Normal Families of Zero-Free Meromorphic Functions

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Let \( a \neq 0 \), \( b \in \mathbb{C} \), and \( n \) and \( k \) be two positive integers such that \( n \geq 2 \). Let \( \mathcal{F} \) be a family of zero-free meromorphic functions defined in a domain \( \mathcal{D} \) such that for each \( f \in \mathcal{F} \),
\[
\frac{f'}{af^n} - b \text{ has at most } nk \text{ zeros, ignoring multiplicity.}
\]
Then \( \mathcal{F} \) is normal in \( \mathcal{D} \).

1. Introduction and Main Results

Let \( \mathcal{D} \) be a domain in \( \mathbb{C} \), and let \( \mathcal{F} \) be a family of meromorphic functions defined in the domain \( \mathcal{D} \). \( \mathcal{F} \) is said to be normal in \( \mathcal{D} \), in the sense of Montel, if for every sequence \( \{f_n\} \subseteq \mathcal{F} \) contains a subsequence \( \{f_{n_j}\} \) such that \( f_{n_j} \) converges spherically uniformly on compact subsets of \( \mathcal{D} \) (see [1, Definition 3.1.1]).

\( \mathcal{F} \) is said to be normal at a point \( z_0 \in \mathcal{D} \) if there exists a neighborhood of \( z_0 \) in which \( \mathcal{F} \) is normal. It is well known that \( \mathcal{F} \) is normal in a domain \( \mathcal{D} \) if and only if it is normal at each of its points (see [1, Theorem 3.3.2]).

Let \( f \) be a meromorphic function in the complex plane. We use the standard notations and results of value distribution theory as presented in [2–4]. In particular, \( T(r, f) \) is Nevanlinna’s characteristic function and \( S(r, f) \) denotes a function with the property \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \) (outside an exceptional set of finite linear measure).

In 1959, Hayman [5] proved the following well-known result.

**Theorem A.** Let \( f \) be a transcendental meromorphic function on the complex plane \( \mathbb{C} \), let \( a \) be a non-zero finite complex number, and let \( n \) be a positive integer. If \( n \geq 5 \), then \( f' + af^n \) assumes each value \( b \in \mathbb{C} \) infinitely often.
There are some examples constructed by Mues [6] which show that Theorem A is not true when \( n = 3, 4 \). Corresponding to Theorem A, Ye [7, Theorem 2.1] proved the following interesting result.

**Theorem B.** Let \( f \) be a transcendental meromorphic function. If \( a \neq 0 \) is a finite complex number and \( n \geq 3 \) is an integer, then \( f + af^n \) assumes all finite complex number infinitely often.

In [7, Theorem 2.2], Ye also obtained the following result, which may be considered as a normal family analogue of Theorem B.

**Theorem C.** Let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( \mathcal{D} \), \( f \neq b \) and \( f + af^n \neq b \) for every \( f \in \mathcal{F} \), where \( n \geq 2 \) is an integer and \( a \neq 0 \), \( b \) are two finite complex numbers. Then, \( \mathcal{F} \) is normal.

Ye [7] asked whether Theorem B remains valid for \( n = 2 \). Recently, Fang and Zalcman showed that Theorem B holds for \( n = 2 \). In [8], the condition in Theorem C that \( f \neq b \) can be relaxed to that all zeros of each function in \( \mathcal{F} \) are of multiplicity at least 2. Actually, they obtained the following results.

**Theorem D.** Let \( f \) be a transcendental meromorphic function. If \( a \neq 0 \) is a finite complex number and \( n \geq 2 \) is an integer, then \( f + af^n \) assumes all finite complex number infinitely often.

**Theorem E.** Let \( \mathcal{F} \) be a family of meromorphic functions on the plane domain \( \mathcal{D} \), let \( n \geq 2 \) be a positive integer, and let \( a \neq 0 \), \( b \) be complex numbers. If, for each \( f \in \mathcal{F} \), all zeros of \( f \) are multiple and \( f + af^n \neq b \) on \( D \), then \( \mathcal{F} \) is normal on \( D \).

