (x_i + y_i)^2} \right) + \mathcal{F} \left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2} \right) - 2 \sum_{i=1}^n (\mathcal{F}(x_i) + \mathcal{F}(y_i)), z \right\|_{\mathcal{X}} \\ &= \lim_{m \rightarrow \infty} \frac{1}{4^{\ell m}} \left\| f \left(\sqrt{4^{\ell m} \sum_{i=1}^n (x_i + y_i)^2} \right) + f \left(\sqrt{4^{\ell m} \sum_{i=1}^n (x_i - y_i)^2} \right) \right. \\ & \quad \left. - 2 \sum_{i=1}^n (f(2^{\ell m}x_i) + f(2^{\ell m}y_i)), z \right\|_{\mathcal{X}} \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{4^{\ell m}} \phi(2^{\ell m}x_1, \dots, 2^{\ell m}x_n, 2^{\ell m}y_1, \dots, 2^{\ell m}y_n, z) = 0 \end{aligned} \quad (2.15)$$

holds for all $x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R}$, and so by Lemma 2.1, the mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{X}$ is quadratic. Taking the limit $r \rightarrow \infty$ in (2.13) with $k = 0$, we find that the mapping $\mathcal{F}$ is quadratic mapping satisfying the inequality (2.7) near the approximate mapping $f : \mathbb{R} \rightarrow \mathcal{X}$ of (1.5). To prove the aforementioned uniqueness, we assume now that there is another quadratic mapping $\mathcal{G} : \mathbb{R} \rightarrow \mathcal{X}$ which satisfies (1.5) and the inequality (2.7). Since the mapping $\mathcal{G} : \mathbb{R} \rightarrow \mathcal{X}$ satisfies (1.5), then

$$\mathcal{G}(2^{\ell}x) = 4^{\ell}\mathcal{G}(x), \quad \mathcal{G}(2^{\ell m}x) = 4^{\ell m}\mathcal{G}(x) \quad (2.16)$$

for all $x \in \mathbb{R}$ and integers $m \geq 1$. Thus, one proves by the last equality and (2.7) that

$$\left\| \frac{1}{4^{\ell m}} f(2^{\ell m}x) - \mathcal{G}(x), y \right\|_{\mathcal{X}} = \frac{1}{4^{\ell m}} \left\| f(2^{\ell m}x) - \mathcal{G}(2^{\ell m}x), y \right\|_{\mathcal{X}} \leq \frac{1}{4^{m\ell+1}n} \Phi(2^{\ell m}x, y), \quad (2.17)$$

for all $x, y \in \mathbb{R}$ and integers $m \geq 1$. Therefore, from $m \rightarrow \infty$, one establishes $\mathcal{F}(x) - \mathcal{G}(x) = 0$ for all $x \in \mathbb{R}$. $\square$

Corollary 2.4. Let $\ell \in \{-1, 1\}$ be fixed. If there exist nonnegative real numbers p_i, q_i, q with $\ell \sum_{i=1}^n (p_i + q_i) < 2\ell$ such that a mapping $f : \mathbb{R} \rightarrow \mathcal{X}$ satisfies the inequality

$$\begin{aligned} & \left\| f \left(\sqrt{\sum_{i=1}^n (x_i + y_i)^2} \right) + f \left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2} \right) - 2 \sum_{i=1}^n (f(x_i) + f(y_i)), z \right\|_{\mathcal{X}} \\ &\leq \theta \prod_{i=1}^n |x_i|^{p_i} |y_i|^{q_i} |z|^q, \end{aligned} \quad (2.18)$$

for all $x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R}$ and some $\theta \geq 0$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{X}$, satisfies (1.5) and the inequality

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{X}} \leq \frac{1}{\ell n(4 - 2^\lambda)} \theta |x|^\lambda |y|^q, \quad (2.19)$$

for all $x, y \in \mathbb{R}$, where $\lambda := \sum_{i=1}^n (p_i + q_i)$.

