**Research Article**

**Strong Convergence Theorems for Maximal Monotone Operators with Nonsparing Mappings in a Hilbert Space**

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We prove the strong convergence theorems for finding a common element of the set of fixed points of a nonspreading mapping $T$ and the solution sets of zero of a maximal monotone mapping and an $\alpha$-inverse strongly monotone mapping in a Hilbert space. Manaka and Takahashi (2011) proved weak convergence theorems for maximal monotone operators with nonspreading mappings in a Hilbert space; there we introduced new iterative algorithms and got some strong convergence theorems for maximal monotone operators with nonspreading mappings in a Hilbert space.

1. **Introduction**

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $C$ be a nonempty closed convex subset of $H$. We denote by $F(T)$ the set of fixed point of $T$. Then, a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The mapping $T : C \to C$ is said to be firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$ for all $x, y \in C$; see, for instance, Browder [1] and Goebel and Kirk [2]. The mapping $T : C \to C$ is said to be firmly nonspreading [3] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2,$$

(1.1)

for all $x, y \in C$. Iemoto and Takahashi [4] proved that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle,$$

(1.2)
for all $x, y \in C$. It is not hard to know that a nonspreading mapping is deduced from a firmly nonexpansive mapping; see [5, 6], and a firmly nonexpansive mapping is a nonexpansive mapping.

Many studies have been done for structuring the fixed point of nonexpansive mapping $T$. In 1953, Mann [7] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = a_n x_n + (1 - a_n)Tx_n,$$  \hspace{1cm} (1.3)

where the initial guess $x_1 \in C$ is arbitrary and $\{a_n\}$ is a real sequence in $[0,1]$. It is known that under appropriate settings, the sequence $\{x_n\}$ converges weakly to a fixed point of $T$. However, even in a Hilbert space, Mann iteration may fail to converge strongly, for example see [8].

Some attempts to construct iteration method guaranteeing the strong convergence have been made. For example, Halpern [9] proposed the following so-called Halpern iteration:

$$x_{n+1} = a_n u + (1 - a_n)Tx_n,$$  \hspace{1cm} (1.4)

where $u, x_1 \in C$ are arbitrary and $\{a_n\}$ is a real sequence in $[0,1]$ which satisfies $a_n \to 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$; see [9, 10].

In 1975, Baillon [11] first introduced the nonlinear ergodic theorem in Hilbert space as follows:

$$S_n x = \sum_{k=0}^{n-1} T^k x$$  \hspace{1cm} (1.5)

converges weakly to a fixed point of $T$ for some $x \in C$.

Recently, in the case when $T : C \rightarrow C$ is a nonexpansive mapping, $A : C \rightarrow H$ is an $\alpha$-inverse strongly monotone mapping, and $B \in H \times H$ is a maximal monotone operator, Takahashi et al. [12] proved a strong convergence theorem for finding a point of $F(T) \cap (A + B)^{-1}(0)$, where $F(T)$ is the set of fixed points of $T$ and $(A + B)^{-1}(0)$ is the set of zero points of $A + B$.

In 2011, Manaka and Takahashi [13] for finding a point of the set of fixed points of $T$ and the set of zero points of $A + B$ in a Hilbert space, they introduced an iterative scheme as follows:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T(J_{\lambda_n}(I - \lambda_n A)x_n),$$  \hspace{1cm} (1.6)

where $T$ is a nonspreading mapping, $A$ is an $\alpha$-inverse strongly monotone mapping, and $B$ is a maximal monotone operator such that $J_\lambda = (I - \lambda B)^{-1}$; $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences which satisfy $0 < c \leq \beta_n \leq d < 1$ and $0 < \alpha \leq \lambda_n \leq b < 2\alpha$. Then they proved that $\{x_n\}$ converges weakly to a point $p = \lim_{n \to \infty} P_{F(T) \cap (A + B)^{-1}(0)}x_n$.