A natural problem arises: what can we say if \( f' \) in Theorems E is replaced by the kth derivative \( f^{(k)} \)? In [9], Xu et al. proved the following result.

**Theorem F.** Let \( a(\neq 0), b \in \mathbb{C} \) and \( n \) and \( k \) be two positive integers such that \( n \geq k + 1 \). Let \( \mathcal{F} \) be a family of meromorphic functions defined on a domain \( \mathcal{D} \). If, for every function \( f \in \mathcal{F} \), \( f \) has only zeros of multiplicity at least \( k + 1 \), and \( f + a(f^{(k)})^n \neq b \) in \( D \), then \( \mathcal{F} \) is normal.

Xu et al. [9] asked whether Theorem F remains valid for \( n = 2 \). We partially answer this question. If \( f \neq 0 \), we generalize Theorem F by allowing \( f + a(f^{(k)})^n \) to have zeros but restricting their numbers.

**Theorem 1.1.** Let \( a(\neq 0), b \in \mathbb{C} \), and \( n \) and \( k \) be two positive integers such that \( n \geq 2 \). Let \( \mathcal{F} \) be a family of zero-free meromorphic functions defined in a domain \( \mathcal{D} \) such that for each \( f \in \mathcal{F} \), \( f + a(f^{(k)})^n \neq b \) has at most nk zeros, ignoring multiplicity. Then, \( \mathcal{F} \) is normal in \( \mathcal{D} \).

**Remark 1.2.** Here, \( f \neq 0 \) can be replaced by \( f \neq c \), where \( c \) is any finite complex numbers.

**Example 1.3.** Let \( \mathcal{D} = \{ z : |z| < 1 \} \). Let \( \mathcal{F} = \{ f_m \} \), where \( f_m := e^{mz} \). Then, \( f_m + af_m^n = (1 + am)e^{mz} \neq 0 \) in \( \mathcal{D} \) for every function \( f \in \mathcal{F} \). However, it is easily obtained that \( \mathcal{F} \) is not normal at the point \( z = 0 \).

**Example 1.4.** Let \( \mathcal{D} = \{ z : |z| < 1 \} \). Let \( \mathcal{F} = \{ f_m \} \), where \( f_m := 1/mz \). Then, \( f_m + a(f_m)^2 = (mz^2 + 1)/m^2z^4 \) has 3 zeros in \( \mathcal{D} \) for every function \( f \in \mathcal{F} \). However, it is easily obtained that \( \mathcal{F} \) is not normal at the point \( z = 0 \).
Example 1.5. Let $\mathcal{D} = \{z : |z| < 1\}$. Let $\mathcal{F} = \{f_m\}$, where $f_m := mz$. It follows that $f_m + a(f_m)^2 = mz + m^2$ has no zero in $\mathcal{D}$ for every function $f \in \mathcal{F}$. However, it is easily obtained that $\mathcal{F}$ is not normal at the point $z = 0$.

Examples 1.3 and 1.4 show that the conditions that $n \geq 2$ and $f + a(f^{(k)})^n - b$ have at most $nk$ distinct zeros in Theorem 1.1 are shape. Example 1.5 shows the condition that $f \neq 0$ cannot be omitted.

2. Some Lemmas

To prove our results, we need some preliminary results.

**Lemma 2.1 ([9], Lemma 2.2).** Let $n \geq 2$, $k$ be positive integers, let $a$ be a nonzero constant and let $P(z)$ be a polynomial. Then, the solution of the differential equation $a(W^{(k)}(z))^n + W(z) = P(z)$ must be polynomial.

**Lemma 2.2.** Let $f$ be a nonzero transcendental meromorphic function. If $a$ be a nonzero finite complex number and let $n \geq 2$ and $k$ be two positive integers. Then, $f + a(f^{(k)})^n$ assumes each value $b \in \mathbb{C}$ infinitely often.

