Research Article

The Equivalence of Convergence Results of Modified Mann and Ishikawa Iterations with Errors without Bounded Range Assumption

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Let $E$ be an arbitrary uniformly smooth real Banach space, let $D$ be a nonempty closed convex subset of $E$, and let $T : D \to D$ be a uniformly generalized Lipschitz generalized asymptotically $\Phi$-strongly pseudocontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be four real sequences in $[0, 1]$ and satisfy the conditions: (i) $a_n + c_n \leq 1$, $b_n + d_n \leq 1$; (ii) $a_n, b_n, d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0, z_0 \in D$, let $\{u_n\}, \{v_n\}, \{w_n\}$ be any bounded sequences in $D$, and let $\{x_n\}, \{z_n\}$ be the modified Ishikawa and Mann iterative sequences with errors, respectively. Then the convergence of $\{x_n\}$ is equivalent to that of $\{z_n\}$.

1. Introduction and Preliminary

Let $E$ be a real Banach space and let $E^*$ be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E, \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that

(i) if $E$ is a smooth Banach space, then the mapping $J$ is single-valued;
(ii) $J(ax) = aJ(x)$ for all $x \in E$ and $a \in \mathbb{R}$;
(iii) if $E$ is a uniformly smooth Banach space, then the mapping $J$ is uniformly continuous on any bounded subset of $E$. Throughout this paper, we denote that
\( j \) is the single-valued normalized duality mapping, \( D \) is a nonempty closed convex subset of \( E, T : D \to D \) is a mapping, and \( T^0 \) is the unit mapping \( I \).

In 1972, Goebel and Kirk [1] introduced the class of asymptotically nonexpansive mappings as follows.

**Definition 1.1.** A mapping \( T \) is said to be asymptotically nonexpansive if for each \( x, y \in D \)

\[
\| T^n x - T^n y \| \leq k_n \| x - y \|, \quad \forall n \geq 0,
\]

where \( \{k_n\} \subset [1, +\infty) \) with \( \lim_{n \to \infty} k_n = 1 \).

Schu [2], in 1991, gave the definition of asymptotically pseudocontractive mappings and proved the equivalence results.

**Definition 1.2.** The mapping \( T \) is called asymptotically pseudocontractive with the sequence \( \{k_n\} \subset [1, +\infty) \) if and only if \( \lim_{n \to \infty} k_n = 1 \), and for all \( n \in N \) and all \( x, y \in D \), there exists \( j(x - y) \in J(x - y) \) such that

\[
\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \| x - y \|^2.
\]

It is easy to find that every asymptotically nonexpansive mapping is asymptotically pseudocontractive. However, the converse is not true in general. See example of [3].

Recently, Colao [4] combined the proof ideas of the papers of Chang [5] and C. E. Chidume and C. O. Chidume [6] and then showed the equivalent theorem results of the convergence between Mann and Ishikawa iterations with errors for generalized strongly asymptotically \( \phi \)-pseudocontractive mapping with bounded range. In fact, he proved the following theorem.

**Theorem 1.3.** Let \( X \) be a uniformly smooth Banach space, and let \( T : X \to X \) be generalized strongly asymptotically \( \phi \)-pseudocontractive mapping with fixed point \( x^* \) and bounded range. Let \( \{x_n\} \) and \( \{z_n\} \) be the sequences defined by (1.4) and (1.5), respectively,

\[
\begin{align*}
    y_n &= (1 - \beta_n - \delta_n) x_n + \beta_n T^n x_n + \delta_n v_n, \quad n \geq 0, \\
    x_{n+1} &= (1 - \alpha_n - \gamma_n) x_n + \alpha_n T^n y_n + \gamma_n u_n, \quad n \geq 0, \\
    z_{n+1} &= (1 - \alpha_n - \gamma_n) z_n + \alpha_n T^n z_n + \gamma_n w_n, \quad n \geq 0,
\end{align*}
\]

where \( \{\alpha_n\}, \{\gamma_n\}, \{\beta_n\}, \{\delta_n\} \subset [0, 1] \) satisfy

(H1) \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \delta_n = 0 \) and \( \gamma_n = o(\alpha_n) \),

(H2) \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

and the sequences \( \{u_n\}, \{v_n\}, \{w_n\} \) are bounded in \( X \), then for any initial point \( z_0, x_0 \in X \), the following two assertions are equivalent.

(1) The modified Ishikawa iteration sequence with errors (1.4) converges to \( x^* \);

(2) The modified Mann iteration sequence with errors (1.5) converges to \( x^* \).
The aim of this paper is to prove the equivalence of convergent results of above Ishikawa and Mann iterations with errors for generalized asymptotically $\Phi$-strongly pseudocontractive mappings without bounded range assumptions in uniformly smooth real Banach spaces. For this, we need the following concepts and lemmas.

**Definition 1.4** (see [4]). The mapping $T$ is called generalized asymptotically $\Phi$-strongly pseudocontractive if

$$\langle T^nx - T^ny, j(x - y) \rangle \leq k_n \|x - y\|^2 - \Phi(\|x - y\|), \quad n \geq 0,$$

where $j(x - y) \in J(x - y), \{k_n\} \subset [1, +\infty)$ is converging to one and $\Phi: [0, +\infty) \to [0, +\infty)$ is strictly increasing continuous function with $\Phi(0) = 0$.

