Research Article

On the Hyers-Ulam Stability of a General Mixed Additive and Cubic Functional Equation in $n$-Banach Spaces

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The objective of the present paper is to determine the generalized Hyers-Ulam stability of the mixed additive-cubic functional equation in $n$-Banach spaces by the direct method. In addition, we show under some suitable conditions that an approximately mixed additive-cubic function can be approximated by a mixed additive and cubic mapping.

1. Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?

If the problem accepts a unique solution, we say the equation is stable (see [1]). The study of stability problems for functional equations is related to a question of Ulam [2] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [3]. The result of Hyers was generalized by Aoki [4] for approximate additive mappings and by Rassias [5] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a generalization of Rassias’ theorem was obtained by Găvruta [6], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. On the other hand, several further interesting discussions, modifications, extensions, and generalizations of the original problem of Ulam have been proposed (see, e.g. [7–12] and the references therein).

Recently, Park [9] investigated the approximate additive mappings, approximate Jensen mappings, and approximate quadratic mappings in 2-Banach spaces and proved the
generalized Hyers-Ulam stability of the Cauchy functional equation, the Jensen functional equation, and the quadratic functional equation in 2-Banach spaces. This is the first result for the stability problem of functional equations in 2-Banach spaces.

In [11, 12], we introduced the following mixed additive-cubic functional equation for fixed integers \( k \) with \( k \neq 0, \pm 1 \):

\[
f(kx + y) + f(kx - y) = kf(x + y) + kf(x - y) + 2f(kx) - 2kf(x),
\]

with \( f(0) = 0 \), and investigated the generalized Hyers-Ulam stability of (1.1) in quasi-Banach spaces and non-Archimedean fuzzy normed spaces, respectively.

In this paper, we investigate, approximate mixed additive-cubic mappings in \( n \)-Banach spaces and non-Archimedean fuzzy normed spaces, respectively.

The concept of 2-normed spaces was initially developed by Gähler [13, 14] in the middle of 1960s, while that of \( n \)-normed spaces can be found in [15, 16]. Since then, many others have studied this concept and obtained various results; see for instance [15, 17–19]. We recall some basic facts concerning \( n \)-normed spaces and some preliminary results.

**Definition 1.1.** Let \( n \in \mathbb{N} \), and let \( X \) be a real linear space with \( \dim X \geq n \) and \( \|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R} \) a function satisfying the following properties:

(N1) \( \|x_1, x_2, \ldots, x_n\| = 0 \) if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent,

(N2) \( \|x_1, x_2, \ldots, x_n\| \) is invariant under permutation,

(N3) \( \|\alpha x_1, x_2, \ldots, x_n\| = |\alpha|\|x_1, x_2, \ldots, x_n\| \),

(N4) \( \|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\| \)

for all \( \alpha \in \mathbb{R} \) and \( x, y, x_1, x_2, \ldots, x_n \in X \). Then the function \( \|\cdot, \ldots, \cdot\| \) is called an \( n \)-norm on \( X \) and the pair \((X, \|\cdot, \ldots, \cdot\|)\) is called an \( n \)-normed space.

**Example 1.2.** For \( x_1, x_2, \ldots, x_n \in \mathbb{R}^n \), the Euclidean \( n \)-norm \( \|x_1, x_2, \ldots, x_n\|_E \) is defined by

\[
\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})| = \text{abs} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix},
\]

where \( x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \ldots, n \).