**Proof.** Set

$$F = f + a\left(f^{(k)}\right)^n - b, \quad (2.1)$$

$$\phi = \frac{F'}{F} = \frac{f' + an(f^{(k)})^{n-1}f^{(k+1)}}{f + a(f^{(k)})^n - b}, \quad (2.2)$$

$$q' = n\frac{f^{(k+1)}}{f^{(k)}} - \frac{F'}{F} = \frac{n(f^{(k+1)}f - bf^{(k)} - f'f^{(k)})}{f^{(k)}(f + a(f^{(k)})^n - b)}. \quad (2.3)$$

We claim that $q \equiv 0$. If $q \equiv 0$, then $F \equiv 0$. We can deduce that $F \equiv c$, where $c$ is a finite complex number. We conclude from (2.1) and Lemma 2.1 that, $f$ must be a polynomial, which is a contradiction.

If $q \equiv 0$, from (2.3), we can obtain

$$c\left(f^{(k)}\right)^n = f + a\left(f^{(k)}\right)^n - b, \quad (2.4)$$

where $c$ is a finite complex number, that is,

$$(a - c)\left(f^{(k)}\right)^n + f = b. \quad (2.5)$$

If $a - c = 0$, we can get that $f \equiv b$, which is a contradiction.

If $a - c \neq 0$, we conclude from (2.5) and Lemma 2.1 that $f$ must be a polynomial, which is a contradiction.
By elementary Nevanlinna theory and (2.1), we have \( T(r, F) = O(T(r, f)) \). Thus, from (2.2) and (2.3), we have

\[
m(r, \phi) = S(r, f), \quad m(r, \varphi) = S(r, f).
\] (2.6)

It follows from (2.2), (2.3) and Nevanlinna’s First Fundamental Theorem that

\[
N\left( r, \frac{1}{\phi} \right) \leq m(r, \phi) + N(r, \phi) - m\left( r, \frac{1}{\phi} \right) + O(1)
\] (2.7)

\[
\leq N(r, \phi) + S(r, f) \leq \overline{N}(r, f) + \overline{N}\left( r, \frac{1}{F} \right) + S(r, f),
\]

\[
N\left( r, \frac{1}{\varphi} \right) \leq m(r, \varphi) + N(r, \varphi) - m\left( r, \frac{1}{\varphi} \right) + O(1)
\] (2.8)

\[
\leq N(r, \varphi) + S(r, f) \leq \overline{N}\left( r, \frac{1}{f^{(k)}} \right) + \overline{N}\left( r, \frac{1}{F} \right) + S(r, f).
\]

By (2.2) and (2.3), we get

\[
\phi(f - b) - f' = a \left( f^{(k)} \right)^n \varphi.
\] (2.9)

We have by (2.6)-(2.7)

\[
T(r, \phi(f - b) - f') = T\left( r, (f - b) \left( \phi - \frac{f'}{f - b} \right) \right)
\]

\[
\leq T(r, f - b) + T\left( r, \phi - \frac{f'}{f - b} \right) + S(r, f)
\]

\[
\leq m(r, f - b) + N(r, f - b) + m\left( r, \phi - \frac{f'}{f - b} \right) + N\left( r, \phi - \frac{f'}{f - b} \right) + S(r, f)
\] (2.10)

\[
\leq m(r, f) + N(r, f) + m(r, \phi) + m\left( r, \frac{f'}{f - b} \right) + N\left( r, \phi - \frac{f'}{f - b} \right) + S(r, f)
\]

\[
\leq T(r, f) + \overline{N}(r, f) + \overline{N}\left( r, \frac{1}{F} \right) + S(r, f).
\]
It follows from (2.6)–(2.10) that

\[ nT(r, f^{(k)}) \leq T(r, \varphi) + T(r, \phi(f - b) - f') + S(r, f) \]

\[ \leq m(r, \varphi) + N(r, \varphi) + T(r, f) + \overline{N}(r, f) + N \left( r, \frac{1}{F} \right) + S(r, f) \]

\[ \leq \overline{N} \left( r, \frac{1}{f^{(k)}} \right) + N \left( r, \frac{1}{F} \right) + m \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f} \right) + \overline{N}(r, f) \]

\[ + \overline{N} \left( r, \frac{1}{F} \right) + S(r, f) \]

\[ \leq \overline{N} \left( r, \frac{1}{f^{(k)}} \right) + 2N \left( r, \frac{1}{F} \right) + m \left( r, \frac{f^{(k)}}{f} \right) + m \left( r, \frac{1}{f^{(k)}} \right) + N \left( r, \frac{1}{f} \right) \]

\[ + \overline{N}(r, f) + S(r, f) \]

\[ \leq T \left( r, \frac{1}{f^{(k)}} \right) + 2N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{f} \right) + \overline{N}(r, f) + S(r, f) \]

\[ \leq T(r, f^{(k)}) + 2N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{f} \right) + \overline{N}(r, f) + S(r, f). \quad (2.11) \]