**Definition 1.5** (see [4]). For arbitrary given $x_0 \in D$, modified Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$y_n = (1 - b_n - d_n)x_n + b_n T^nx_n + d_n w_n, \quad n \geq 0,$$
$$x_{n+1} = (1 - a_n - c_n)x_n + a_n T^ny_n + c_n v_n, \quad n \geq 0,$$

where $\{v_n\}, \{w_n\}$ are any bounded sequences in $D; \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are four real sequences in $[0, 1]$ and satisfy $a_n + c_n \leq 1, b_n + d_n \leq 1$, for all $n \geq 0$. If $b_n = d_n = 0$, we define modified Mann iterative process with errors $\{z_n\}$ by

$$z_{n+1} = (1 - a_n - c_n)z_n + a_n T^nz_n + c_n u_n, \quad n \geq 0,$$

where $\{u_n\}$ is any bounded sequence in $D$.

**Lemma 1.6** (see [7]). Let $E$ be a uniformly smooth real Banach space and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle,$$

for all $x, y \in E$.

**Lemma 1.7** (see [8]). Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality:

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n, \quad n \geq 0,$$

where $\lambda_n \in [0, 1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty, \sigma_n = o(\lambda_n)$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Main Results

First of all, we give a new concept.
Definition 2.1. A mapping $T : D \to D$ is called uniformly generalized Lipschitz if there exists a constant $L > 0$ such that
\[ \|T^nx - T^ny\| \leq L(1 + \|x - y\|), \quad \forall x, y \in D, \forall n \geq 0. \] (2.1)

It is mentioned to notice that if $T$ has bounded range, then it is uniformly generalized Lipschitz. In fact, since $R(T^n) \subseteq R(T)$, then $\sup_{x \in D} \|T^nx\| \leq \sup_{x \in D} \|Tx\| = M_1$, thus $\|T^nx - T^ny\| \leq 2M_1 \leq L(1 + \|x - y\|)$, where $L = 2M_1$. On the contrary, it is not true in general (See [6]).

In the following, we prove the main theorems of this paper.

Theorem 2.2. Let $E$ be an arbitrary uniformly smooth real Banach space, let $D$ be a nonempty closed convex subset of $E$, and let $T : D \to D$ be a uniformly generalized Lipschitz generalized asymptotically $\Phi$-strongly pseudocontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be four real sequences in $[0,1]$ and satisfy the following conditions:

(i) $a_n + c_n \leq 1, b_n + d_n \leq 1$;

(ii) $a_n, b_n, d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$;

(iii) $\sum_{n=0}^\infty a_n = \infty$.

For some $x_0, z_0 \in D$, let $\{u_n\}, \{v_n\}, \{w_n\}$ be any bounded sequences in $D$, and let $\{x_n\}$ and $\{z_n\}$ be Ishikawa and Mann iterative sequences with errors defined by (1.7) and (1.8), respectively. Then the following conclusions are equivalent:

(1) $\{x_n\}$ converges strongly to the unique fixed point $q$ of $T$;

(2) $\{z_n\}$ converges strongly to the unique fixed point $q$ of $T$.

Proof. $(1) \Rightarrow (2)$ is obvious, that is, let $b_n = d_n = 0$, (1.7) turns into (1.8). We only need to show that $(2) \Rightarrow (1)$. Since $T : D \to D$ is a uniformly generalized Lipschitz generalized asymptotically $\Phi$-strongly pseudocontractive mapping, then there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that
\[ \langle T^nx - T^ny, J(x - y) \rangle \leq k_n \|x - y\|^2 - \Phi(\|x - y\|), \] (2.2)

that is,
\[ \langle (k_nI - T^n)x - (k_nI - T^n)y, J(x - y) \rangle \geq \Phi(\|x - y\|), \] (2.3)
\[ \|T^nx - T^ny\| \leq L(1 + \|x - y\|), \] (2.4)

for any $x, y \in D$. For convenience, denote $k = \sup_n \{k_n\}$.

Step 1. There exists $x_0 \in D$ and $x_0 \neq Tx_0$ such that $r_0 = (k + L)\|x_0 - q\|^2 + L\|x_0 - q\| \in R(\Phi)(\text{range of } \Phi)$.

Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup \{\Phi(r) : r \in [0, +\infty)\} = r_1 < +\infty$ with $r_1 < r_0$, then, for $q \in D$, there exists a sequence $\{v_n\}$ in $D$ such that $v_n \to q$ as $n \to \infty$ with $v_n \neq q$. Furthermore, there exists a natural number $n_0$ such that $(k + L)\|v_n - q\|^2 + L\|v_n - q\| <
Next, we want to prove that \(x_n\) where \(n \geq n_0\), then we redefine \(x_n, r_0\) such that \(x_0 = v_{n_0}, r_0 = (k+L)\|x_0-q\|^2+L\|x_0-q\| \in R(\Phi)\). Hence, it is to ensure that \(\Phi^{-1}(r_0)\) is well defined.