Motivated by above authors, we generalize and modify the iterative algorithms (1.5) and (1.6) for finding a common element of the set of fixed points of a nonspreading mapping $T$ and the set of zero points of monotone operator $A + B$ ($A$ is an $\alpha$-inverse strongly monotone
mapping, and $B$ is a maximal monotone operator). First, we prove that the sequence generated by our iterative method is weak convergence under the property conditions. Then, we prove that the strong convergence in a Hilbert space. As expected, we get some weak and strong convergence theorems about the common element of the set of fixed points of a nonsplaying mapping and the set of zero points of an $\alpha$-inverse strongly monotone mapping and a maximal monotone operator in a Hilbert space.

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $C$ be a nonempty closed convex subset of $H$. A set-valued mapping $B : D(B) \subseteq H \to H$ is said to be monotone if for any $x, y \in D(B)$ and $x^* \in Bx$ and $y^* \in By$, it holds that

$$\langle x - y, x^* - y^* \rangle \geq 0.$$  

A monotone operator $B$ on $H$ is said to be maximal if $B$ has no monotone extension, that is, its graph is not properly contained in the graph of any other monotone operator on $H$. For a maximal monotone operator $B$ on $H$ and $r > 0$, we may define a single-valued operator

$$J_r = (I + rB)^{-1} : H \to D(B),$$

which is called the resolvent of $B$ for $r > 0$. Let $B$ be a maximal monotone operator on $H$, and let $B^{-1}(0) = \{ x \in H : 0 \in Bx \}$. For a constant $\alpha > 0$, the mapping $A : C \to H$ is said to be an $\alpha$-inverse strongly monotone if for any $x, y \in C$,

$$\langle x - y, Ax - Ay \rangle \geq \alpha \| Ax - Ay \|^2.$$  

Remark 2.1. It is not hard to know that if $A$ is an $\alpha$-inverse strongly monotone mapping, then it is $1/\alpha$-Lipschitzian and hence uniformly continuous. Clearly, the class of monotone mappings include the class of an $\alpha$-inverse strongly monotone mappings.

Remark 2.2. It is well known that if $T : C \to C$ is a nonexpansive mapping, then $I - T$ is $1/2$-inverse strongly monotone, where $I$ is the identity mapping on $H$; see, for instance, [14]. It is known that the resolvent $J_r$ is firmly nonexpansive and $B^{-1}(0) = F(J_r)$ for all $r > 0$.

For a single-valued mapping $T$, a point $p$ is called a fixed point of $T$ if $p = Tp$. For a multivalued mapping $T$, a point $p$ is called a fixed point of $T$ if $p \in Tp$. The set of fixed points of $T$ is denoted by $F(T)$.

Let $E$ be a uniformly convex real Banach space, $K$ be a nonempty closed convex subset of $E$. A multivalued mapping $T : K \to CB(K)$ is said to be as follows.

(i) Contraction if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k \| x - y \|, \quad \forall x, y \in K. \quad (2.3)$$

(ii) Nonexpansive if

$$H(Tx, Ty) \leq \| x - y \|, \quad \forall x, y \in K. \quad (2.4)$$
(iii) Quasinonexpansive if \( F(T) \neq \emptyset \) and

\[
H(Tx, Tp) \leq \|x - p\|, \quad \forall x \in K, \forall p \in F(T).
\]

(2.5)

It is well known that every nonexpansive multivalued mapping \( T \) with \( F(T) \neq \emptyset \) is multivalued quasinonexpansive. But there exist multivalued quasi-nonexpansive mappings that are not multivalued nonexpansive. It is clear that if \( T \) is a quasi-nonexpansive multivalued mapping, then \( F(T) \) is closed.

A Banach space \( E \) is said to satisfy Opial’s condition if whenever \( \{x_n\} \) is a sequence in \( E \) which converges weakly to \( x \), then

\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|y_n - y\|, \quad \forall y \in E, \ x \neq y.
\]

(2.6)

**Lemma 2.3** (Manaka and Takahashi [13]). Let \( H \) be a real Hilbert space, and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \alpha > 0 \). Let \( A \) be an \( \alpha \)-inverse strongly monotone mapping of \( C \) into \( H \), and let \( B \) be a maximal monotone operator on \( H \) such that the domain of \( B \) is included in \( C \). Let \( J_\lambda = (I + \lambda B)^{-1} \) be the resolvent of \( B \) for any \( \lambda > 0 \). Then, the following hold

(i) if \( u, v \in (A + B)^{-1}(0) \), then \( Au = Av \);

(ii) for any \( \lambda > 0 \), \( u \in (A + B)^{-1}(0) \) if and only if \( u = J_\lambda(A - \lambda u)u \).