**Example 1.3.** The standard \( n \)-norm on \( X \), a real inner product space of dimension \( \dim X \geq n \), is as follows:

\[
\|x_1, x_2, \ldots, x_n\|_S = \sqrt{\left(\langle x_1, x_1 \rangle \cdots \langle x_1, x_n \rangle \right)^{1/2}} = \left( \begin{array}{c} \langle x_1, x_1 \rangle \\ \vdots \\ \langle x_n, x_1 \rangle \end{array} \right)^{1/2},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( X \). If \( X = \mathbb{R}^n \), then this \( n \)-norm is exactly the same as the Euclidean \( n \)-norm \( \|x_1, x_2, \ldots, x_n\|_E \) mentioned earlier. For \( n = 1 \), this \( n \)-norm is the usual norm \( \|x_1\| = (x_1, x_1)^{1/2} \).
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Definition 1.4. A sequence \( \{x_k\} \) in an \( n \)-normed space \( X \) is said to converge to some \( x \in X \) in the \( n \)-norm if

\[
\lim_{k \to \infty} \|x_k - x, y_2, \ldots, y_n\| = 0, \tag{1.4}
\]

for every \( y_2, \ldots, y_n \in X \).

Definition 1.5. A sequence \( \{x_k\} \) in an \( n \)-normed space \( X \) is said to be a Cauchy sequence with respect to the \( n \)-norm if

\[
\lim_{k,l \to \infty} \|x_k - x_l, y_2, \ldots, y_n\| = 0, \tag{1.5}
\]

for every \( y_2, \ldots, y_n \in X \). If every Cauchy sequence in \( X \) converges to some \( x \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be an \( n \)-Banach space.

Now we state the following results as lemma (see [9] for the details).

Lemma 1.6. Let \( X \) be an \( n \)-normed space. Then,

1. For \( x_i \in X (i = 1, \ldots, n) \) and \( \gamma \), a real number,

\[
\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i, \gamma x_i, \ldots, x_n\| \quad \text{for all } 1 \leq i \neq j \leq n, \tag{1.6}
\]

2. \( \|x, y_2, \ldots, y_n\| - \|y, y_2, \ldots, y_n\| \leq \|x - y, y_2, \ldots, y_n\| \) for all \( x, y, y_2, \ldots, y_n \in X \),

3. if \( \|x, y_2, \ldots, y_n\| = 0 \) for all \( y_2, \ldots, y_n \in X \), then \( x = 0 \),

4. for a convergent sequence \( \{x_j\} \) in \( X \),

\[
\lim_{j \to \infty} \|x_j, y_2, \ldots, y_n\| = \|\lim_{j \to \infty} x_j, y_2, \ldots, y_n\| \quad \text{for all } y_2, \ldots, y_n \in X. \tag{1.7}
\]

2. Approximate Mixed Additive-Cubic Mappings

In this section, we investigate the generalized Hyers-Ulam stability of the generalized mixed additive-cubic functional equation in \( n \)-Banach spaces. Let \( X \) be a linear space and \( Y \) an \( n \)-Banach space. For convenience, we use the following abbreviation for a given mapping \( f : X \to Y \):

\[
Df(x, y) := f(kx + y) + f(kx - y) - kf(x + y) - kf(x - y) - 2f(kx) + 2kf(x) \quad \tag{2.1}
\]

for all \( x, y \in X \).
Theorem 2.1. Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that

\[
\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, u_2, \ldots, u_n) < \infty, \tag{2.2}
\]

\[
\|D f(x, y), u_2, \ldots, u_n\| \leq \varphi(x, y, u_2, \ldots, u_n) \tag{2.3}
\]

for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique additive mapping $A : X \to Y$ such that

\[
\|f(2x) - 8f(x) - A(x), u_2, \ldots, u_n\| \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \tilde{\varphi}(2^j x, u_2, \ldots, u_n) \tag{2.4}
\]