**Step 2.** For any \(n \geq 0\), \(\{x_n\}\) is a bounded sequence.

Set \(R = \Phi^{-1}(r_0)\). From (2.3), we have

\[
\langle k_n(x_0 - q) - (T^n x_0 - q), J(x_0 - q) \rangle \geq \Phi(\|x_0 - q\|),
\]

that is, \((k+L)\|x_0 - q\|^2 + L\|x_0 - q\| \geq \Phi(\|x_0 - q\|).\) Thus, we obtain that \(\|x_0 - q\| \leq R\). Denote

\[
\begin{align*}
B_1 &= \{ x \in D : \|x - q\| \leq R \}, \\
B_2 &= \{ x \in D : \|x - q\| \leq 2R \}, \\
M &= \sup_n \{ \|v_n - q\| \} + \sup_n \{ \|w_n - q\| \}.
\end{align*}
\]

Next, we want to prove that \(x_n \in B_1\) for any \(n \geq 0\) by induction. If \(n = 0\), then \(x_0 \in B_1\). Now we assume that it holds for some \(n\), that is, \(x_n \in B_1\). We prove that \(x_{n+1} \in B_1\). Suppose that it is not the case, then \(\|x_{n+1} - q\| > R\). Since \(J\) is uniformly continuous on bounded subset of \(E\), then, for \(\epsilon_0 = \Phi(R/4)/24L(1+2R)\), there exists \(\delta > 0\) such that \(\|Jx - Jy\| < \epsilon_0\) when \(\|x - y\| < \delta\), for all \(x, y \in B_2\). Now denote

\[
\tau_0 = \min \left\{ \frac{R}{2[L(1+2R) + 2R + M]}, \frac{R}{4[L(1+R) + 2R + M]'}, \frac{\delta}{2[L(1+2R) + 2R + M]'}, \frac{\Phi(R/4)}{24R^2}, \frac{\Phi(R/4)}{24L(1+2R)'}, \frac{\Phi(R/4)}{48MR} \right\}.
\]

Since \(a_n, b_n, c_n, d_n \to 0\) as \(n \to \infty\), and \(c_n = o(a_n)\), without loss of generality, we assume that \(0 \leq a_n, b_n, c_n, d_n \leq \tau_0, c_n < a_n\tau_0\) for any \(n \geq 0\). Then we obtain the following estimates:

\[
\begin{align*}
\|T^n x_n - q\| &\leq L(1 + \|x_n - q\|) \\
&\leq L(1 + R), \\
\|y_n - q\| &\leq (1 - b_n - d_n)\|x_n - q\| + b_n\|T^n x_n - q\| + d_n\|w_n - q\| \\
&\leq R + b_nL(1 + \|x_n - q\|) + d_nM \\
&\leq R + b_nL(1 + R) + d_nM \\
&\leq R + \tau_0[L(1 + R) + M] \\
&\leq 2R, \\
\|T^n y_n - q\| &\leq L(1 + \|y_n - q\|) \\
&\leq L(1 + 2R),
\end{align*}
\]
\[ \| x_n - T^n x_n \| \leq \| x_n - q \| + \| T^n x_n - q \| \]
\[ \leq L + (1 + L) \| x_n - q \| \]
\[ \leq L + (1 + L) R, \]
\[ \| (x_n - q) - (y_n - q) \| \leq b_n \| x_n - T^n x_n \| + d_n \| \| w_n - q \| + \| x_n - q \| \|
\[ \leq b_n [L + (1 + L) R] + d_n (M + R) \]
\[ \leq \tau_0 [L(1 + R) + 2R + M] \]
\[ \leq \tau_0 [L(1 + 2R) + 2R + M] \]
\[ \leq \delta \]

\[ \left\| x_n - q \right\| \geq \left\| x_{n+1} - q \right\| - a_n \left\| T^n y_n - x_n \right\| - c_n \left\| v_n - x_n \right\| \]
\[ \geq \left\| x_{n+1} - q \right\| - a_n \left\| T^n y_n - q \right\| + \| x_n - q \| - c_n \left\| x_n - q \right\| + \| v_n - q \| \]
\[ \geq R - a_n [L(1 + 2R) + R] - c_n (R + M) \]
\[ \geq R - \tau_0 [L(1 + 2R) + M + 2R] \]
\[ \geq R - \frac{R}{2} - \frac{R}{4} = \frac{R}{4} \]

\[ \| y_n - q \| \geq \| x_n - q \| - b_n \| T^n x_n - x_n \| - d_n \| x_n - w_n \| \]
\[ \geq \| x_n - q \| - b_n [L + (1 + L) R] - d_n \| x_n - q \| - w_n - q \| \]
\[ \geq \| x_n - q \| - b_n [L + (1 + L) R] - d_n (R + M) \]
\[ \geq \| x_n - q \| - \tau_0 [L(1 + R) + 2R + M] \]
\[ \geq \frac{R}{2} - \frac{R}{4} = \frac{R}{4} \]

\[ \| x_{n+1} - q \| \leq (1 - a_n - c_n) \| x_n - q \| + a_n \| T^n y_n - q \| + c_n \| v_n - q \| \]
\[ \leq R + \tau_0 [L(1 + 2R) + M] \]
\[ \leq 2R, \]

\[ \| (x_{n+1} - q) - (x_n - q) \| \leq a_n \| T^n y_n - x_n \| + c_n \| u_n - x_n \| \]
\[ \leq a_n \left[ \| T^n y_n - q \| + \| x_n - q \| \| + c_n \| v_n - q \| + \| x_n - q \| \right] \]
\[ \leq a_n [L(1 + 2R) + R] + c_n (M + R) \]
\[ \leq \tau_0 [L(1 + 2R) + 2R + M] \]
\[ \leq \frac{\delta}{2} < \delta. \]

