Research Article

The Univalence Conditions of Some Integral Operators

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We introduce new integral operators of analytic functions f and g defined in the open unit disk $\mathbb{U}$. For these operators, we discuss some univalence conditions.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \quad (1.2)$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0. \quad (1.3)$$

Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f , which are univalent in $\mathbb{U}$ (see, for details [1]; see also [2, 3]).

In [4, 5], Pescar gave the following univalence conditions for the functions $f \in \mathcal{A}$.

Theorem 1.1 (see [4]). Let α be a complex number, $\operatorname{Re} \alpha > 0$, and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f(z) = z + \dots$ a regular function in $\mathbb{U}$. If

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \quad (1.4)$$

for all $z \in \mathbb{U}$, then the function

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t) dt \right)^{1/\alpha} = z + \dots \quad (1.5)$$

is regular and univalent in $\mathbb{U}$.

Theorem 1.2 (see [5]). Let α be a complex number, $\operatorname{Re} \alpha > 0$, and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$. If

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c| \quad (1.6)$$

for all $z \in \mathbb{U}$, then for any complex number β , $\operatorname{Re} \beta \geq \operatorname{Re} \alpha$, the function

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t) dt \right)^{1/\beta} \quad (1.7)$$

is in the class $\mathcal{S}$.

On the other hand, for the functions $f \in \mathcal{A}$, Ozaki and Nunokawa [6] proved another univalence condition asserted by Theorem 1.3.

Theorem 1.3 (see [6]). Let $f \in \mathcal{A}$ satisfy the condition

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1 \quad (z \in \mathbb{U}). \quad (1.8)$$

Then f is univalent in $\mathbb{U}$.

In the paper [7], Pescar determined some univalence conditions for the following integral operators.

Theorem 1.4 (see [7]). Let the function g satisfy (1.8), M a positive real number fixed, and c a complex number. If $\alpha \in [(2M+1)/(2M+2), (2M+1)/2M]$,

$$|c| \leq 1 - \left| \frac{\alpha-1}{\alpha} \right| (2M+1), \quad c \neq -1, \quad (1.9)$$

$$|g(z)| \leq M$$

for all $z \in \mathbb{U}$, then the function

$$G_\alpha(z) = \left(\alpha \int_0^z [g(t)]^{\alpha-1} dt \right)^{1/\alpha} \quad (1.10)$$

is in the class $\mathcal{S}$.

Theorem 1.5 (see [7]). Let $g \in \mathcal{A}$, α a real number, $\alpha \geq 1$, and c a complex number, $|c| \leq 1/\alpha$, $c \neq -1$. If

$$\left| \frac{g''(z)}{g'(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad (1.11)$$

then the function

$$H_\alpha(z) = \left(\alpha \int_0^z [tg'(t)]^{\alpha-1} dt \right)^{1/\alpha} \quad (1.12)$$

is in the class $\mathcal{S}$.

Theorem 1.6 (see [7]). Let $g \in \mathcal{A}$ satisfies (1.8), α a complex number, $M > 1$ fixed, $\operatorname{Re} \alpha > 0$, and c a complex number, $|c| < 1$. If $|g(z)| \leq M$ for all $z \in \mathbb{U}$, then for any complex number β

$$\operatorname{Re} \beta \geq \operatorname{Re} \alpha \geq \frac{2M+1}{|\alpha|(1-|c|)} \quad (1.13)$$

the function

$$H_\beta(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{g(t)}{t} \right)^{1/\alpha} dt \right)^{1/\beta} \quad (1.14)$$

is in the class $\mathcal{S}$.

In this paper, we introduce the following integral operators as follows:

$$F_1(f, g)(z) = \left(\alpha \int_0^z (f(t)e^{g(t)})^{\alpha-1} dt \right)^{1/\alpha} \quad (f, g \in \mathcal{A}; \alpha \in \mathbb{C}), \quad (1.15)$$

$$G_1(f, g)(z) = \left(\alpha \int_0^z (tf'(t)e^{g(t)})^{\alpha-1} dt \right)^{1/\alpha} \quad (f, g \in \mathcal{A}; \alpha \in \mathbb{C}), \quad (1.16)$$

$$H_1(f, g)(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} e^{g(t)} \right)^{1/\alpha} dt \right)^{1/\beta} \quad (f, g \in \mathcal{A}; \alpha, \beta \in \mathbb{C} - \{0\}). \quad (1.17)$$

Remark 1.7. For $e^{g(z)} = 1$ and $f(z) = g(z)$, the integral operators (1.15), (1.16), and (1.17) would reduce to the integral operators (1.10), (1.12), and (1.14).

