Research Article

The Univalence Conditions of Some Integral Operators

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We introduce new integral operators of analytic functions $f$ and $g$ defined in the open unit disk $U$. For these operators, we discuss some univalence conditions.

1. Introduction and Preliminaries

Let $A$ denote the class of all functions of the form

$$ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1) $$

which are analytic in the open unit disk

$$ U = \{ z \in \mathbb{C} : |z| < 1 \} \quad (1.2) $$

and satisfy the following usual normalization condition:

$$ f(0) = f'(0) - 1 = 0. \quad (1.3) $$

Also, let $S$ denote the subclass of $A$ consisting of functions $f$, which are univalent in $U$ (see, for details [1]; see also [2, 3]).

In [4, 5], Pescar gave the following univalence conditions for the functions $f \in A$. 
Theorem 1.1 (see [4]). Let \( \alpha \) be a complex number, \( \Re \alpha > 0 \), and \( c \) a complex number, \( |c| \leq 1 \), \( c \neq -1 \), and \( f(z) = z + \cdots \) a regular function in \( U \). If

\[
|cz^{2\alpha} + (1-|z|^{2\alpha})\frac{zf''(z)}{f'(z)}| \leq 1, \tag{1.4}
\]

for all \( z \in U \), then the function

\[
F_\alpha(z) = \left( \alpha \int_0^z t^{\alpha-1}f'(t)dt \right)^{1/\alpha} = z + \cdots
\]

is regular and univalent in \( U \).

Theorem 1.2 (see [5]). Let \( \alpha \) be a complex number, \( \Re \alpha > 0 \), and \( c \) a complex number, \( |c| \leq 1 \), \( c \neq -1 \), and \( f \in \mathcal{A} \). If

\[
\frac{1-|z|^{2\Re \alpha}}{\Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c| \tag{1.6}
\]

for all \( z \in U \), then for any complex number \( \beta \), \( \Re \beta \geq \Re \alpha \), the function

\[
F_\beta(z) = \left( \beta \int_0^z t^{\beta-1}f'(t)dt \right)^{1/\beta}
\]

is in the class \( \mathcal{S} \).

On the other hand, for the functions \( f \in \mathcal{A} \), Ozaki and Nunokawa [6] proved another univalence condition asserted by Theorem 1.3.

Theorem 1.3 (see [6]). Let \( f \in \mathcal{A} \) satisfy the condition

\[
\left| \frac{z^2f''(z)}{[f(z)]^2} - 1 \right| < 1 \quad (z \in U). \tag{1.8}
\]

Then \( f \) is univalent in \( U \).

In the paper [7], Pescar determined some univalence conditions for the following integral operators.

Theorem 1.4 (see [7]). Let the function \( g \) satisfy (1.8), \( M \) a positive real number fixed, and \( c \) a complex number. If \( \alpha \in [(2M+1)/(2M+2), (2M+1)/2M] \),

\[
|c| \leq 1 - \frac{\alpha - 1}{\alpha} \frac{1}{(2M+1)}, \quad c \neq -1, \quad |g(z)| \leq M \tag{1.9}
\]
for all \( z \in \mathbb{U} \), then the function

\[
G_\alpha(z) = \left( \alpha \int_0^z [g(t)]^{\alpha-1} dt \right)^{1/\alpha} \tag{1.10}
\]

is in the class \( \mathcal{S} \).

**Theorem 1.5** (see [7]). Let \( g \in \mathcal{A} \), \( \alpha \) a real number, \( \alpha \geq 1 \), and \( c \) a complex number, \( |c| \leq 1/\alpha, \ c \neq -1 \). If

\[
\left| \frac{g''(z)}{g'(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \tag{1.11}
\]

then the function

\[
H_\alpha(z) = \left( \alpha \int_0^z [tg'(t)]^{\alpha-1} dt \right)^{1/\alpha} \tag{1.12}
\]

is in the class \( \mathcal{S} \).

**Theorem 1.6** (see [7]). Let \( g \in \mathcal{A} \) satisfies (1.8), \( \alpha \) a complex number, \( M > 1 \) fixed, \( \text{Re} \alpha > 0 \), and \( c \) a complex number, \( |c| < 1 \). If \( |g(z)| \leq M \) for all \( z \in \mathbb{U} \), then for any complex number \( \beta \)

\[
\text{Re} \beta \geq \text{Re} \alpha \geq \frac{2M + 1}{\alpha(1 - |c|)} \tag{1.13}
\]

the function

\[
H_\beta(z) = \left( \beta \int_0^z t^{\beta-1} \left( \frac{g(t)}{t} \right)^{1/\alpha} dt \right)^{1/\beta} \tag{1.14}
\]

is in the class \( \mathcal{S} \).