Hence, \( \| f(x_n - q) - f(y_n - q) \| < \varepsilon_0; \| f(x_{n+1} - q) - f(x_n - q) \| < \varepsilon_0. \)
Using Lemma 1.6 and formulas above, we obtain

\[
\|x_{n+1} - q\|^2 \leq (1 - a_n - c_n)^2 \|x_n - q\|^2 + 2a_n \langle T^x y_n - q, J(x_{n+1} - q) \rangle \\
+ 2c_n \langle u_n - q, J(x_{n+1} - q) \rangle \\
\leq (1 - a_n)^2 \|x_n - q\|^2 + 2a_n \langle T^x y_n - q, J(x_{n+1} - q) - J(x_n - q) \rangle \\
+ 2a_n \langle T^x y_n - q, J(x_n - q) - J(y_n - q) \rangle \\
+ 2a_n \langle T^x y_n - q, J(y_n - q) \rangle + 2c_n \langle u_n - q, J(x_{n+1} - q) \rangle \\
\leq (1 - a_n)^2 \|x_n - q\|^2 + 2a_n \|T^x y_n - q\| \cdot \|J(x_{n+1} - q) - J(x_n - q)\| \\
+ 2a_n \|T^x y_n - q\| \cdot \|J(x_n - q) - J(y_n - q)\| \\
+ 2a_n \left( \|y_n - q\|^2 - \Phi(\|y_n - q\|) \right) + 2c_n \|u_n - q\| \cdot \|x_{n+1} - q\| \\
\leq (1 - a_n)^2 R^2 + 4a_n L(1 + 2R)e_0 + 2a_n \left( \|y_n - q\|^2 - \Phi(\|y_n - q\|) \right) \\
+ 4c_n MR,
\]

\[
\|y_n - q\|^2 \leq (1 - b_n - d_n)^2 \|x_n - q\|^2 + 2b_n \langle T^x x_n - q, J(y_n - q) \rangle \\
+ 2d_n \langle w_n - q, J(y_n - q) \rangle \\
\leq \|x_n - q\|^2 + 2b_n \langle T^x x_n - q, J(y_n - q) - J(x_n - q) \rangle \\
+ 2b_n \langle T^x x_n - q, J(x_n - q) \rangle + 2d_n \|w_n - q\| \cdot \|y_n - q\| \\
\leq \|x_n - q\|^2 + 2b_n \|T^x - q\| \cdot \|J(y_n - q) - J(x_n - q)\| \\
+ 2b_n \left( \|x_n - q\|^2 - \Phi(\|x_n - q\|) \right) + 2d_n \|w_n - q\| \cdot \|y_n - q\| \\
\leq R^2 + 2b_n L(1 + R)e_0 + 2b_n R^2 + 4d_n MR.
\]

Substitute (2.10) into (2.9)

\[
\|x_{n+1} - q\|^2 \leq (1 - a_n)^2 R^2 + 4a_n L(1 + 2R)e_0 + 2a_n \left[ R^2 + 2b_n L(1 + R)e_0 + 2b_n R^2 + 4d_n MR \right] \\
- 2a_n \Phi(\|y_n - q\|) + 4c_n MR \\
\leq R^2 + a_n^2 R^2 + 4a_n L(1 + 2R)e_0 + 2a_n \left[ 2b_n L(1 + R)e_0 + 2b_n R^2 + 4d_n MR \right] \\
- 2a_n \Phi \left( \frac{R}{4} \right) + 4c_n MR
\]
this is a contradiction. Thus \( x_{n+1} \in B_1 \), that is, \( \{ x_n \} \) is a bounded sequence. So \( \{ y_n \} \), \( \{ T^n y_n \}, \{ T^n x_n \} \) are all bounded sequences. Since \( \| z_n - q \| \to 0 \) as \( n \to \infty \), without loss of generality, we let \( \| z_n - q \| \leq 1 \). Therefore, \( \| x_n - z_n \| \) is also bounded.

**Step 3.** We want to prove \( \| x_n - z_n \| \to 0 \) as \( n \to \infty \).

Set \( M_0 = \max \{ \sup_n \| T^n y_n - T^n z_n \|, \sup_n \| v_n - u_n \|, \sup_n \| x_n - z_n \|, \sup_n \| T^n x_n - x_n \|, \sup_n \| w_n - x_n \|, \sup_n \| y_n - z_n \|, \sup_n \| v_n - x_n \| \} \).

Again using Lemma 1.6, we have

\[
\| x_{n+1} - z_{n+1} \|^2 \leq (1 - a_n - c_n)^2 \| x_n - z_n \|^2 + 2a_n \langle T^n y_n - T^n z_n, J(x_{n+1} - z_{n+1}) \rangle \\
+ 2c_n \langle v_n - u_n, J(x_{n+1} - z_{n+1}) \rangle \\
\leq (1 - a_n)^2 \| x_n - z_n \|^2 + 2a_n \langle T^n y_n - T^n z_n, J(x_{n+1} - z_{n+1}) - J(x_n - z_n) \rangle \\
+ 2a_n \langle T^n y_n - T^n z_n, J(x_n - z_n) \rangle \\
+ 2a_n \langle T^n y_n - T^n z_n, J(y_n - z_n) \rangle + 2c_n \| v_n - u_n \| \cdot \| x_{n+1} - z_{n+1} \| \\
\leq (1 - a_n)^2 \| x_n - z_n \|^2 + 2a_n M_0 A_n + 2a_n M_0 B_n \\
+ 2a_n \left[ \| y_n - z_n \|^2 - \Phi(\| y_n - z_n \|) \right] + 2c_n M_0^2
\]

(2.12)

\[
\| y_n - z_n \|^2 \leq \| x_n - z_n \|^2 + 2b_n \langle T^n x_n - x_n, J(y_n - z_n) \rangle \\
+ 2d_n \langle w_n - x_n, J(y_n - z_n) \rangle \\
\leq \| x_n - z_n \|^2 + 2b_n M_0^2 + 2d_n M_0^2
\]