So, we have

\[ (n - 1)T \left( r, f^{(k)} \right) \leq 2N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{f} \right) + \overline{N}(r, f) + S(r, f). \quad (2.12) \]

We have

\[ (n - 1)T \left( r, f^{(k)} \right) \geq (n - 1)N \left( r, f^{(k)} \right) \geq (n - 1)N(r, f) + (n - 1)\overline{N}(r, f). \quad (2.13) \]

Since \( f \neq 0 \), if \( f + a(f^{(k)})^n \) assumes the value \( b \) only finitely often, we by (2.12) can get

\[ N(r, f) = S(r, f). \quad (2.14) \]

Hence,

\[ (n - 1)T \left( r, f^{(k)} \right) \leq 2N \left( r, \frac{1}{F} \right) + S(r, f). \quad (2.15) \]

So \( f + a(f^{(k)})^n \) assumes each value \( b \in \mathbb{C} \) infinitely often.

We complete the proof of Lemma 2.2. \( \Box \)
Using the method of Chang [10, Lemma 4], we obtain the following lemma.

**Lemma 2.3.** Let \( f \) be a nonconstant zero-free rational function, \( n \geq 2 \), let \( k \) be two positive integers, and \( a \neq 0 \), \( b \) be two complex constants. Then, the function \( f + a(f^{(k)})^n - b \) has at least \( nk + 1 \) distinct zeros in \( \mathbb{C} \).

**Proof.** Since \( f(z) \) is a nonconstant zero-free rational function, \( f(z) \) is not a polynomial, and hence it has at least one finite pole. Thus, we can write

\[
f(z) = \frac{C_1}{\prod_{i=1}^{m}(z + z_i)^{p_i}},
\]

(2.16)

where \( C_1 \) is a nonzero constant, \( m \) and \( p_i \) are positive integers, the \( z_i \) (when \( 1 \leq i \leq m \)) are distinct complex numbers, and denote \( p = \sum_{i=1}^{m} p_i \).

By induction, we deduce from (2.16) that

\[
f^{(k)}(z) = \frac{P_{(m-1)k}}{\prod_{i=1}^{m}(z + z_i)^{p_i+k}},
\]

(2.17)

where \( P_{(m-1)k} \) is polynomial of degree \( (m-1)k \).

So the degree of numerator of the function \( f + a(f^{(k)})^n \) is equal to \( \sum_{i=1}^{m} (n-1)p_i + nk \).

By calculation, \( f + a(f^{(k)})^n - b \) has at least one zero in \( \mathbb{C} \). Thus, we can write

\[
f + a\left((f^{(k)})^n\right) - b = \frac{C_2\prod_{i=1}^{m}(z + \alpha_i)^{l_i}}{\prod_{i=1}^{m}(z + z_i)^{n(p_i+k)}}.
\]

(2.18)

where \( C_2 \) is a nonzero constant, \( l_i \) are positive integers, \( \alpha_i \) (when \( 1 \leq i \leq s \)), and \( z_i \) (when \( 1 \leq i \leq m \)) are distinct complex numbers. Thus, by (2.16), (2.17), and (2.18), we get

\[
C_1\prod_{i=1}^{m}(z + z_i)^{(n-1)p_i+nk} + a(P_{(m-1)k})^n = b\prod_{i=1}^{m}(z + z_i)^{(n-1)p_i+nk} + C_2\prod_{i=1}^{s}(z + \alpha_i)^{l_i}.
\]

(2.19)