Corollary 2.5. Let $\ell \in \{-1, 1\}$ be fixed. If there exist nonnegative real numbers s, t with $\ell s < 2\ell$ such that a mapping $f : \mathbb{R} \rightarrow \mathcal{X}$ satisfies the inequality

$$\begin{aligned} & \left\| f\left(\sqrt{\sum_{i=1}^n (x_i + y_i)^2}\right) + f\left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2}\right) - 2 \sum_{i=1}^n (f(x_i) + f(y_i)), z \right\|_{\mathcal{X}} \\ & \leq \theta \sum_{i=1}^n (|x_i|^s + |y_i|^s) |z|^t, \end{aligned} \quad (2.20)$$

for all $x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R}$ and some $\theta \geq 0$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{X}$ satisfies (1.5) and the inequality

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{X}} \leq \frac{1 + 2^{s-2}}{\ell(2 - 2^{s-1})} \theta |x|^s |y|^t, \quad (2.21)$$

for all $x, y \in \mathbb{R}$.

Now, we are going to establish the modified Hyers-Ulam stability of (1.5).

Theorem 2.6. Let $\ell \in \{-1, 1\}$ be fixed, let $\mathcal{Y}$ be a p -2-Banach space, and, $f : \mathbb{R} \rightarrow \mathcal{Y}$ be a ϕ -approximatively radical quadratic function with $f(0) = 0$. Assume that the map ϕ is contractively subadditive if $\ell = 1$ and is expansively superadditive if $\ell = -1$ with a constant L satisfying $2^{\ell(1-3p)}L < 1$, where $3\ell p \leq \ell$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{Y}$ which satisfies (1.5) and the inequality

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{Y}} \leq \frac{1}{\ell(4^p - 2^{1-p}L^\ell)} \Psi(x, y), \quad (2.22)$$

for all $x, y \in \mathbb{R}$, where

$$\Psi(x, y) := \frac{1}{n^p} \phi\left(\overbrace{x, \dots, x}^{2n}, y\right) + \frac{1}{(2n)^p} \phi\left(\overbrace{2x, \dots, 2x}^n, \overbrace{0, \dots, 0}^n, y\right). \quad (2.23)$$

Proof. Using the same method as in the proof of Theorem 2.3, we have

$$\begin{aligned} \left\| f(x) - \frac{1}{4}f(2x), y \right\|_y &\leq \frac{1}{4^p} \Psi(x, y), \\ \left\| f(x) - 4f\left(\frac{x}{2}\right), y \right\|_y &\leq 2^p \Psi\left(\frac{x}{2}, \frac{y}{2}\right), \end{aligned} \quad (2.24)$$

for all $x, y \in \mathbb{R}$. Hence

$$\begin{aligned} \left\| \frac{1}{4^{\ell k}} f(2^{\ell k} x) - \frac{1}{4^{\ell r}} f(2^{\ell r} x), y \right\|_y &\leq \frac{1}{4^p} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} \frac{1}{2^{3\ell p j}} \Psi(2^{\ell j} x, 2^{\ell j} y) \\ &\leq \frac{1}{4^p} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} \frac{(2^\ell L)^j}{2^{3\ell p j}} \Psi(x, y) \\ &= \frac{\Psi(x, y)}{4^p} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} (2^{\ell(1-3p)} L)^j, \end{aligned} \quad (2.25)$$

for all $x, y \in \mathbb{R}$ and integers $r > k \geq 0$. Thus, $\{(1/4^{\ell m})f(2^{\ell m}x)\}$ is a Cauchy sequence in the p -2-Banach space $\mathcal{Y}$. Hence, we can define a mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{Y}$ by $\mathcal{F}(x) := \lim_{n \rightarrow \infty} (1/4^{\ell n})f(2^{\ell n}x)$, for all $x \in \mathbb{R}$. Also