(2.13)

where \( A_n = \| J(x_{n+1} - z_{n+1}) - J(x_n - z_n) \|, B_n = \| J(x_n - z_n) - J(y_n - z_n) \|, \) and \( A_n, B_n \to 0 \) as \( n \to \infty \).
\[\|x_{n+1} - z_{n+1}\|^2 \leq (1 - a_n)^2\|x_n - z_n\|^2 + 2a_n M_0 A_n + 2a_n M_0 B_n \]
\[+ 2a_n \left[\|x_n - z_n\|^2 + 2b_n M_0^2 + 2a_n M_0^2 - \Phi(\|y_n - z_n\|)\right] + 2c_n M_0^2 \]
\[\leq \|x_n - z_n\|^2 + 2a_n^2 M_0^2 + 2a_n M_0 A_n + 2a_n M_0 B_n + 4a_n b_n M_0^2 + 4a_n d_n M_0^2 \tag{2.14} \]
\[- 2a_n \Phi(\|y_n - z_n\|) + 2c_n M_0^2 \]
\[= \|x_n - z_n\|^2 + 2a_n \left[C_n - 2a_n \Phi(\|y_n - z_n\|)\right], \]
where \(C_n = a_n M_0^2 / 2 + M_0 A_n + M_0 B_n + 2b_n M_0^2 + 2a_n M_0^2 + c_n M_0^2 / a_n \to 0 \) as \(n \to \infty\).

Set \(\inf_{n \geq 0} \Phi(\|y_n - z_n\|)(1 + \|x_{n+1} - z_{n+1}\|^2) = \lambda\), then \(\lambda = 0\). If it is not the case, we assume that \(\lambda > 0\). Let \(0 < \gamma < \min\{1, \lambda\}\), then \(\Phi(\|y_n - z_n\|)(1 + \|x_{n+1} - z_{n+1}\|^2) \geq \gamma\), that is, \(\Phi(\|y_n - z_n\|) \geq \gamma \|x_{n+1} - z_{n+1}\|^2 \geq \gamma \|x_n - z_n\|^2\). Thus, from (2.14) that
\[\|x_{n+1} - z_{n+1}\|^2 \leq \|x_n - z_n\|^2 + 2a_n \left(C_n - \gamma \|x_{n+1} - z_{n+1}\|^2\right), \tag{2.15} \]
which implies that
\[\|x_{n+1} - z_{n+1}\|^2 \leq \frac{1}{1 + 2a_n \gamma} \|x_n - z_n\|^2 + \frac{2a_n C_n}{1 + 2a_n \gamma} \]
\[= \left(1 - \frac{2a_n \gamma}{1 + 2a_n \gamma}\right) \|x_n - z_n\|^2 + \frac{2a_n C_n}{1 + 2a_n \gamma}. \tag{2.16} \]
Let \(\rho_n = \|x_n - z_n\|^2\), \(\lambda_n = 2a_n \gamma / (1 + 2a_n \gamma)\), \(\sigma_n = 2a_n C_n / (1 + 2a_n \gamma)\). Then we get that
\[\rho_{n+1} \leq (1 - \lambda_n) \rho_n + \sigma_n. \tag{2.17} \]

Applying Lemma 1.7, we get that \(\rho_n \to 0\) as \(n \to \infty\). This is a contradiction and so \(\lambda = 0\). Therefore, there exists an infinite subsequence such that \(\Phi(\|y_n - z_n\|)(1 + \|x_{n+1} - z_{n+1}\|^2) \to 0\) as \(i \to \infty\). Since \(0 \leq \Phi(\|y_n - z_n\|) \leq 2\Phi(\|y_n - z_n\|)(1 + \|x_{n+1} - z_{n+1}\|^2)\), then \(\Phi(\|y_n - z_n\|) \to 0\) as \(i \to \infty\). In view of the strictly increasing and continuity of \(\Phi\), we have \(\|y_n - z_n\| \to 0\) as \(i \to \infty\). From (1.7), we have
\[\|x_n - z_n\| \leq \|y_n - z_n\| + b_n \|x_n - T x_n\| + c_n \|x_n - w_n\| \to 0, \tag{2.18} \]
as \(i \to \infty\). Next we want to prove \(\|x_n - z_n\| \to 0\) as \(n \to \infty\). Let for all \(\varepsilon \in (0, 1)\), there exists \(n_i \) such that \(\|x_n - z_n\| < \varepsilon, a_n, a_n < \min\{\varepsilon / 4L(1 + M_0), \varepsilon / 8M_0\}, c_n, c_n < \varepsilon / 16M_0, b_n, d_n, b_n, d_n < \varepsilon / 8M_0, C_n, C_n < \Phi(\varepsilon / 4) / 2\), for any \(n, n \geq n_i\). First, we want to prove \(\|x_{n+1} - z_{n+1}\| < \varepsilon\).
Suppose it is not this case, then \(\|x_{n+1} - z_{n+1}\| \geq \epsilon\). Using (1.7), we may get the following estimates:

\[
\|x_n - z_n\| \geq \|x_{n+1} - z_{n+1}\| - a_n \|T^ny_n - T^nz_n\| - a_n \|x_n - z_n\|
- \ c_n \|v_n - u_n\| - c_n \|x_n - z_n\|
\geq \epsilon - a_n L(1 + M_0) - (a_n + 2c_n) M_0
\geq \frac{\epsilon}{2},
\]

\[
\|y_n - z_n\| \geq \|x_n - z_n\| - b_n \|T^nx_n - x_n\| - d_n \|v_n - x_n\|
\geq \frac{\epsilon}{2} - (b_n + d_n) M_0
\geq \frac{\epsilon}{4}.
\]