for all $x, u_2, \ldots, u_n \in X$, where

\[
\tilde{\varphi}(x, u_2, \ldots, u_n) := \frac{1}{|k^3 - k|} \left\{ (|k| + 1) \left[ \varphi(x, (2k + 1)x, u_2, \ldots, u_n) + \varphi(x, (2k - 1)x, u_2, \ldots, u_n) \right] \\
+ \varphi(3x, x, u_2, \ldots, u_n) + (8k^2 + 1) \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
+ \varphi(x, kx, u_2, \ldots, u_n) + k^2 \varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) \\
+ 2\varphi(x, (k + 1)x, u_2, \ldots, u_n) + 2\varphi(x, (k - 1)x, u_2, \ldots, u_n) + 2\varphi(2x, x, u_2, \ldots, u_n) \\
+ 2\varphi(2x, kx, u_2, \ldots, u_n) + 8\varphi \left( \frac{x}{2}, \frac{kx}{2}, u_2, \ldots, u_n \right) \\
+ 8k|k| \varphi \left( \frac{x}{2}, \frac{(2k - 1)x}{2}, u_2, \ldots, u_n \right) + 8k|k| \varphi \left( \frac{x}{2}, \frac{(2k + 1)x}{2}, u_2, \ldots, u_n \right) \\
+ 8\varphi \left( \frac{x}{2}, \frac{3kx}{2}, u_2, \ldots, u_n \right) + \frac{|k| + 1}{|k - 1|} \varphi(0, (k + 1)x, u_2, \ldots, u_n) \\
+ \frac{8k^2 + 1}{|k - 1|} \varphi(0, (k - 1)x, u_2, \ldots, u_n) + \frac{2}{|k - 1|} \varphi(0, x, u_2, \ldots, u_n) \\
+ \frac{|k|}{|k - 1|} \varphi(0, (3k - 1)x, u_2, \ldots, u_n) + \frac{k^2}{|k - 1|} \varphi(0, 2(k - 1)x, u_2, \ldots, u_n) \\
+ \frac{k^2 + |k| - 1}{|k - 1|} \varphi(0, 2kx, u_2, \ldots, u_n) \\
+ \frac{8|k|}{|k - 1|} \varphi \left( 0, \frac{(3k - 1)x}{2}, u_2, \ldots, u_n \right) + \frac{8|k|}{|k - 1|} \varphi \left( 0, \frac{(k + 1)x}{2}, u_2, \ldots, u_n \right) \\
+ \frac{8k^2 + 2|k| - 8}{|k - 1|} \varphi(0, kx, u_2, \ldots, u_n) \right\}. \tag{2.5}
\]
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Proof. Letting $x = 0$ in (2.3), we get

$$\|f(y) + f(-y), u_2, \ldots, u_n\|_Y \leq \frac{1}{|k-1|} \varphi(0, y, u_2, \ldots, u_n)$$  \hspace{1cm} (2.6)

for all $y, u_2, \ldots, u_n \in X$. Putting $y = x$ in (2.3), we have

$$\|f((k+1)x) + f((k-1)x) - kf(2x) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\|_Y \leq \varphi(x, x, u_2, \ldots, u_n)$$  \hspace{1cm} (2.7)

for all $x, u_2, \ldots, u_n \in X$. Thus

$$\|f(2(k+1)x) + f(2(k-1)x) - kf(4x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n\|_Y$$
$$\leq \varphi(2x, 2x, u_2, \ldots, u_n)$$  \hspace{1cm} (2.8)

for all $x, u_2, \ldots, u_n \in X$. Letting $y = kx$ in (2.3), we get

$$\|f(2kx) - kf((k+1)x) - kf((k-1)x) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\|_Y \leq \varphi(x, kx, u_2, \ldots, u_n)$$  \hspace{1cm} (2.9)

for all $x, u_2, \ldots, u_n \in X$. Letting $y = (k+1)x$ in (2.3), we have

$$\|f((2k+1)x) + f(-x) - kf((k+2)x) - kf(-kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\|_Y$$
$$\leq \varphi(x, (k+1)x, u_2, \ldots, u_n)$$  \hspace{1cm} (2.10)