**Case 1.** If \( b = 0 \), it follows that \( \sum_{i=1}^{m} [(n-1)p_i + nk] = \sum_{i=1}^{s} l_i \) and \( C_1 = C_2 \). Thus, it follows from (2.19) that

\[
\prod_{i=1}^{m}(1 + z_i t)^{(n-1)p_i+nk} - \prod_{i=1}^{s}(1 + \alpha_i t)^{l_i} = t^{(n-1)p+nk}Q(t),
\]

(2.20)

where \( Q(t) = (-a/C_1)t^{(m-1)nk}(P_{(m-1)k}(1/t))^n \) is a polynomial. Then, \( Q(t) \) is a polynomial of degree less than \( (m-1)nk \), and it follows that

\[
\frac{\prod_{i=1}^{m}(1 + z_i t)^{(n-1)p_i+nk}}{\prod_{i=1}^{s}(1 + \alpha_i t)^{l_i}} = 1 + \frac{t^{(n-1)p+nk}Q(t)}{\prod_{i=1}^{s}(1 + \alpha_i t)^{l_i}} = 1 + O\left(t^{(n-1)p+nk}\right)
\]

(2.21)

as \( t \to 0 \).
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Logarithmic differentiation of both sides of (2.21) shows that

$$\sum_{i=1}^{m} \frac{(n-1)p_i + nk}{1 + z_i t} z_i - \sum_{i=1}^{s} l_i \alpha_i = O\left(t^{(n-1)p + nk - 1}\right)$$  \hspace{1cm} (2.22)

as $t \to 0$.

Comparing the coefficient of (2.22) for $t^j, j = 0, 1, \ldots, (n-1)p + nk - 2$, we have

$$\sum_{i=1}^{m} ((n-1)p_i + nk) z_i^j - \sum_{i=1}^{s} l_i \alpha_i^j = 0$$  \hspace{1cm} (2.23)

for $j = 1, \ldots, (n-1)p + nk - 1$.

Set $z_{m+i} = -\alpha_i$ when $1 \leq i \leq s$. Noting that $\sum_{i=1}^{m} [(n-1)p_i + nk] = \sum_{i=1}^{s} l_i$, then it follows from (2.23) that the system of linear equations,

$$\sum_{i=1}^{m+s} z_i^j x_i = 0,$$  \hspace{1cm} (2.24)

where $0 \leq j \leq (n-1)p + nk - 1$, has a nonzero solution

$$(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+s}) = ((n-1)p_1 + nk, \ldots, (n-1)p_m + nk, l_1, \ldots, l_s).$$  \hspace{1cm} (2.25)

If $(n-1)p + nk \geq m + s$, then the determinant $\det(z_i^j)_{(m+s)\times(m+s)}$ of the coefficients of the system of (2.24), where $0 \leq j \leq (n-1)p + nk - 1$, is equal to zero, by Cramer’s rule (see, e.g., [11]). However, the $z_i$ are distinct complex numbers when $1 \leq i \leq m + s$, and the determinant is a Vandermonde determinant, so it cannot be 0 (see [11]), which is a contradiction.

Hence, we conclude that $(n-1)p + nk < m + s$. Noting that $n \geq 2$, it follows from this and $p = \sum_{i=1}^{m} p_i \geq m$ that $s \geq nk + 1$.

Case 2. If $b \neq 0$, set

$$b \prod_{i=1}^{m} (z + z_i) n(p_i + k) - C_1 \prod_{i=1}^{m} (z + z_i) (n-1)p_i + nk = b \prod_{i=1}^{m} (z + z_i) (n-1)p_i + nk \prod_{i=1}^{q} (z + \beta_i)^{t_i},$$  \hspace{1cm} (2.26)

where $t_i$ are positive integers. It follows that $\beta_i$ (when $1 \leq i \leq q$) and $z_i$ (when $1 \leq i \leq m$) are distinct complex numbers, and $\sum_{i=1}^{q} t_i = p$.