In this paper, we introduce the following integral operators as follows:

\[
F_1(f, g)(z) = \left( \alpha \int_0^z \left( f(t)e^{g(t)} \right)^{\alpha-1} dt \right)^{1/\alpha} \quad (f, g \in \mathcal{A}; \ \alpha \in \mathbb{C}), \tag{1.15}
\]

\[
G_1(f, g)(z) = \left( \alpha \int_0^z \left( tf'(t)e^{g(t)} \right)^{\alpha-1} dt \right)^{1/\alpha} \quad (f, g \in \mathcal{A}; \ \alpha \in \mathbb{C}), \tag{1.16}
\]

\[
H_1(f, g)(z) = \left( \beta \int_0^z t^{\beta-1} \left( \frac{f(t)}{t} e^{g(t)} \right)^{1/\alpha} dt \right)^{1/\beta} \quad (f, g \in \mathcal{A}; \ \alpha, \beta \in \mathbb{C} - \{0\}). \tag{1.17}
\]

**Remark 1.7.** For \( e^{g(z)} = 1 \) and \( f(z) = g(z) \), the integral operators (1.15), (1.16), and (1.17) would reduce to the integral operators (1.10), (1.12), and (1.14).
In this paper, we generalize the integral operators given by Pescar [7], and we study the univalence conditions for the integral operators defined by (1.15), (1.16), and (1.17).

For this purpose, we need the following result.

Lemma 1.8 (General Schwarz Lemma [8]). Let the function $f$ be regular in the disk $U_R = \{ z \in \mathbb{C} : |z| < R \}$, with $|f(z)| < M$ for fixed $M$. If $f$ has one zero with multiplicity order bigger than $m$ for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m}|z|^m \quad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{\iota \theta} \frac{M}{R^m} z^m,$$

where $\theta$ is constant.

## 2. Main Results

Theorem 2.1. Let $f, g \in \mathcal{A}$, where $g$ satisfies the condition (1.8), $M_1$ and $M_2$ are real positive numbers, and $\alpha$ a complex number, $\text{Re}\alpha > 0$. If

$$\left| \frac{f'(z)}{f(z)} \right| \leq M_1 \quad (z \in U), \quad |g(z)| \leq M_2 \quad (z \in U),$$

$$|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (M_1 + 2M_2^2 + 1), \quad c \in \mathbb{C}, \; c \neq -1,$$

then the integral operator $F_1(f, g)(z)$ defined by (1.15) is in the class $\mathcal{S}$.

Proof. From (1.15), we have

$$F_1(f, g)(z) = \left( \alpha \int_0^z t^{\alpha-1} \left( \frac{f(t)}{t} e^{g(t)} \right)^{a-1} dt \right)^{1/\alpha}.$$  

Let us consider the function

$$h(z) = \int_0^z \left( \frac{f(t)}{t} e^{g(t)} \right)^{a-1} dt.$$
The function $h$ is regular in $U$. From (2.4), we get

$$h'(z) = \left( \frac{f(z)}{z} e^{g(z)} \right)^{a-1},$$

$$h''(z) = (a-1) \left( \frac{f(z)}{z} e^{g(z)} \right)^{a-2} \left( \frac{zf''(z) - f(z)}{z^2} e^{g(z)} + \frac{f(z)}{z} g'(z)e^{g(z)} \right).$$

(2.5)

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (a-1) \left[ \frac{zf'(z)}{f(z)} - 1 \right] + zg'(z),$$

(2.6)

which readily shows that

$$\left| c |z|^{2a} + \left( 1 - |z|^{2a} \right) \frac{zh''(z)}{ah'(z)} \right| = \left| c |z|^{2a} + \left( 1 - |z|^{2a} \right) \frac{\alpha-1}{\alpha} \left( \frac{zf'(z)}{f(z)} - 1 \right) + zg'(z) \right|$$

$$\leq \left| c \right| + \left| \frac{\alpha-1}{\alpha} \right| \left( \left( \frac{zf'(z)}{f(z)} + 1 \right) + \left| \frac{z^2 g'(z)}{(g(z))^2} \right| \left| \frac{g(z)^2}{z} \right| \right) \quad (z \in U).$$