for all $x, u_2, \ldots, u_n \in X$. Letting $y = (k-1)x$ in (2.3), we have

$$\|f((2k-1)x) - (k+2)f(kx) - kf(-k-2)x) + (2k+1)f(x), u_2, \ldots, u_n\|_Y$$
$$\leq \varphi(x, (k-1)x, u_2, \ldots, u_n)$$  \hspace{1cm} (2.11)

for all $x, u_2, \ldots, u_n \in X$. Replacing $x$ and $y$ by $2x$ and $x$ in (2.3), respectively, we get

$$\|f((2k+1)x) + f((2k-1)x) - 2f(2kx) - kf(3x) + 2kf(2x) - kf(x), u_2, \ldots, u_n\|_Y$$
$$\leq \varphi(2x, x, u_2, \ldots, u_n)$$  \hspace{1cm} (2.12)

for all $x, u_2, \ldots, u_n \in X$. Replacing $x$ and $y$ by $3x$ and $x$ in (2.3), respectively, we get

$$\|f((3k+1)x) + f((3k-1)x) - 2f(3kx) - kf(4x) - kf(2x) + 2kf(3x), u_2, \ldots, u_n\|_Y$$
$$\leq \varphi(3x, x, u_2, \ldots, u_n)$$  \hspace{1cm} (2.13)
for all \( x, u_2, \ldots, u_n \in X \). Replacing \( x \) and \( y \) by \( 2x \) and \( kx \) in (2.3), respectively, we have

\[
\| f(3kx) + f(kx) - k f((k + 2)x) - k f(-(k - 2)x) - 2 f(2kx) + 2k f(x), u_2, \ldots, u_n \|_Y \\
\leq \phi(2x, kx, u_2, \ldots, u_n)
\]  

(2.14)

for all \( x, u_2, \ldots, u_n \in X \). Setting \( y = (2k + 1)x \) in (2.3), we have

\[
\| f((3k + 1)x) + f(-(k + 1)x) - k f(2(k + 1)x) - k f(-2kx) - 2 f(kx) + 2k f(x), u_2, \ldots, u_n \|_Y \\
\leq \phi(x, (2k + 1)x, u_2, \ldots, u_n)
\]  

(2.15)

for all \( x, u_2, \ldots, u_n \in X \). Setting \( y = (2k - 1)x \) in (2.3), we have

\[
\| f((3k - 1)x) + f(-(k - 1)x) - k f(-2(k - 1)x) - k f(2kx) - 2 f(kx) + 2k f(x), u_2, \ldots, u_n \|_Y \\
\leq \phi(x, (2k - 1)x, u_2, \ldots, u_n)
\]  

(2.16)

for all \( x, u_2, \ldots, u_n \in X \). Setting \( y = 3kx \) in (2.3), we have

\[
\| f(4kx) + f(-2kx) - k f((3k + 1)x) - k f(-(3k - 1)x) - 2 f(kx) + 2k f(x), u_2, \ldots, u_n \|_Y \\
\leq \phi(x, 3kx, u_2, \ldots, u_n)
\]  

(2.17)

for all \( x, u_2, \ldots, u_n \in X \). By (2.6), (2.7), (2.13), (2.15), and (2.16), we get

\[
\| kf(2k + 1)x + k f(-2(k - 1)x) + 6 f(kx) - 2 f(3kx) - k f(4kx) + 2k f(x), u_2, \ldots, u_n \|_Y \\
\leq \phi(x, (2k + 1)x, u_2, \ldots, u_n) + \phi(x, (2k - 1)x, u_2, \ldots, u_n) + \phi(3x, x, u_2, \ldots, u_n) \\
+ \phi(x, x, u_2, \ldots, u_n) + \frac{1}{|k - 1|}\phi(0, (k + 1)x, u_2, \ldots, u_n) \\
+ \frac{1}{|k - 1|}\phi(0, (k - 1)x, u_2, \ldots, u_n) + \frac{|k|}{|k - 1|}\phi(0, 2kx, u_2, \ldots, u_n)
\]  