**Lemma 2.4** (Schu [15]). Suppose that \( E \) is a uniformly convex Banach space and \( 0 < p \leq t_n \leq q < 1 \) for all positive integers \( n \). Also suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences of \( E \) such that \( \limsup_{n \to \infty} \|x_n\| \leq r \), \( \limsup_{n \to \infty} \|y_n\| \leq r \), and \( \lim_{n \to \infty} \|t_n x_n + (1 - t_n)y_n\| = r \) hold for some \( r \geq 0 \). Then, \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

**Lemma 2.5** (Liu [16] and Xu [17]). Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the property as follows

\[
a_{n+1} \leq (1 - t_n)a_n + b_n + t_n c_n,
\]

(2.7)

where \( \{t_n\}, \{b_n\}, \) and \( \{c_n\} \) satisfy the restrictions as follows

(i) \( \sum_{n=0}^{\infty} t_n = \infty \),

(ii) \( \sum_{n=0}^{\infty} b_n < \infty \),

(iii) \( \limsup_{n \to \infty} c_n \leq 0 \).

Then, \( \{a_n\} \) converges to zero as \( n \to \infty \).

### 3. Strong Convergence Theorem

In this section, we prove the strong convergence theorems for finding a common element in common set of the fixed sets of a nonspreading mapping and the solution sets of zero of a maximal monotone operator and an \( \alpha \)-inverse strongly monotone operator and in a Hilbert space.
Theorem 3.1. Let \( C \) be a nonempty convex closed subset of a real Hilbert space \( H \), let \( A : C \to H \) be an \( \alpha \)-inverse strongly monotone, let \( B : D(B) \subseteq C \to 2^H \) be maximal monotone, let \( J_1 = (I + \lambda B)^{-1} \) be the resolvent of \( B \) for any \( \lambda > 0 \), and let \( T : C \to C \) be a nonspreading mapping. Assume that \( F := F(T) \cap (A + B)^{-1}(0) \neq \emptyset \). We define

\[
\begin{align*}
\alpha_n &= \text{arbitrarily}, \\
z_n &= J_{\lambda_n}(I - \lambda_n A)x_n, \\
y_n &= \frac{1}{n} \sum_{k=1}^{n} T^k z_n, \\
x_{n+1} &= \alpha_n u + (1 - \alpha_n)y_n,
\end{align*}
\]

where \( \{\alpha_n\} \) is sequences in \([0, 1]\) such that \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \). There exists \( a, b \) such that \( 0 < a \leq \lambda_n \leq b < 2a \) for each \( n \in \mathbb{N} \). Then, \( \{x_n\} \) converges strongly to \( Pu \), and \( P \) is the metric projection of \( H \) onto \( F \).

Proof. First, we prove that \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} \|x_n - p\| \) exists for each \( p \in F(T) \). In fact, from Lemma 2.3, we have \( p = J_{\alpha_n}(I - \lambda_n A)p \), together with (3.1) and \( A \) is an \( \alpha \)-inverse strongly monotone, we get that

\[
\begin{align*}
\|z_n - p\|^2 &= \|J_{\alpha_n}(I - \lambda_n A)x_n - J_{\alpha_n}(I - \lambda_n A)p\|^2 \\
&\leq \|z_n - p\|^2 - 2\lambda_n \langle x_n - p, Ax_n - Ap \rangle + \lambda_n^2 \|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2\lambda_n \lambda_n \|Ax_n - Ap\|^2 + \lambda_n^2 \|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 - \lambda_n(2\lambda - \lambda_n) \|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2.
\end{align*}
\]

From the definition of \( y_n \) and \( T \) is nonspreading mapping, we obtain that

\[
\begin{align*}
\|y_n - p\| &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k z_n - p \right\| \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k z_n - p\| \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \|z_n - p\| \\
&= \|z_n - p\| \leq \|x_n - p\|.
\end{align*}
\]