(2.7)

From the hypothesis of Theorem 2.1, we have

$$\left| \frac{f'(z)}{f(z)} \right| \leq M_1 \quad (z \in U), \quad |g(z)| \leq M_2 \quad (z \in U),$$

(2.8)

then by General Schwarz Lemma for the function $g$, we obtain

$$|g(z)| \leq M_2 |z| \quad (z \in U).$$

(2.9)

Using the inequality (2.7), we have

$$\left| c |z|^{2a} + \left( 1 - |z|^{2a} \right) \frac{zh''(z)}{ah'(z)} \right| \leq \left| c \right| + \left| \frac{\alpha-1}{\alpha} \right| \left( M_1 + 1 \right) + \left( 1 \right) M_2^2.$$  

(2.10)

From (2.10) and since $g$ satisfies the condition (1.8), we have

$$\left| c |z|^{2a} + \left( 1 - |z|^{2a} \right) \frac{zh''(z)}{ah'(z)} \right| \leq |c| + \left| \frac{\alpha-1}{\alpha} \right| \left( M_1 + 2M_2^2 + 1 \right),$$

(2.11)

from which, by (2.2), we get

$$\left| c |z|^{2a} + \left( 1 - |z|^{2a} \right) \frac{zh''(z)}{ah'(z)} \right| \leq 1 \quad (z \in U).$$

(2.12)
Applying Theorem 1.1, we conclude that the integral operator $F_1(f, g)(z)$ defined by (1.15) is in the class $S$. 

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

**Corollary 2.2.** Let $f, g \in A$, where $g$ satisfies the condition (1.8) and $\alpha$ a complex number, $\text{Re} \alpha > 0$. If

\[
\left| \frac{f'(z)}{f(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad \left| g(z) \right| \leq 1 \quad (z \in \mathbb{U}),
\]

\[
\left| c \right| \leq 1 - \left| \frac{4(\alpha - 1)}{\alpha} \right|, \quad c \in \mathbb{C}, \ c \neq -1,
\]

then the integral operator $F_1(f, g)(z)$ defined by (1.15) is in the class $S$.

**Theorem 2.3.** Let $f, g \in A$, where $g$ satisfies the inequality $\left| g(z) \right| \leq M$, $M \geq 1$. Also, let $\alpha$ be a real number, $\alpha \geq 1$, and $c$ a complex number with

\[
\left| c \right| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (M + 1), \quad c \neq -1.
\]

If

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad \left| \frac{g'(z)}{g(z)} \right| \leq 1 \quad (z \in \mathbb{U}),
\]

then the integral operator $G_1(f, g)(z)$ defined by (1.16) is in the class $S$.

**Proof.** We observe that

\[
G_1(f, g)(z) = \left( \alpha \int_0^z t^{\alpha - 1} \left( f'(t)e^{g(t)} \right)^{\alpha - 1} dt \right)^{1/\alpha}.
\]

Let us consider the function

\[
h(z) = \int_0^z \left( f'(t)e^{g(t)} \right)^{\alpha - 1} dt.
\]

The function $h$ is regular in $\mathbb{U}$. From (2.17), we have

\[
h'(z) = \left( f'(z)e^{g(z)} \right)^{\alpha - 1},
\]

\[
\frac{zh''(z)}{h'(z)} = (\alpha - 1) \left( \frac{zf''(z)}{f'(z)} + zg'(z) \right),
\]

\[
\frac{zg''(z)}{g'(z)} - \frac{zf''(z)}{f'(z)} - (\alpha - 1) zg'(z) = 0.
\]
which readily shows that

\[ |c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{ah'(z)} \leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left( \left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{zg'(z)}{g(z)} \right| |g(z)| \right) \quad (z \in \mathbb{U}). \tag{2.19} \]

From (2.19) and the conditions of Theorem 2.3, we get

\[ |c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{ah'(z)} \leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| (1 + M) \leq 1 \quad (z \in \mathbb{U}). \tag{2.20} \]

Applying Theorem 1.1, we conclude that the integral operator \( G_1(f, g)(z) \) defined by (1.16) is in the class \( S \).

Setting \( M = 1 \) in Theorem 2.3, we obtain the following consequence of Theorem 2.3.