(2.18)

for all \( x, u_2, \ldots, u_n \in X \). By (2.6), (2.10), and (2.11), we have

\[
\| f((2k + 1)x) + f((2k - 1)x) - k f((k + 2)x) - k f(-(k - 2)x) - 4 f(kx) + 4k f(x), u_2, \ldots, u_n \|_Y \\
\leq \phi(x, (k + 1)x, u_2, \ldots, u_n) + \phi(x, (k - 1)x, u_2, \ldots, u_n) + \frac{1}{|k - 1|}\phi(0, x, u_2, \ldots, u_n) \\
+ \frac{k}{|k - 1|}\phi(0, kx, u_2, \ldots, u_n)
\]  

(2.19)
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for all \( x, u_2, \ldots, u_n \in X \). It follows from (2.12) and (2.19) that

\[
\|kf((k+2)x) + kf(-(k-2)x) - 2f(2kx) + 4f(kx) - kf(3x) + 2kf(2x) - 5kf(x), u_2, \ldots, u_n\|_Y
\leq \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \varphi(2x, x, u_2, \ldots, u_n)
\]

\[
+ \frac{1}{k-1} \varphi(0, x, u_2, \ldots, u_n) + \frac{k}{k-1} \varphi(0, kx, u_2, \ldots, u_n)
\]

(2.20)

for all \( x, u_2, \ldots, u_n \in X \). By (2.14) and (2.20), we have

\[
\|f(3kx) - 4f(2kx) + 5f(kx) - kf(3x) + 2kf(2x) - 5kf(x), u_2, \ldots, u_n\|_Y
\leq \varphi(x, (k+1)x, u_2, \ldots, u_n) + \varphi(x, (k-1)x, u_2, \ldots, u_n) + \varphi(2x, x, u_2, \ldots, u_n)
\]

\[
+ \varphi(2x, kx, u_2, \ldots, u_n) + \frac{1}{k-1} \varphi(0, x, u_2, \ldots, u_n) + \frac{k}{k-1} \varphi(0, kx, u_2, \ldots, u_n)
\]

(2.21)

for all \( x, u_2, \ldots, u_n \in X \). By (2.6), (2.15), (2.16), and (2.17), we have

\[
\|kf(-(k+1)x) - kf(-(k-1)x) - k^2f(2(k+1)x) + k^2f(-2(k-1)x)
\]

\[
+ k^2f(2kx) - (k^2 - 1)f(-2kx) + f(4kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n\|_Y
\leq |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(3kx, u_2, \ldots, u_n)
\]

\[
+ \frac{k}{k-1} \varphi(0, (3k-1)x, u_2, \ldots, u_n)
\]

(2.22)

for all \( x, u_2, \ldots, u_n \in X \). It follows from (2.6), (2.8), (2.9), and (2.22) that

\[
\|f(4kx) - 2f(2kx) - k^3f(4x) + 2k^3f(2x), u_2, \ldots, u_n\|_Y
\leq |k|\varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k|\varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(3kx, u_2, \ldots, u_n)
\]

\[
+ \varphi(x, kx, u_2, \ldots, u_n) + k^2\varphi(2x, 2x, u_2, \ldots, u_n) + \frac{k}{k-1} \varphi(0, (3k-1)x, u_2, \ldots, u_n)
\]

\[
+ \frac{k}{k-1} \varphi(0, (k+1)x, u_2, \ldots, u_n) + \frac{k^2}{k-1} \varphi(0, 2(k-1)x, u_2, \ldots, u_n)
\]

\[
+ \frac{k^2 - 1}{k-1} \varphi(0, 2kx, u_2, \ldots, u_n)
\]