Together with (3.1), we have that

\[
\begin{align*}
x_{n+1} - p &= \alpha_n u + (1 - \alpha_n)y_n - p \\
&\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|y_n - p\| \\
&\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|.
\end{align*}
\]
Hence, we get that

\[ \|x_{n+1} - p\| \leq \max\{\|u - p\|, \|x_n - p\|\}, \]  

(3.5)

for all \( n \in \mathbb{N} \). This means that \( \{x_n - p\} \) is bounded, so \( \{x_n\} \) is bounded. From \( T \) is nonspreading, (3.3), and (3.2), we get that \( \{y_n\}, \{z_n\}, \) and \( \{T^n z_n\} \) are all bounded.

Since \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \lim_{k \to \infty} \|x_{n_k} - p\| \) exists. Since \( \{x_{n_k}\} \) is bounded, there exists a subsequence \( \{x_{n_{k_i}}\} \) of \( \{x_{n_k}\} \) such that \( x_{n_{k_i}} \to w \in C \) as \( i \to \infty \). Now, we prove that \( w \in F(T) \). Since \( \|x_{n+1} - y_n\| = \alpha_n\|u - y_n\| \), replacing \( n \) by \( n_{k_i} \), we have \( \|x_{n_{k+1}} - y_{n_{k_i}}\| = \alpha_{n_{k_i}}\|u - y_{n_{k_i}}\| \). Together with \( \alpha_{n_i} \to 0 \) and \( \{y_n\} \) is bounded, we obtain that \( \lim_{i \to \infty} \|x_{n_{k+1}} - y_{n_{k_i}}\| = 0 \), so we have \( y_{n_{k_i}} \to w \).

Let \( n \in \mathbb{N} \). Since \( T \) is nonspreading, we have that for all \( y \in C \) and \( k = 0, 1, 2, \ldots, n - 1 \),

\[
\left\| T^{k+1}z_n - Ty \right\|^2 \leq \left\| T^k z_n - y \right\|^2 + 2 \left\langle T^k z_n - T^{k+1}z_n, y - Ty \right\rangle
\]

(3.6)

\[
= \left\| T^k z_n - Ty \right\|^2 + \left\| Ty - y \right\|^2 + 2 \left\langle T^k z_n - Ty, Ty - y \right\rangle
\]

Summing these inequalities from \( k = 0 \) to \( n - 1 \) and dividing by \( n \), we have

\[
\frac{1}{n} \left( \left\| T^n z_n - Ty \right\|^2 - \left\| z_n - Ty \right\|^2 \right) \leq \left\| Ty - y \right\|^2 + 2 \left\langle y_n - Ty, Ty - y \right\rangle + \frac{2}{n} \left\langle z_n - T^n z_n, y - Ty \right\rangle.
\]  

(3.7)

Replacing \( n \) by \( n_{k_i} \), we have

\[
\frac{1}{n_{k_i}} \left( \left\| T^{n_{k_i}} z_{n_{k_i}} - Ty \right\|^2 - \left\| z_{n_{k_i}} - Ty \right\|^2 \right)
\leq \left\| Ty - y \right\|^2 + 2 \left\langle y_{n_{k_i}} - Ty, Ty - y \right\rangle
\]  

(3.8)

\[
+ \frac{2}{n_{k_i}} \left\langle z_{n_{k_i}} - T^{n_{k_i}} z_{n_{k_i}}, y - Ty \right\rangle.
\]

Since \( \{z_n\} \) and \( \{T^n z_n\} \) are bounded, we have that

\[
0 \leq \left\| Ty - y \right\|^2 + 2 \left\langle w - Ty, Ty - y \right\rangle
\]  

(3.9)

as \( i \to \infty \). Putting \( y = w \), we have

\[
0 \leq \left\| Tw - w \right\|^2 + 2 \left\langle w - Tw, Tw - w \right\rangle = -\left\| Tw - w \right\|^2.
\]  

(3.10)