In this paper, we generalize the integral operators given by Pescar [7], and we study the univalence conditions for the integral operators defined by (1.15), (1.16), and (1.17).

For this purpose, we need the following result.

Lemma 1.8 (General Schwarz Lemma [8]). *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M . If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R). \quad (1.18)$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m, \quad (1.19)$$

where θ is constant.

2. Main Results

Theorem 2.1. *Let $f, g \in \mathcal{A}$, where g satisfies the condition (1.8), M_1 and M_2 are real positive numbers, and α a complex number, $\operatorname{Re} \alpha > 0$. If*

$$\left| \frac{f'(z)}{f(z)} \right| \leq M_1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq M_2 \quad (z \in \mathbb{U}), \quad (2.1)$$

$$|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (M_1 + 2M_2^2 + 1), \quad c \in \mathbb{C}, \quad c \neq -1, \quad (2.2)$$

then the integral operator $F_1(f, g)(z)$ defined by (1.15) is in the class $\mathcal{S}$.

Proof. From (1.15), we have

$$F_1(f, g)(z) = \left(\alpha \int_0^z t^{\alpha-1} \left(\frac{f(t)}{t} e^{g(t)} \right)^{\alpha-1} dt \right)^{1/\alpha}. \quad (2.3)$$

Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t} e^{g(t)} \right)^{\alpha-1} dt. \quad (2.4)$$

The function h is regular in $\mathbb{U}$. From (2.4), we get

$$\begin{aligned} h'(z) &= \left(\frac{f(z)}{z} e^{g(z)} \right)^{\alpha-1}, \\ h''(z) &= (\alpha-1) \left(\frac{f(z)}{z} e^{g(z)} \right)^{\alpha-2} \left(\frac{zf'(z) - f(z)}{z^2} e^{g(z)} + \frac{f(z)}{z} g'(z) e^{g(z)} \right). \end{aligned} \quad (2.5)$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha-1) \left[\left(\frac{zf'(z)}{f(z)} - 1 \right) + zg'(z) \right], \quad (2.6)$$

which readily shows that

$$\begin{aligned} \left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zh''(z)}{\alpha h'(z)} \right| &= \left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{\alpha-1}{\alpha} \left(\left(\frac{zf'(z)}{f(z)} - 1 \right) + zg'(z) \right) \right| \\ &\leq |c| + \left| \frac{\alpha-1}{\alpha} \right| \left(\left(\left| \frac{zf'(z)}{f(z)} \right| + 1 \right) + \left| \frac{z^2 g'(z)}{[g(z)]^2} \right| \left| \frac{[g(z)]^2}{z} \right| \right) \quad (z \in \mathbb{U}). \end{aligned} \quad (2.7)$$

From the hypothesis of Theorem 2.1, we have

$$\left| \frac{f'(z)}{f(z)} \right| \leq M_1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq M_2 \quad (z \in \mathbb{U}), \quad (2.8)$$

then by *General Schwarz Lemma* for the function g , we obtain

$$|g(z)| \leq M_2 |z| \quad (z \in \mathbb{U}). \quad (2.9)$$

Using the inequality (2.7), we have

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zh''(z)}{\alpha h'(z)} \right| \leq |c| + \left| \frac{\alpha-1}{\alpha} \right| \left(M_1 + 1 + \left(\left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| + 1 \right) M_2^2 \right). \quad (2.10)$$

From (2.10) and since g satisfies the condition (1.8), we have

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zh''(z)}{\alpha h'(z)} \right| \leq |c| + \left| \frac{\alpha-1}{\alpha} \right| (M_1 + 2M_2^2 + 1), \quad (2.11)$$

from which, by (2.2), we get

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{zh''(z)}{\alpha h'(z)} \right| \leq 1 \quad (z \in \mathbb{U}). \quad (2.12)$$