(2.23)
for all \( x, u_2, \ldots, u_n \in X \). Hence,

\[
\begin{align*}
&\| f(2kx) - 2f(kx) - k^3 f(2x) + 2k^3 f(x), u_2, \ldots, u_n \|_Y \\
\leq & \ |k| \varphi \left( x, 2 \frac{(2k+1)x}{2}, u_2, \ldots, u_n \right) + |k| \varphi \left( x, 2 \frac{(2k-1)x}{2}, u_2, \ldots, u_n \right) + \varphi \left( x, 3 \frac{kx}{2}, u_2, \ldots, u_n \right) \\
&+ \varphi \left( x, \frac{kx}{2}, u_2, \ldots, u_n \right) + k^2 \varphi(x, x, u_2, \ldots, u_n) + \left| \frac{k}{k-1} \right| \varphi(0, (k-1)x, u_2, \ldots, u_n) \\
&+ \frac{k^2-1}{|k-1|} \varphi(0, kx, u_2, \ldots, u_n)
\end{align*}
\]

\[ (2.24) \]

for all \( x, u_2, \ldots, u_n \in X \). By (2.9), we have

\[
\begin{align*}
&\| f(4kx) - k f(2(k+1)x) - k f(-2(k-1)x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n \|_Y \\
\leq & \ \varphi(2x, 2kx, u_2, \ldots, u_n)
\end{align*}
\]

\[ (2.25) \]

for all \( x, u_2, \ldots, u_n \in X \). From (2.23) and (2.25), we have

\[
\begin{align*}
&\| kf(2(k+1)x) + kf(-2(k-1)x) - k^3 f(4x) + \left( 2k^3 - 2k \right) f(2x) \|_Y \\
\leq & \ |k| \varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k| \varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
&+ \varphi(x, kx, u_2, \ldots, u_n) + \frac{k^2-1}{|k-1|} \varphi(0, 2(k-1)x, u_2, \ldots, u_n)
\end{align*}
\]

\[ (2.26) \]

for all \( x, u_2, \ldots, u_n \in X \). Also, from (2.18) and (2.26), we get

\[
\begin{align*}
&\| 2f(3kx) - 6f(kx) + \left( k - k^3 \right) f(4x) - 2kf(3x) + \left( 2k^3 - 2k \right) f(2x) + 6kf(x), u_2, \ldots, u_n \|_Y \\
\leq & \ (|k| + 1) \left[ \varphi(x, (2k+1)x, u_2, \ldots, u_n) + \varphi(x, (2k-1)x, u_2, \ldots, u_n) \right] + \varphi(3x, x, x, u_2, \ldots, u_n) \\
&+ \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) + \varphi(x, kx, u_2, \ldots, u_n) \\
&+ k^2 \varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) + \frac{|k| + 1}{|k-1|} \varphi(0, (k+1)x, u_2, \ldots, u_n)
\end{align*}
\]
for all \(x, u_2, \ldots, u_n \in X\).

On the other hand, it follows from (2.21) and (2.27) that

\[
\left\| \frac{8}{k} f(2kx) - 16 f(kx) + (k - k^3) f(4x) + (2k^3 - 10k) f(2x) + 16k f(x), u_2, \ldots, u_n \right\|_Y \\
\leq \left(\frac{[k] + 1}{[k] - 1} \right) \left[ \varphi(x, (2k + 1)x, u_2, \ldots, u_n) + \varphi(x, (2k - 1)x, u_2, \ldots, u_n) \right] + 2^2 \varphi(2x, x, u_2, \ldots, u_n) + 2 \varphi(x, (k + 1)x, u_2, \ldots, u_n)
\]

(2.28)

for all \(x, u_2, \ldots, u_n \in X\). Therefore by (2.24) and (2.28), we get

\[
\left\| f(4x) - 10 f(2x) + 16 f(x), u_2, \ldots, u_n \right\|_Y \\
\leq \frac{1}{[k^3 - k]} \times \left( (k + 1) \left[ \varphi(x, (2k + 1)x, u_2, \ldots, u_n) + \varphi(x, (2k - 1)x, u_2, \ldots, u_n) \right] \\
+ \varphi(3x, x, u_2, \ldots, u_n) + \left(8k^2 + 1\right) \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n)
\]

(2.29)
for all \( x, u_2, \ldots, u_n \in X \).