Since \(\Phi\) is strictly increasing, then (2.20) leads to \(\Phi(\|y_n - z_n\|) \geq \Phi(\epsilon/4)\). From (2.14), we have

\[
\|x_{n+1} - z_{n+1}\|^2 \leq \|x_n - z_n\|^2 + 2a_n \left[C_n - \Phi(\|y_n - z_n\|)\right]
\leq \epsilon^2 + 2a_n \left[\frac{1}{2} \Phi\left(\frac{\epsilon}{4}\right) - \Phi\left(\frac{\epsilon}{4}\right)\right]
\leq \epsilon^2
\leq \epsilon^2,
\]

is a contradiction. Hence, \(\|x_{n+1} - z_{n+1}\| < \epsilon\). Suppose that \(\|x_{n+m} - z_{n+m}\| < \epsilon\) holds. Repeating the above course, we can easily prove that \(\|x_{n+m+1} - z_{n+m+1}\| < \epsilon\) holds. Therefore, for any \(m\) and \(n_i \geq n_0\), we obtain that \(\|x_{n+m} - z_{n+m}\| < \epsilon\), which means \(\|x_n - z_n\| \to 0\) as \(n \to \infty\). This completes the proof.

In order to make the existence of Theorem 2.2 more meaningful, we give the following theorem.

**Theorem 2.3.** Let \(E\) be an arbitrary uniformly smooth real Banach space, let \(D\) be a nonempty closed convex subset of \(E\), and let \(T : D \to D\) be a uniformly generalized Lipschitz generalized asymptotically \(\Phi\)-strongly pseudocontractive mapping with \(q \in F(T) \neq \emptyset\). Let \(\{a_n\}, \{c_n\}\) be two real sequences in \([0, 1]\) and satisfy the conditions (i) \(a_n + c_n \leq 1\); (ii) \(a_n \to 0\) as \(n \to \infty\) and \(c_n = o(a_n)\); (iii) \(\sum_{n=0}^{\infty} a_n = \infty\). For some \(z_0 \in D\), let \(\{u_n\}\) be any bounded sequence in \(D\) and let \(\{z_n\}\) be modified Mann iterative sequence with errors defined by (1.8). Then \(\{z_n\}\) converges strongly to the unique fixed point \(q\) of \(T\).
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**Proof.** Since $T : D \to D$ is a uniformly generalized Lipschitz generalized asymptotically $\Phi$-strongly pseudocontractive mapping, then there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

\[
\langle (k_n I - T^n)x - (k_n I - T^n)y, J(x - y) \rangle \geq \Phi(\|x - y\|),
\]

\[
\|T^n x - T^n y\| \leq L(1 + \|x - y\|),
\]

for any $x, y \in D$.

**Step 1.** There exists $z_0 \in D$ and $z_0 \notin Tz_0$ such that $r_0 = (k + L)\|z_0 - q\|^2 + L\|z_0 - q\| \in R(\Phi)$, where $k = \sup_n \|k_n\|$. In fact, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_1 \in R(\Phi)$; if $\sup \{\Phi(r) : r \in [0, +\infty)\} = \min_{r \in [0, +\infty)} \{\Phi(r)\} < +\infty$, then, for $r < r_1$, there exists a sequence $\{n_k\}$ in $D$ such that $\phi_n \to q$ as $n \to +\infty$ with $\phi_n \neq q$. Furthermore, there exists a natural number $n_0$ such that $(k + L)\|\phi_n - q\|^2 + L\|\phi_n - q\| < (r_1/2)$ for $n \geq n_0$, then we redefine $z_0, r_0$ such that $\phi_0 = \phi_n, r_0 = (k + L)\|\phi_0 - q\|^2 + L\|\phi_0 - q\| \in R(\Phi)$.

**Step 2.** For any $n \geq 0$, $\{z_n\}$ is bounded.

Set $r = \Phi^{-1}(r_0)$, we have $\|x_0 - q\| \leq R$. Let $B'_1 = \{z \in D : \|z - q\| \leq r\}$, $B'_2 = \{z \in D : \|z - q\| \leq 2r\}$, $M' = \sup_n \{\|u_n - q\|\}$. Next, we prove that $z_n \in B'_1$ for any $n \geq 0$ by induction. First $z_0 \in B'_1$ is obvious. Suppose that $z_n \in B'_1$ holds. We prove that $z_{n+1} \in B'_1$. If it is not the case, then $\|z_{n+1} - q\| > r$. By uniformly continuity of $J$ on bounded subset, we choose $\varepsilon_0 = \Phi(r/2)/16L(1 + 2r)$, there exists $\delta > 0$ such that $\|Jx - Jy\| < \varepsilon_0$ when $\|x - y\| < \delta$, for all $x, y \in B'_2$. Now denote

\[
\tau_0 = \min \left\{ \frac{r}{2[L(1 + r) + 2r + M']}, \frac{\Phi(r/2)}{2L(1 + r) + 2r + M'}, \frac{\Phi(r/2)}{8r^2}, \frac{\Phi(r/2)}{24L(1 + 2r)} \right\}. \tag{2.24}
\]