Applying Theorem 1.1, we conclude that the integral operator $F_1(f, g)(z)$ defined by (1.15) is in the class $\mathcal{S}$. $\square$

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

Corollary 2.2. *Let $f, g \in \mathcal{A}$, where g satisfies the condition (1.8) and α a complex number, $\operatorname{Re} \alpha > 0$. If*

$$\begin{aligned} \left| \frac{f'(z)}{f(z)} \right| &\leq 1 \quad (z \in \mathbb{U}), & |g(z)| &\leq 1 \quad (z \in \mathbb{U}), \\ |c| &\leq 1 - \left| \frac{4(\alpha - 1)}{\alpha} \right|, & c &\in \mathbb{C}, \quad c \neq -1, \end{aligned} \quad (2.13)$$

then the integral operator $F_1(f, g)(z)$ defined by (1.15) is in the class $\mathcal{S}$.

Theorem 2.3. *Let $f, g \in \mathcal{A}$, where g satisfies the inequality $|g(z)| \leq M$, $M \geq 1$. Also, let α be a real number, $\alpha \geq 1$, and c a complex number with*

$$|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (M + 1), \quad c \neq -1. \quad (2.14)$$

If

$$\left| \frac{f''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad \left| \frac{g'(z)}{g(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad (2.15)$$

then the integral operator $G_1(f, g)(z)$ defined by (1.16) is in the class $\mathcal{S}$.

Proof. We observe that

$$G_1(f, g)(z) = \left(\alpha \int_0^z t^{\alpha-1} \left(f'(t) e^{g(t)} \right)^{\alpha-1} dt \right)^{1/\alpha}. \quad (2.16)$$

Let us consider the function

$$h(z) = \int_0^z \left(f'(t) e^{g(t)} \right)^{\alpha-1} dt. \quad (2.17)$$

The function h is regular in $\mathbb{U}$. From (2.17), we have

$$\begin{aligned} h'(z) &= \left(f'(z) e^{g(z)} \right)^{\alpha-1}, \\ \frac{zh''(z)}{h'(z)} &= (\alpha - 1) \left(\frac{zf''(z)}{f'(z)} + zg'(z) \right), \end{aligned} \quad (2.18)$$

which readily shows that

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zh''(z)}{\alpha h'(z)} \right| \leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left(\left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{zg'(z)}{g(z)} \right| |g(z)| \right) \quad (z \in \mathbb{U}). \quad (2.19)$$

From (2.19) and the conditions of Theorem 2.3, we get

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zh''(z)}{\alpha h'(z)} \right| \leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| (1 + M) \leq 1 \quad (z \in \mathbb{U}). \quad (2.20)$$

Applying Theorem 1.1, we conclude that the integral operator $G_1(f, g)(z)$ defined by (1.16) is in the class $\mathcal{S}$. $\square$

Setting $M = 1$ in Theorem 2.3, we obtain the following consequence of Theorem 2.3.

Corollary 2.4. *Let $f, g \in \mathcal{A}$, where g satisfies the condition $|g'(z)/g(z)| \leq 1$, α a real number, $\alpha \geq 1$, and c a complex number with $|c| \leq 1 - |2(\alpha - 1)/\alpha|$, $c \neq -1$. If*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq 1 \quad (z \in \mathbb{U}), \quad (2.21)$$

then the integral operator $G_1(f, g)(z)$ defined by (1.16) is in the class $\mathcal{S}$.

Theorem 2.5. *Let $f, g \in \mathcal{A}$, where g satisfies the condition (1.8), α a complex number, $\operatorname{Re} \alpha > 0$, M_1 and M_2 are real positive numbers, and c a complex number, $|c| < 1$. If*

$$\left| \frac{f'(z)}{f(z)} \right| \leq M_1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq M_2 \quad (z \in \mathbb{U}), \quad (2.22)$$

then for any complex number β ,

$$\operatorname{Re} \beta \geq \operatorname{Re} \alpha \geq \frac{M_1 + 2M_2^2 + 1}{|\alpha|(1 - |c|)}, \quad (2.23)$$

the integral operator $H_1(f, g)(z)$ defined by (1.17) is in the class $\mathcal{S}$.