Hence, \( w \in F(T) \).
Next, we prove that \( w \in (A + B)^{-1}(0) \). From (3.2) and (3.3) we have that

\[
\|x_{n+1} - p\|^2 \leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \|y_n - p\|^2 \\
\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 \\
\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \left( \|x_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax_n - Ap\|^2 \right) \\
= \alpha_n \left( \|u - p\|^2 - \|x_n - p\|^2 \right) + \|x_n - p\|^2 - (1 - \alpha_n)\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ap\|^2.
\]

(3.11)

We rewrite above inequality as follows:

\[
(1 - \alpha_n)\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ap\|^2 \leq \alpha_n \left( \|u - p\|^2 - \|x_n - p\|^2 \right) + \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
\]

(3.12)

Replacing \( n \) by \( n_k \), we have

\[
(1 - \alpha_{n_k})\lambda_{n_k}(2\alpha - \lambda_{n_k})\|Ax_{n_k} - Ap\|^2 \\
\leq \alpha_{n_k} \left( \|u - p\|^2 - \|x_{n_k} - p\|^2 \right) \\
+ \|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2.
\]

(3.13)

Together with \( \lim_{n \to \infty} \alpha_n = 0 \), \( 0 < a \leq \lambda_n \leq b < 2\alpha \) and since \( \lim_{k \to \infty} \|x_{n_k} - p\| \) exists, we obtain that

\[
\lim_{k \to \infty} \|Ax_{n_k} - Ap\| = 0.
\]

(3.14)

Since \( J_{\lambda_n} \) is firmly nonexpansive, and from (3.2), we have that

\[
\|z_n - p\|^2 = \|J_{\lambda_n}(I - \lambda_n A)x_n - J_{\lambda_n}(I - \lambda_n A)p\|^2 \\
\leq \langle z_n - p, (I - \lambda_n A)x_n - (I - \lambda_n A)p \rangle \\
= \frac{1}{2} \left\{ \|z_n - p\|^2 + \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\|^2 \\
- \|z_n - p - (I - \lambda_n A)x_n + (I - \lambda_n A)p\|^2 \right\} \\
\leq \frac{1}{2} \left\{ \|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - p - (I - \lambda_n A)x_n + (I - \lambda_n A)p\|^2 \right\} \\
= \frac{1}{2} \left\{ \|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\lambda_n \langle z_n - x_n, Ax_n - Ap \rangle - \lambda_n^2\|Ax_n - Ap\|^2 \right\}.
\]

(3.15)
This means that
\[
\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\lambda_n \langle z_n - x_n , A x_n - A p \rangle - \lambda_n^2 \| A x_n - A p \|^2.
\] (3.16)

Together with (3.1) and (3.3), we have
\[
\|x_{n+1} - p\|^2 \leq \alpha_n \| u - p \|^2 + (1 - \alpha_n) \| y_n - p \|^2
\leq \alpha_n \| u - p \|^2 + (1 - \alpha_n) \| z_n - p \|^2
\leq \alpha_n \| u - p \|^2 + (1 - \alpha_n) \times \left\{ \| x_n - p \|^2 - \| z_n - x_n \|^2 - 2\lambda_n \langle z_n - x_n , A x_n - A p \rangle - \lambda_n^2 \| A x_n - A p \|^2 \right\}
\leq \alpha_n \| u - p \|^2 + \| x_n - p \|^2 - \| z_n - x_n \|^2
- 2\lambda_n \langle z_n - x_n , A x_n - A p \rangle - \lambda_n^2 \| A x_n - A p \|^2.
\] (3.17)

Therefore, we have
\[
\|z_n - x_n\|^2 \leq \alpha_n \| u - p \|^2 + \| x_n - p \|^2 - \| x_{n+1} - p \|^2
- 2\lambda_n \langle z_n - x_n , A x_n - A p \rangle - \lambda_n^2 \| A x_n - A p \|^2.
\] (3.18)

Replacing \( n \) by \( n_k \), we have
\[
\|z_{n_k} - x_{n_k}\|^2 \leq \alpha_{n_k} \| u - p \|^2 + \| x_{n_k} - p \|^2 - \| x_{n_{k+1}} - p \|^2
- 2\lambda_{n_k} \langle z_{n_k} - x_{n_k} , A x_{n_k} - A p \rangle - \lambda_{n_k}^2 \| A x_{n_k} - A p \|^2.
\] (3.19)