Since $a_n, c_n, k_n - 1 \to 0$ as $n \to +\infty$, and $c_n = o(a_n)$, without loss of generality, let $0 \leq a_n, c_n, k_n - 1 \leq \tau_0, c_n < a_n r_0$ for any $n \geq 0$. Then we have the following estimates from (1.8):

\[
\|z_n - T^n z_n\| \leq \|z_n - q\| + \|T^n z_n - q\|
\]

\[
\leq r + L(1 + r),
\]

\[
\|z_n - q\| \geq \|z_{n+1} - q\| - a_n \|T^n z_n - z_n\| - c_n \|u_n - z_n\|
\]

\[
> r - a_n \left[r + L(1 + r)\right] - c_n \left(r + M'\right)
\]

\[
\geq r - \tau_0 \left[L(1 + r) + 2r + M'\right]
\]

\[
\geq \frac{r}{2},
\]
\[ \|z_{n+1} - q\| \leq (1 - a_n - c_n) \|z_n - q\| + a_n \|T^nz_n - q\| + c_n \|u_n - q\| \]
\[ \leq r + \tau_0 [L(1 + r) + M'] \]
\[ \leq 2r, \]
\[ \|(z_{n+1} - q) - (z_n - q)\| \leq a_n \|T^n z_n - z_n\| + c_n \|u_n - z_n\| \]
\[ \leq a_n [r + L(1 + r)] + c_n (r + M') \]
\[ \leq \tau_0 [L(1 + r) + 2r + M'] \]
\[ \leq \frac{\delta}{2} < \delta. \] (2.25)

Therefore, \( \|J(z_{n+1} - q) - J(z_n - q)\| < \epsilon_0. \)
Using Lemma 1.6 and formulas above, we obtain

\[ \|z_{n+1} - q\|^2 \leq (1 - a_n)^2 \|z_n - q\|^2 + 2a_n \langle T^n z_n - q, J(z_{n+1} - q) - J(z_n - q) \rangle \]
\[ + 2a_n \langle T^n z_n - q, J(z_n - q) \rangle + 2c_n \langle u_n - q, J(z_{n+1} - q) \rangle \]
\[ \leq (1 - a_n)^2 \|z_n - q\|^2 + 2a_n \|T^n z_n - q\| \cdot \|J(z_{n+1} - q) - J(z_n - q)\| \]
\[ + 2a_n [k_n \|z_n - q\|^2 - \Phi(\|z_n - q\|)] + 2c_n \|u_n - q\| \cdot \|z_{n+1} - q\| \]
\[ \leq (1 - a_n)^2 r^2 + 4a_n L(1 + 2r) \epsilon_0 \]
\[ + 2a_n [k_n \|z_n - q\|^2 - \Phi(\|z_n - q\|)] + 4c_n M' r \] (2.26)
\[ \leq (1 - a_n)^2 r^2 + 4a_n L(1 + 2r) \epsilon_0 + 2a_n [k_n r^2 - \Phi\left(\frac{r}{2}\right)] + 4c_n M' r \]
\[ = r^2 + 2a_n \left[ \frac{a_n}{2} r^2 + 2L(1 + 2r) \epsilon_0 + (k_n - 1) r^2 + \frac{2c_n M' r}{a_n} \right] - 2a_n \Phi\left(\frac{r}{2}\right) \]
\[ \leq r^2 + 2a_n \left[ \frac{\Phi(r/2)}{2} - \Phi\left(\frac{r}{2}\right) \right] \]
\[ \leq r^2 - a_n \Phi\left(\frac{r}{2}\right) \]
\[ \leq r^2, \]

this is a contradiction. Thus \( z_{n+1} \in B'_r \), that is, \( \{z_n\} \) is a bounded sequence, so \( \{T^n z_n\} \) is also bounded. Denote \( M_0 = \sup_n \|z_n - q\| + \sup_n \|T^n z_n - q\| + \sup_n \|u_n - q\| \).

Step 3. We prove \( \|z_n - q\| \to 0 \) as \( n \to \infty. \)
Again using Lemma 1.6, we have

\[
\|z_{n+1} - q\|^2 \leq (1 - a_n - c_n)^2 \|z_n - q\|^2 + 2a_n \langle T^n z_n - q, J(z_{n+1} - q) \rangle \\
+ 2c_n \langle u_n - q, J(z_{n+1} - q) \rangle \\
\leq (1 - a_n)^2 \|z_n - q\|^2 + 2a_n \langle T^n z_n - q, J(z_{n+1} - q) - J(z_n - q) \rangle \\
+ 2c_n \langle u_n - q, \|z_{n+1} - q\| : \|z_n - q\| \rangle \\
\leq (1 - a_n)^2 \|z_n - q\|^2 + 2a_n M_0 D_n \\
+ 2a_n \left[ k_n \|z_n - q\|^2 - \Phi(\|z_n - q\|) \right] + 2c_n M_0^2 \\
\leq \|z_n - q\|^2 + 2a_n \left[ (k_n - 1) M_0^2 + \frac{a_n M_0^2}{2} + M_0 D_n + \frac{c_n M_0^2}{a_n} - \Phi(\|z_n - q\|) \right] \\
\leq \|z_n - q\|^2 + 2a_n \left[ E_n - \Phi(\|z_n - q\|) \right],
\]

where

\[
D_n = \|J(z_{n+1} - q) - J(z_n - q)\|, \quad E_n = (k_n - 1) M_0^2 + \frac{a_n M_0^2}{2} + M_0 D_n + \frac{c_n M_0^2}{a_n},
\]

and \(D_n, E_n \to 0\) as \(n \to \infty\).