$$\begin{aligned} &\left\| \mathcal{F}\left(\sqrt{\sum_{i=1}^n (x_i + y_i)^2}\right) + \mathcal{F}\left(\sqrt{\sum_{i=1}^n (x_i - y_i)^2}\right) - 2 \sum_{i=1}^n (\mathcal{F}(x_i) + \mathcal{F}(y_i)), z \right\|_y \\ &= \lim_{m \rightarrow \infty} \left\| \frac{1}{4^{\ell m}} f\left(\sqrt{4^{\ell m} \sum_{i=1}^n (x_i + y_i)^2}\right) + \frac{1}{4^{\ell m}} f\left(\sqrt{4^{\ell m} \sum_{i=1}^n (x_i - y_i)^2}\right) \right. \\ &\quad \left. - \frac{2}{4^{\ell m}} \sum_{i=1}^n (f(2^{\ell m} x_i) + f(2^{\ell m} y_i)), z \right\|_y \\ &\leq \lim_{m \rightarrow \infty} \frac{1}{2^{3\ell p m}} \phi(2^{\ell m} x_1, \dots, 2^{\ell m} x_n, 2^{\ell m} y_1, \dots, 2^{\ell m} y_n, 2^{\ell m} z) \\ &\leq \lim_{m \rightarrow \infty} (2^{\ell(1-3p)} L)^m \phi(x_1, \dots, x_n, y_1, \dots, y_n, z) = 0 \end{aligned} \quad (2.26)$$

holds for all $x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R}$, and so by Lemma 2.1, the mapping $\mathcal{F} : \mathbb{R} \rightarrow \mathcal{Y}$ is quadratic. Taking the limit $r \rightarrow \infty$ in (2.25) with $k = 0$, we find that the mapping $\mathcal{F}$ is quadratic mapping satisfying the inequality (2.22) near the approximate mapping $f : \mathbb{R} \rightarrow \mathcal{Y}$ of (1.5). The remaining assertion goes through in a similar way to the corresponding part of Theorem 2.3. $\square$

References

- [1] G. Pólya and G. Szeg, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Springer, Berlin, Germany, 1925.
- [2] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960.
- [3] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, NY, USA, 1964.
- [4] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [5] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
- [6] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [7] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
- [8] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
- [9] P. Gavruta, "An answer to a question of John M. Rassias concerning the stability of Cauchy equation," in *Advances in Equations and Inequalities*, Hadronic Mathematics, pp. 67–71, 1999.
- [10] J. M. Rassias and H.-M. Kim, "Generalized Hyers-Ulam stability for general additive functional equations in quasi- β -normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 302–309, 2009.
- [11] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
- [12] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
- [13] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
- [14] G. Z. Eskandani, P. Gavruta, J. M. Rassias, and R. Zarghami, "Generalized Hyers-Ulam stability for a general mixed functional equation in quasi- β -normed spaces," *Mediterranean Journal of Mathematics*, vol. 8, no. 3, pp. 331–348, 2011.
- [15] T. Z. Xu, J. M. Rassias, and W. X. Xu, "Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces," *Journal of Mathematical Physics*, vol. 51, no. 9, Article ID 093508, 19 pages, 2010.
- [16] T. Z. Xu, J. M. Rassias, and W. X. Xu, "Intuitionistic fuzzy stability of a general mixed additive-cubic equation," *Journal of Mathematical Physics*, vol. 51, no. 6, Article ID 063519, 21 pages, 2010.
- [17] T. Z. Xu, J. M. Rassias, and W. X. Xu, "On the stability of a general mixed additive-cubic functional equation in random normed spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 328473, 16 pages, 2010.
- [18] M. Eshaghi Gordji, M. B. Ghaemi, Y. J. Cho, and H. Majani, "A general system of Euler-Lagrange-type quadratic functional equations in Menger probabilistic non-Archimedean 2-normed spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 208163, 21 pages, 2011.
- [19] M. Eshaghi Gordji, "Nearly ring homomorphisms and nearly ring derivations on non-Archimedean Banach algebras," *Abstract and Applied Analysis*, vol. 2010, Article ID 393247, 12 pages, 2010.
- [20] M. Eshaghi Gordji and Z. Alizadeh, "Stability and superstability of ring homomorphisms on non-Archimedean Banach algebras," *Abstract and Applied Analysis*, vol. 2011, Article ID 123656, 10 pages, 2011.
- [21] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, Article ID 042303, 9 pages, 2009.
- [22] C. Park and M. E. Gordji, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 51, no. 4, Article ID 044102, 7 pages, 2010.
- [23] M. Eshaghi Gordji, S. Abbaszadeh, and C. Park, "On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces," *Journal of Inequalities and Applications*, vol. 2009, Article ID 153084, 26 pages, 2009.
- [24] M. Eshaghi Gordji, A. Ebadian, and S. Zolfaghari, "Stability of a functional equation deriving from cubic and quartic functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 801904, 17 pages, 2008.
- [25] F. Rahbarnia, T. M. Rassias, R. Saadati, and G. Sadeghi, "Forti's approach in fixed point theory and the stability of a functional equation on metric and ultra metric spaces," *Journal of Computational Analysis and Applications*, vol. 13, no. 3, pp. 458–462, 2011.