Since \( \lim_{k \to \infty} \| x_{n_k} - p \| \) exists, from (3.14) and \( \lim_{n \to \infty} \alpha_n = 0 \), we obtain
\[
\lim_{n \to \infty} \| z_{n_k} - x_{n_k} \| = 0.
\] (3.20)

Since \( A \) is Lipschitz continuous, we also obtain
\[
\lim_{n \to \infty} \| A z_{n_k} - A x_{n_k} \| = 0.
\] (3.21)
Abstract and Applied Analysis

By the definition of $J_{\lambda_n}$ and (3.1), we have that

$$z_n = (I - \lambda_n B)^{-1}(I - \lambda_n A)x_n$$

$$\iff (I - \lambda_n A)x_n \in (I - \lambda_n B)z_n = z_n + \lambda_n Bz_n$$

$$\iff x_n - z_n - \lambda_n Ax_n \in \lambda_n Bz_n$$

$$\iff \frac{1}{\lambda_n}(x_n - z_n - \lambda_n Ax_n) \in Bz_n. \tag{3.22}$$

Since $B$ is monotone, so for $(e, f) \in B$, we have that

$$\left\langle z_n - e, \frac{1}{\lambda_n}(x_n - z_n - \lambda_n Ax_n) - f \right\rangle \geq 0, \tag{3.23}$$

and hence

$$\left\langle z_n - e, x_n - z_n - \lambda_n (Ax_n + f) \right\rangle \geq 0. \tag{3.24}$$

Replacing $n$ by $n_k$, we have that

$$\left\langle z_{n_k} - e, x_{n_k} - z_{n_k} - \lambda_{n_k} (Ax_{n_k} + f) \right\rangle \geq 0. \tag{3.25}$$

Since $A$ is an $\alpha$-inverse strongly monotone, we have

$$\left\langle x_{n_k} - w, Ax_{n_k} - Aw \right\rangle \geq \alpha \|Ax_{n_k} - Aw\|^2. \tag{3.26}$$

This means that $Ax_{n_k} \to Aw$ as $i \to \infty$. From (3.20) and $x_{n_k} \to w$, we get that $z_{n_k} \to w$, together with (3.25), we have that

$$\left\langle w - e, -Aw - f \right\rangle \geq 0. \tag{3.27}$$

Since $B$ is maximal monotone, so $(-Aw) \in Bw$. That is, $w \in (A + B)^{-1}(0)$.

Now, we prove that $x_n \to Pu$ as $n \to \infty$. Without loss of generality, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n \to \infty} \langle u - Pu, x_{n+1} - Pu \rangle = \lim_{i \to \infty} \langle u - Pu, x_{n_i+1} - Pu \rangle. \tag{3.28}$$

Since $P$ is the metric projection of $H$ onto $F$ and $x_{n_i+1} \to w \in F$, we have

$$\lim_{i \to \infty} \langle u - Pu, x_{n_i+1} - Pu \rangle = \langle u - Pu, w - Pu \rangle \leq 0. \tag{3.29}$$
This implies that
\[
\lim_{n \to \infty} \langle u - Pu, x_{n+1} - Pu \rangle \leq 0. \tag{3.30}
\]

From (2.1), (3.1), and (3.3), we have
\[
\|x_{n+1} - Pu\|^2 = \| (1 - \alpha_n) (y_n - Pu) + \alpha_n (u - Pu) \|^2 \\
\leq (1 - \alpha_n)^2 \| y_n - Pu \|^2 + 2 \alpha_n \langle u - Pu, x_{n+1} - Pu \rangle \\
\leq (1 - \alpha_n) \| x_n - Pu \|^2 + 2 \alpha_n \langle u - Pu, x_{n+1} - Pu \rangle. \tag{3.31}
\]

From Lemma 2.5 and (3.30), we have
\[
\lim_{n \to \infty} \| x_n - Pu \| = 0. \tag{3.32}
\]

This means that \(x_n \to Pu\) as \(n \to \infty\). \(\square\)

References