By (2.19), we have

$$b \prod_{i=1}^{m} (z + z_i) (n-1)p_i + nk \prod_{i=1}^{q} (z + \beta_i)^{t_i} + C_2 \prod_{i=1}^{s} (z + \alpha_i)^{t_i} = a (P_{(m-1),k})^n.$$  \hspace{1cm} (2.27)
It follows that
\[
\sum_{i=1}^{m} [(n-1)p_i + nk] + \sum_{i=1}^{q} t_i = np + nmk = \sum_{i=1}^{s} l_i, 
\]
and \(C_2 = -b\). Thus, by (2.27),
\[
\frac{\prod_{i=1}^{m} (1 + z_i t)^{(n-1)p_i + nk} \prod_{i=1}^{q} (1 + \beta_i t)^{l_i} - \prod_{i=1}^{s} (1 + \alpha_i t)^{l_i}}{\prod_{i=1}^{s} (1 + \alpha_i t)^{l_i}} = t^{n(p+k)} Q(t),
\]
where \(Q(t) = (a/b)t^{(m-1)nk(P_{(m-1)k}(1/t))^n}\) is a polynomial. Then, \(Q(t)\) is a polynomial of degree less than \((m-1)nk\), and it follows that
\[
\frac{\prod_{i=1}^{m} (1 + z_i t)^{(n-1)p_i + nk} \prod_{i=1}^{q} (1 + \beta_i t)^{l_i}}{\prod_{i=1}^{s} (1 + \alpha_i t)^{l_i}} = 1 + t^{n(p+k)} Q(t) = O\left(t^{n(p+k)}\right)
\]
as \(t \to 0\).
Thus, by taking logarithmic derivatives of both sides of (2.12), we get
\[
\sum_{i=1}^{m} \left(\frac{(n-1)p_i + nk}{1 + z_i t}\right) z_i + \sum_{i=1}^{q} \frac{t_i \beta_i}{1 + \beta_i t} - \sum_{i=1}^{s} \frac{l_i \alpha_i}{1 + \alpha_i t} = O\left(t^{n(p+k)-1}\right).
\]

We consider two cases.

**Subcase 2.1** \((\{\alpha_1, \ldots, \alpha_s\} \cap \{\beta_1, \ldots, \beta_q\} = \emptyset)\). Applying the reasoning of Case 1 and noting that \(p \geq q\), we deduce that \(s \geq nk\).

**Subcase 2.2** \((\{\alpha_1, \ldots, \alpha_s\} \cap \{\beta_1, \ldots, \beta_q\} \neq \emptyset)\). Without loss of generality, we may assume that \(\alpha_{q-i} = \beta_i\) for \((1 \leq i \leq M)\). Denote
\[
z_i = \begin{cases} 
\beta_i & \text{for } 1 \leq i \leq m, \\
\alpha_{i-M} & \text{for } m+1 \leq i \leq m+q, \\
\alpha_{M+i-q} & \text{for } m+q+1 \leq i \leq m+q+s-M, 
\end{cases}
\]
and
\[
N_i = \begin{cases} 
(n-1)p_i + nk & \text{for } 1 \leq i \leq m, \\
t_{i-M} & \text{for } m+1 \leq i \leq m+s-M, \\
t_{i-M} - l_{i-M-s+M} & \text{for } m+s-M+1 \leq i \leq m+q, \\
l_{i-M-q+M} & \text{for } m+q+1 \leq i \leq m+q+s-M.
\end{cases}
\]

The formula (2.31) can be rewritten:
\[
\sum_{i=1}^{m+q+s-M} \frac{N_i z_i}{1 + z_i t} = O\left(t^{n(p+k)-1}\right).
\]
Applying the reasoning of Case 1, and noting that \( p \geq q \), we deduce that \( s \geq nk + 1 \). This completes the proof of Lemma 2.3.

**Lemma 2.4** ([10], Lemma 4). Let \( f \) be a nonconstant zero-free rational function, let \( a \neq 0 \) be a complex constant, and let \( k \) be a positive integer. Then \( f^{(k)} - a \) has at least \( k + 1 \) distinct zeros in \( \mathbb{C} \).