Set \(\inf_{n \geq 0} \Phi(\|z_n - q\|)/(1 + \|z_n - q\|^2) = \lambda\), then \(\lambda = 0\). If it is not the case, we assume that \(\lambda > 0\). Let \(0 < \gamma < \min\{1, \lambda\}\), then \(\Phi(\|z_n - q\|)/(1 + \|z_n - q\|^2) \geq \gamma\), that is, \(\Phi(\|z_n - q\|) \geq \gamma + \gamma \|z_{n+1} - q\|^2 \geq \gamma \|z_{n+1} - q\|^2\). Thus, from (2.14) that

\[
\|z_{n+1} - q\|^2 \leq \|z_n - q\|^2 + 2a_n \left( E_n - \gamma \|z_{n+1} - q\|^2 \right),
\]

which implies that

\[
\|z_{n+1} - q\|^2 \leq \frac{1}{1 + 2a_n \gamma} \|z_n - q\|^2 + \frac{2a_n E_n}{1 + 2a_n \gamma} \\
= \left( 1 - \frac{2a_n \gamma}{1 + 2a_n \gamma} \right) \|z_n - q\|^2 + \frac{2a_n E_n}{1 + 2a_n \gamma}.
\]

Let \(\rho_n = \|z_n - q\|^2, \lambda_n = 2a_n \gamma/(1 + 2a_n \gamma), \sigma_n = 2a_n E_n/(1 + 2a_n \gamma)\). Then we get that

\[
\rho_{n+1} \leq (1 - \lambda_n) \rho_n + \sigma_n.
\]

Applying Lemma 1.7, we get that \(\rho_n \to 0\) as \(n \to \infty\). This is a contradiction and so \(\lambda = 0\). Therefore, there exists an infinite subsequence such that \(\Phi(\|z_n - q\|)/(1 + \|z_n - q\|^2) \to 0\) as \(i \to \infty\). Since \(0 \leq \Phi(\|z_n - q\|)/(1 + M_0^2) \leq \Phi(\|z_n - q\|)/(1 + \|z_n - q\|^2)\), then \(\Phi(\|z_n - q\|) \to 0\)
as \( i \to \infty \). In view of the strictly increasing and continuity of \( \Phi \), we have \( \|z_n - q\| \to 0 \) as \( i \to \infty \). Let \( \varepsilon \in (0,1) \) be any given, there exists \( n_i \) such that \( \|z_n - q\| < \varepsilon, a_n, a_n < \min\{\varepsilon/4L(1 + M_0), \varepsilon/8M_0\} \), \( c_n, c_n < \varepsilon/16M_0 \), \( E_n, E_n < \Phi(\varepsilon/2)/2 \), for any \( n_i, n \geq n_i \). First, we want to prove \( \|z_{n+1} - q\| < \varepsilon \). Suppose it is not this case, then \( \|z_{n+1} - q\| \geq \varepsilon \). Using (1.8), we may get the following estimates:

\[
\|z_n - q\| \geq \|z_{n+1} - q\| - a_n\|T^n z_n - q\| - a_n\|z_n - q\| - c_n\|u_n - q\|
\]

\[
geq \varepsilon - a_nL(1 + M_0) - (a_n + 2c_n)M_0
\]

\[
> \frac{\varepsilon}{2}.
\]

Since \( \Phi \) is strictly increasing, then (2.32) leads to \( \Phi(\|z_n - q\|) \geq \Phi(\varepsilon/2) \). From (2.27), we have

\[
\|z_{n+1} - q\|^2 \leq \|z_n - q\|^2 + 2a_n[E_n - \Phi(\|z_n - q\|)]
\]

\[
< \varepsilon^2 + 2a_n\left[\frac{1}{2}\Phi\left(\frac{\varepsilon}{2}\right) - \Phi\left(\frac{\varepsilon}{2}\right)\right]
\]

\[
\leq \varepsilon^2 - \Phi\left(\frac{\varepsilon}{2}\right)a_n
\]

\[
\leq \varepsilon^2,
\]

is a contradiction. Hence, \( \|z_{n+1} - q\| < \varepsilon \). Suppose that \( \|z_{n+1} - q\| < \varepsilon \) holds. Repeating the above course, we can easily prove that \( \|z_{n+m+1} - q\| < \varepsilon \) holds. Therefore, for any \( m \) and \( n_i \geq n_i \), we obtain that \( \|z_{n+m} - q\| < \varepsilon \), which means \( \|z_n - q\| \to 0 \) as \( n \to \infty \). This completes the proof. \( \square \)

**Theorem 2.4.** Let \( E \) be an arbitrary uniformly smooth real Banach space, let \( D \) be a nonempty closed convex subset of \( E \), and let \( T : D \to D \) be a uniformly generalized Lipschitz generalized asymptotically \( \Phi \)-strongly pseudocontractive mapping with \( q \in F(T) \neq \emptyset \). Let \( \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \) be four real sequences in \([0,1]\) and satisfy the conditions (i) \( a_n + c_n \leq 1, b_n + d_n \leq 1 \); (ii) \( a_n, b_n, d_n \to 0 \) as \( n \to \infty \) and \( c_n = o(a_n) \); (iii) \( \sum_{m=0}^{\infty} c_m = \infty \). For some \( x_0 \in D \), let \( \{\nu_n\}, \{\omega_n\} \) be two arbitrary bounded sequences in \( D \), and let \( \{x_n\} \) be Ishikawa iterative sequence with errors defined by (1.7). Then (1.7) converges strongly to the unique fixed point \( q \) of \( T \).

**Proof.** By Theorems 2.3 and 2.2, we obtain directly the result of Theorem 2.4. \( \square \)

**Remark 2.5.** Our Theorem 2.2 extends and improves Theorem 3.1 of [4] from the bounded range of \( T \) to uniformly generalized Lipschitz mapping, and the proof course of Theorem 2.2 is quite different from that of [4].

**References**


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