**Corollary 2.4.** Let \( f, g \in \mathcal{A} \), where \( g \) satisfies the condition \( |g'(z)/g(z)| \leq 1 \), \( \alpha \) a real number, \( \alpha \geq 1 \), and \( c \) a complex number with \( |c| \leq 1 - |2(\alpha - 1)/\alpha|, \ c \neq -1 \). If

\[ \left| \frac{f''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq 1 \quad (z \in \mathbb{U}), \tag{2.21} \]

then the integral operator \( G_1(f, g)(z) \) defined by (1.16) is in the class \( S \).

**Theorem 2.5.** Let \( f, g \in \mathcal{A} \), where \( g \) satisfies the condition (1.8), \( \alpha \) a complex number, \( \text{Re} \alpha > 0 \), \( M_1 \) and \( M_2 \) are real positive numbers, and \( c \) a complex number, \( |c| < 1 \). If

\[ \left| \frac{f'(z)}{f(z)} \right| \leq M_1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq M_2 \quad (z \in \mathbb{U}), \tag{2.22} \]

then for any complex number \( \beta \),

\[ \text{Re} \beta \geq \text{Re} \alpha \geq \frac{M_1 + 2M_2^2 + 1}{|\alpha(1 - |c|)|}, \tag{2.23} \]

the integral operator \( H_1(f, g)(z) \) defined by (1.17) is in the class \( S \).

**Proof.** Let us consider the function

\[ h(z) = \int_0^z \left( \frac{f(t)}{\frac{t}{1 + e^{\delta t}}} \right)^{1/\alpha} \ dt. \tag{2.24} \]
The function $h$ is regular in U. From (2.24), we have

$$h'(z) = \left( \frac{f(z)}{z} \right)^{1/\alpha},$$

which readily shows that

$$1 - \left| z \right|^{2Re\alpha} \leq 1 - \left| z \right|^{2Re\alpha} \left( \left| \frac{z f'(z)}{f(z)} \right| + 1 + \left| \frac{z^2 g'(z)}{g(z)} \right| \right).$$

(2.26)

By the General Schwarz Lemma for the function $g$, we obtain

$$\left| g(z) \right| \leq M_2 |z| \quad (z \in U),$$

(2.27)

and using the inequality (2.26), we have

$$\frac{1 - \left| z \right|^{2Re\alpha} \left| zh''(z) \right|}{\left| h'(z) \right|} \leq \frac{1 - \left| z \right|^{2Re\alpha}}{|\alpha| Re\alpha} \left( M_1 + 1 + \left( \frac{z^2 g'(z)}{g(z)} - 1 \right) \right).$$

(2.28)

From (2.28) and since $g$ satisfies the condition (1.8), we get

$$\frac{1 - \left| z \right|^{2Re\alpha} \left| zh''(z) \right|}{\left| h'(z) \right|} \leq \frac{1 - \left| z \right|^{2Re\alpha}}{|\alpha| Re\alpha} \left( M_1 + 1 + \left( \frac{z^2 g'(z)}{g(z)} - 1 \right) \right).$$

(2.29)

From (2.23), we have

$$\frac{M_1 + 2M_2^2 + 1}{|\alpha| Re\alpha} \leq 1 - |c|,$$

(2.30)

and using (2.29), we obtain

$$\frac{1 - \left| z \right|^{2Re\alpha} \left| zh''(z) \right|}{\left| h'(z) \right|} \leq 1 - |c| \quad (z \in U).$$

(2.31)

Applying Theorem 1.2, we conclude that the integral operator $H_1(f, g)(z)$ defined by (1.17) is in the class $S$. □
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Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.5, we obtain the following corollary.

**Corollary 2.6.** Let $f, g \in \mathcal{A}$, where $g$ satisfies the condition (1.8), $\alpha$ a complex number, $\text{Re}\, \alpha > 0$, and $c$ a complex number, $|c| < 1$. If

$$
\left| \frac{f'(z)}{f(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq 1 \quad (z \in \mathbb{U}),
$$

(2.32)

then for any complex number $\beta$,

$$
\text{Re}\, \beta \geq \text{Re}\, \alpha \geq \frac{4}{|\alpha|(1 - |c|)},
$$

(2.33)

the integral operator $H_1(f, g)(z)$ defined by (1.17) is in the class $\mathcal{S}$.

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**References**

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