**Lemma 2.5** (see [12], Lemma 2, Zalcman’s lemma). Let \( \mathcal{F} \) be a family of functions meromorphic on a domain \( \mathcal{D} \), all of whose zeros have multiplicity at least \( k \). Suppose that there exists \( A \geq 1 \) such that \( |f^{(k)}(z)| \leq A \) whenever \( f(z) = 0 \). Then, if \( \mathcal{F} \) is not normal at \( z_0 \in \mathcal{D} \), there exist, for each \( 0 \leq \alpha \leq k \),

(a) points \( z_n, z_n \to z_0 \);
(b) functions \( f_n \in \mathcal{F} \);
(c) positive numbers \( \rho_n \to 0^+ \);

such that \( \rho^{-\alpha}_n f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi) \) locally uniformly with respect to the spherical metric, where \( g(\xi) \) is a nonconstant meromorphic function on \( \mathbb{C} \), all of whose zeros of \( g(\xi) \) are of multiplicity at least \( k \), such that \( g^k(\xi) \leq g^k(0) = kA + 1 \).

Here, as usual, \( g^k(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2) \) is the spherical derivative.

### 3. Proof of Theorem

Suppose that \( \mathcal{F} \) is not normal in \( \mathcal{D} \). Then, there exists at least one point \( z_0 \) such that \( \mathcal{F} \) is not normal at the point \( z_0 \in \mathcal{D} \). Without loss of generality, we assume that \( z_0 = 0 \). We consider two cases.

**Case 1** \((b = 0)\). By Zalcman’s lemma, there exist:

(a) points \( z_n, z_n \to z_0 \);
(b) functions \( f_n \in \mathcal{F} \);
(c) positive numbers \( \rho_n \to 0^+ \);

such that

\[
g_j(\xi) = \rho_j^{-nk/(n-1)} f_j(z_j + \rho_j \xi) \to g(\xi),
\]

spherically uniformly on compact subsets of \( \mathbb{C} \), where \( g(\xi) \) is a nonconstant meromorphic function in \( \mathbb{C} \). Since \( f_j \neq 0 \), by Hurwitz’s theorem, it implies that \( g(\xi) \neq 0 \).

On every compact subset of \( \mathbb{C} \) which contains no poles of \( g \), from (3.1), we get

\[
g_j(\xi) + a\left(g_j^k(\xi)\right)^n = \rho_j^{-nk/(n-1)} \left(f_j(z_j + \rho_j \xi) + a\left(f_j^k(z_j + \rho_j \xi)\right)^n\right) \to g(\xi) + a\left(g^k(\xi)\right)^n,
\]

also locally uniformly with respect to the spherical metric.
We claim that \( g(\xi) + a(g^k(\xi))^n \) has at most \( nk \) distinct zeros.

Suppose that \( g(\xi) + a(g^k(\xi))^n \) has \( nk + 1 \) distinct zeros \( \xi_i, 1 \leq i \leq nk + 1 \), and choose \( \delta(>0) \) small enough such that \( \bigcap_{i=1}^{nk+1} D(\xi_i, \delta) = \emptyset \), where \( D(\xi_0, \delta) = \{ \xi : |\xi - \xi_0| < \delta \} \).

From (3.2), by Hurwitz’s theorem, there exist points \( \xi_i^j \in D(\xi_i, \delta) \) (1 \( \leq i \leq nk + 1 \)) such that for sufficiently large \( j \),

\[
f_j(z_j + \rho_j \xi_i^j) + a\left(f_j^k(z_j + \rho_j \xi_i^j)\right)^n = 0, \tag{3.3}
\]

for \( 1 \leq i \leq nk + 1 \).

Since \( z_j \to 0 \) and \( \rho_j \to 0^+ \), we have \( z_j + \rho_j \xi_i^j \in D(0, \sigma) \) (\( \sigma \) is a positive constant) for sufficiently large \( j \), so \( f_j(z) + a(f_j^k(z))^n \) has \( nk + 1 \) distinct zeros, which contradicts the fact that \( f_j(z) + a(f_j^k(z))^n \) has at most \( nk \) zero.

However, by Lemmas 2.2 and 2.3, there do not exist nonconstant meromorphic functions that have the above properties. This contradiction shows that \( \mathcal{F} \) is normal in \( \mathbb{D} \).