Now, let \( g : X \to Y \) be the mapping defined by \( g(x) = f(2x) - 8f(x) \) for all \( x, u_2, \ldots, u_n \in X \). Then, \((2.29)\) means that

\[
\|f(4x) - 10f(2x) + 16f(x), u_2, \ldots, u_n\|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n)
\]

for all \( x, u_2, \ldots, u_n \in X \). Also, we get

\[
\|g(2x) - 2g(x), u_2, \ldots, u_n\|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n)
\]

for all \( x \in X \). Replacing \( x \) by \( 2^j x \) in \((2.31)\) and dividing both sides of \((2.31)\) by \( 2^{j+1} \), we get

\[
\left\| \frac{1}{2^j} g(2^j x) - \frac{1}{2^{j+1}} g(2^{j+1} x), u_2, \ldots, u_n \right\|_Y \leq \frac{1}{2^{j+1}} \tilde{\varphi}(2^j x, u_2, \ldots, u_n)
\]

for all \( x, u_2, \ldots, u_n \in X \) and all integers \( j \geq 0 \). For all integers \( l, m \) with \( 0 \leq l < m \), we have

\[
\left\| \frac{1}{2^l} g(2^l x) - \frac{1}{2^m} g(2^m x), u_2, \ldots, u_n \right\|_Y \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} g(2^j x) - \frac{1}{2^{j+1}} g(2^{j+1} x), u_2, \ldots, u_n \right\|_Y
\]

\[
\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \tilde{\varphi}(2^j x, u_2, \ldots, u_n)
\]

for all \( x, u_2, \ldots, u_n \in X \). So, we get

\[
\lim_{l,m \to \infty} \left\| \frac{1}{2^l} g(2^l x) - \frac{1}{2^m} g(2^m x), u_2, \ldots, u_n \right\|_Y = 0
\]
for all \(x, u_2, \ldots, u_n \in X\). This shows that the sequence \(\{(1/2^j)g(2^j x)\}\) is a Cauchy sequence in \(Y\). Since \(Y\) is an \(n\)-Banach space, the sequence \(\{(1/2^j)g(2^j x)\}\) converges. So, we can define a mapping \(A : X \to Y\) by

\[
A(x) := \lim_{j \to \infty} \frac{1}{2^j} g \left( \frac{2^j x}{2^j} \right)
\]  

(2.35)

for all \(x \in X\). Putting \(l = 0\), then passing the limit \(m \to \infty\) in (2.33), and using Lemma 1.6(4), we get

\[
\|g(x) - A(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{2^j} \|g\left(\frac{2^j x}{2^j}, u_2, \ldots, u_n\right)\|_Y
\]  

(2.36)

for all \(x, u_2, \ldots, u_n \in X\).

Now we show that \(A\) is additive. By Lemma 1.6, (2.2), (2.32), and (2.35), we have

\[
\|A(2x) - 2A(x), u_2, \ldots, u_n\|_Y = \lim_{j \to \infty} \left\| \frac{1}{2^j} g \left( \frac{2^j x}{2^j} \right) - \frac{1}{2^j} g \left( \frac{2^j x}{2^j} \right), u_2, \ldots, u_n \right\|_Y
\]

\[
= 2 \lim_{j \to \infty} \left\| \frac{1}{2^{j+1}} g \left( \frac{2^{j+1} x}{2^{j+1}} \right) - \frac{1}{2^j} g \left( \frac{2^j x}{2^j} \right), u_2, \ldots, u_n \right\|_Y
\]

\[
\leq \lim_{j \to \infty} \frac{1}{2^j} \|g\left(\frac{2^j x}{2^j}, u_2, \ldots, u_n\right)\| = 0
\]  