Proof. Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t} e^{g(t)} \right)^{1/\alpha} dt. \quad (2.24)$$

The function h is regular in $\mathbb{U}$. From (2.24), we have

$$\begin{aligned} h'(z) &= \left(\frac{f(z)}{z} e^{g(z)} \right)^{1/\alpha}, \\ \frac{zh''(z)}{h'(z)} &= \frac{1}{\alpha} \left(\left(\frac{zf'(z)}{f(z)} - 1 \right) + zg'(z) \right), \end{aligned} \quad (2.25)$$

which readily shows that

$$\begin{aligned} \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{|\alpha| \operatorname{Re} \alpha} \left(\left| \frac{zf'(z)}{f(z)} \right| + 1 + \left| \frac{z^2 g'(z)}{[g(z)]^2} \right| \left| \frac{[g(z)]^2}{z} \right| \right) \\ &\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{|\alpha| \operatorname{Re} \alpha} \left(\left| \frac{zf'(z)}{f(z)} \right| + 1 + \left(\left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| + 1 \right) \left| \frac{[g(z)]^2}{z} \right| \right). \end{aligned} \quad (2.26)$$

By the *General Schwarz Lemma* for the function g , we obtain

$$|g(z)| \leq M_2 |z| \quad (z \in \mathbb{U}), \quad (2.27)$$

and using the inequality (2.26), we have

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{|\alpha| \operatorname{Re} \alpha} \left(M_1 + 1 + \left(\left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| + 1 \right) M_2^2 \right). \quad (2.28)$$

From (2.28) and since g satisfies the condition (1.8), we get

$$\begin{aligned} \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \frac{(M_1 + 2M_2^2 + 1)}{|\alpha|} \\ &\leq \frac{M_1 + 2M_2^2 + 1}{|\alpha| \operatorname{Re} \alpha}. \end{aligned} \quad (2.29)$$

From (2.23), we have

$$\frac{M_1 + 2M_2^2 + 1}{|\alpha| \operatorname{Re} \alpha} \leq 1 - |c|, \quad (2.30)$$

and using (2.29), we obtain

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1 - |c| \quad (z \in \mathbb{U}). \quad (2.31)$$

Applying Theorem 1.2, we conclude that the integral operator $H_1(f, g)(z)$ defined by (1.17) is in the class $\mathcal{S}$. $\square$

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let $f, g \in \mathcal{A}$, where g satisfies the condition (1.8), α a complex number, $\operatorname{Re} \alpha > 0$, and c a complex number, $|c| < 1$. If*

$$\left| \frac{f'(z)}{f(z)} \right| \leq 1 \quad (z \in \mathbb{U}), \quad |g(z)| \leq 1 \quad (z \in \mathbb{U}), \quad (2.32)$$

then for any complex number β ,

$$\operatorname{Re} \beta \geq \operatorname{Re} \alpha \geq \frac{4}{|\alpha|(1 - |c|)}, \quad (2.33)$$

the integral operator $H_1(f, g)(z)$ defined by (1.17) is in the class $\mathcal{S}$.

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References

- [1] D. Blezu, "On univalence criteria," *General Mathematics*, vol. 14, pp. 77–84, 2006.
- [2] D. Blezu and R. N. Pascu, "Univalence criteria for integral operators," *Glasnik Matematički*, vol. 36(56), no. 2, pp. 241–245, 2001.
- [3] V. Pescar, "Univalence of certain integral operators," *Acta Universitatis Apulensis*, no. 12, pp. 43–48, 2006.
- [4] V. Pescar, "A new generalization of Ahlfors's and Becker's criterion of univalence," *Bulletin of Malaysian Mathematical Society*, vol. 19, no. 2, pp. 53–54, 1996.
- [5] V. Pescar, "Univalence criteria of certain integral operators," *Acta Ciencia Indica, Mathematics*, vol. 29, no. 1, pp. 135–138, 2003.
- [6] S. Ozaki and M. Nunokawa, "The Schwarzian derivative and univalent functions," *Proceedings of the American Mathematical Society*, vol. 33, pp. 392–394, 1972.
- [7] V. Pescar, "On the univalence of some integral operators," *General Mathematics*, vol. 14, no. 2, pp. 77–84, 2006.
- [8] Z. Nehari, *Conformal Mapping*, Dover, New York, NY, USA, 1975.