- [26] R. Saadati, S. M. Vaezpour, and C. Park, "The stability of the cubic functional equation in various spaces," *Mathematical Communications*, vol. 16, no. 1, pp. 131–145, 2011.
- [27] R. Saadati, M. M. Zohdi, and S. M. Vaezpour, "Nonlinear $\mathcal{L}$ -random stability of an ACQ functional equation," *Journal of Inequalities and Applications*, vol. 2011, Article ID 194394, 23 pages, 2011.
- [28] R. Saadati and C. Park, "Non-Archimedean $\mathcal{L}$ -fuzzy normed spaces and stability of functional equations," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2488–2496, 2010.
- [29] M. M. Pourpasha, J. M. Rassias, R. Saadati, and S. M. Vaezpour, "A fixed point approach to the stability of Pexider quadratic functional equation with involution," *Journal of Inequalities and Applications*, vol. 2010, Article ID 839639, 18 pages, 2010.
- [30] L. Gavruta and P. Gavruta, "On a problem of John M. Rassias concerning the stability in Ulam sense of Euler-Lagrange equation," in *Functional Equations, Difference Inequalities*, pp. 47–53, Nova Sciences, 2010.
- [31] Z. Gajda and R. Ger, "Subadditive multifunctions and Hyers-Ulam stability," in *General Inequalities*, vol. 5 of *International Series of Numerical Mathematics*, Birkhäuser, Basel, Switzerland, 1987.
- [32] Y. J. Cho, C. Park, and R. Saadati, "Functional inequalities in non-Archimedean Banach spaces," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1238–1242, 2010.
- [33] M. Eshaghi Gordji and H. Khodaei, *Stability of Functional Equations*, Lap Lambert Academic, 2010.
- [34] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, Switzerland, 1998.
- [35] S.-M. Jung and P. K. Sahoo, "Stability of a functional equation for square root spirals," *Applied Mathematics Letters*, vol. 15, no. 4, pp. 435–438, 2002.
- [36] M. Eshaghi Gordji, "Stability of a functional equation deriving from quartic and additive functions," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 491–502, 2010.
- [37] M. Eshaghi Gordji, "Stability of an additive-quadratic functional equation of two variables in F-spaces," *Journal of Nonlinear Science and its Applications*, vol. 2, no. 4, pp. 251–259, 2009.
- [38] M. Eshaghi Gordji, M. B. Ghaemi, and H. Majani, "Generalized Hyers-Ulam-Rassias theorem in Menger probabilistic normed spaces," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 162371, 11 pages, 2010.
- [39] M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, "Generalized Hyers-Ulam stability of generalized (n, k) -derivations," *Abstract and Applied Analysis*, vol. 2009, Article ID 437931, 8 pages, 2009.
- [40] M. Eshaghi Gordji, S. Kaboli Gharetapeh, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type additive, quadratic, and cubic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.
- [41] M. E. Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, "Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 417473, 14 pages, 2009.
- [42] C. Park and A. Najati, "Homomorphisms and derivations in C^* -algebras," *Abstract and Applied Analysis*, vol. 2007, Article ID 80630, 12 pages, 2007.
- [43] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, UK, 1989.
- [44] P. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.
- [45] Y. J. Cho, P. C. S. Lin, S. S. Kim, and A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science, New York, NY, USA, 2001.
- [46] S. Gähler, "Lineare 2-normierte Räume," *Mathematische Nachrichten*, vol. 28, pp. 1–43, 1964.
- [47] S. Gähler, "Über 2-banach-Räume," *Mathematische Nachrichten*, vol. 42, pp. 335–347, 1969.
- [48] W.-G. Park, "Approximate additive mappings in 2-banach spaces and related topics," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 193–202, 2011.
- [49] A. G. White, "2-banach spaces," *Mathematische Nachrichten*, vol. 42, pp. 43–60, 1969.