**Case 2 (\( b \neq 0 \)).** By Zalcman’s lemma, there exist:

(a) points \( z_n, z_n \to z_0 \);  
(b) functions \( f_n \in \mathcal{F} \);  
(c) positive numbers \( \rho_n \to 0^+ \);

such that

\[
g_j(\xi) = \rho_j^k f_j(z_j + \rho_j \xi) \to g(\xi) \tag{3.4}
\]

spherically uniformly on compact subsets of \( \mathbb{C} \), where \( g(\xi) \) is a nonconstant meromorphic function in \( \mathbb{C} \). Since \( f_j \neq 0 \), by Hurwitz’s theorem, it implies that \( g(\xi) \neq 0 \).

On every compact subset of \( \mathbb{C} \) which contains no poles of \( g \), from (3.4), we get

\[
\rho_j^k g_j(\xi) + a\left(g_j^k(\xi)\right)^n - b \to a\left(g^k(\xi)\right)^n - b \tag{3.5}
\]

also locally uniformly with respect to the spherical metric.

Noting that

\[
\rho_j^k g_j(\xi) + a\left(g_j^k(\xi)\right)^n - b = f_j(z_j + \rho_j \xi) + a\left(f_j^k(z_j + \rho_j \xi)\right)^n - b, \tag{3.6}
\]

we claim that \( a(g^k(\xi))^n - b \) has at most \( nk \) distinct zeros.

Suppose that \( g(\xi) + a(g^k(\xi))^n - b \) has \( nk + 1 \) distinct zeros \( \xi_i, 1 \leq i \leq nk + 1 \), and choose \( \delta(>0) \) small enough such that \( \bigcap_{i=1}^{nk+1} D(\xi_i, \delta) = \emptyset \), where \( D(\xi_0, \delta) = \{ \xi : |\xi - \xi_0| < \delta \} \).

From (3.2), by Hurwitz’s theorem, there exist points \( \xi_i^j \in D(\xi_i, \delta) \) (1 \( \leq i \leq nk + 1 \)) such that for sufficiently large \( j \)

\[
f_j(z_j + \rho_j \xi_i^j) + a\left(f_j^k(z_j + \rho_j \xi_i^j)\right)^n - b = 0, \tag{3.7}
\]

for \( 1 \leq i \leq nk + 1 \).
Since \( z_j \to 0 \) and \( \rho_j \to 0^+ \), we have \( z_j + \rho_j g_j^i \in D(0, \sigma) \) (\( \sigma \) is a positive constant) for sufficiently large \( j \), so \( f_j(z) + a(f_j^n(z))^n - b \) has \( nk + 1 \) distinct zeros, which contradicts the fact that \( f_j(z) + a(f_j^n(z))^n - b \) has at most \( nk \) zero.

Denote \( c_1, c_2, \ldots, c_n \) by the different roots of \( \omega^n = b/a \), then

\[
a(g^k(\xi))^n - b = a \prod_{i=1}^{n} (g^k(\xi) - c_i). \tag{3.8}
\]

**Subcase 2.1** (If \( g(\xi) \) is a rational function). By Lemma 2.4 and (3.8), we can deduce that \( a(g^k(\xi))^n - b \) has at least \( nk + n \) distinct zeros. This contradicts the claim that \( a(g^k(\xi))^n - b \) has at most \( nk \) distinct zeros.

**Subcase 2.2** (If \( g(\xi) \) is a transcendental meromorphic function). By Nevanlinnas second main theorem, we have

\[
T(r, g^{(k)}) \leq N(r, g^{(k)}) + \sum_{i=1}^{n} N(r, \frac{1}{g^{(k)} - c_i}) + S(r, g^{(k)})
\]

\[
= N(r, g^{(k)}) + N(r, \frac{1}{a(g^{(k)} - b)} + S(r, g^{(k)})
\]

\[
\leq \frac{1}{k+1} N(r, g^{(k)}) + S(r, g^{(k)})
\]

\[
\leq \frac{1}{k+1} T(r, g^{(k)}) + S(r, g^{(k)}).
\]

It follows that \( T(r, g^{(k)}) \leq S(r, g^{(k)}) \), which is a contradiction. This contradiction shows that \( \mathfrak{F} \) is normal in \( \mathfrak{D} \).

Hence, Theorem 1.1 is proved.

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**References**

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