(2.37)

for all \(x, u_2, \ldots, u_n \in X\). By Lemma 1.6(3), \(A(2x) = 2A(x)\) for all \(x \in X\). Also, by Lemma 1.6(4), (2.2), (2.3), and (2.35), we get

\[
\left\|DA(x, y), u_2, \ldots, u_n\right\|_Y
\]

\[
= \lim_{j \to \infty} \frac{1}{2^j} \left\|Dg \left( \frac{2^j x}{2^j}, \frac{2^j y}{2^j} \right), u_2, \ldots, u_n \right\|_Y
\]

\[
= \lim_{j \to \infty} \frac{1}{2^j} \left\|Df \left( \frac{2^j x}{2^j}, \frac{2^j y}{2^j} \right) - 8Df \left( \frac{2^j x}{2^j}, \frac{2^j y}{2^j} \right), u_2, \ldots, u_n \right\|_Y
\]

\[
\leq \lim_{j \to \infty} \frac{1}{2^j} \left[ \left\|Df \left( \frac{2^j x}{2^j}, \frac{2^j y}{2^j} \right), u_2, \ldots, u_n \right\|_Y + 8 \left\|Df \left( \frac{2^j x}{2^j}, \frac{2^j y}{2^j} \right), u_2, \ldots, u_n \right\|_Y \right]
\]

\[
\leq \lim_{j \to \infty} \frac{1}{2^j} \left[ \varphi \left( \frac{2^j x}{2^j}, \frac{2^j y}{2^j}, u_2, \ldots, u_n \right) + 8 \varphi \left( \frac{2^j x}{2^j}, \frac{2^j y}{2^j}, u_2, \ldots, u_n \right) \right] = 0
\]  

(2.38)

for all \(x, y, u_2, \ldots, u_n \in X\). By Lemma 1.6(3), \(DA(x, y) = 0\) for all \(x, y \in X\). Hence, the mapping \(A\) satisfies (1.1). By [11, Lemma 2.3], the mapping \(x \to A(2x) - 8A(x)\) is additive. Therefore, \(A(2x) = 2A(x)\) implies that the mapping \(A\) is additive.
The proof is similar to the proof of Theorem 2.1. Let

\[ A(x) = B(x), u_2, \ldots, u_n \] \quad \text{for all } x, u_2, \ldots, u_n \in X, \text{ and } l \in \mathbb{N}. \]

By Lemma 1.6, we can conclude that

\[ \|A(x) - B(x), u_2, \ldots, u_n\|_Y = \left\| A(2^l x) - B(2^l x), u_2, \ldots, u_n \right\|_Y \]

\[ \leq \frac{1}{2^l} \left[ f \left( 2^{l+1} x \right) - 8f \left( 2^l x \right) - A \left( 2^l x, u_2, \ldots, u_n \right) \right] \]

\[ + \left\| B \left( 2^l x \right) - f \left( 2^{l+1} x \right) + 8f \left( 2^l x \right), u_2, \ldots, u_n \right\|_Y \]

\[ \leq \frac{1}{2^l} \sum_{j=0}^{\infty} 2^j \tilde{p} \left( 2^{l+j} x, u_2, \ldots, u_n \right) \]

\[ \leq \frac{1}{2^l} \sum_{j=0}^{\infty} 2^j \tilde{p} \left( 2^l x, u_2, \ldots, u_n \right) = \sum_{j=0}^{\infty} 2^j \tilde{p} \left( 2^j x, u_2, \ldots, u_n \right) \]

for all \( x, u_2, \ldots, u_n \in X, \text{ and } l \in \mathbb{N}. \) By (2.2), we see that the right-hand side of the above inequality tends to 0 as \( l \to \infty. \) Therefore, \( \|A(x) - B(x), u_2, \ldots, u_n\|_Y = 0 \) for all \( u_2, \ldots, u_n \in X. \) By Lemma 1.6, we can conclude that \( A(x) = B(x) \) for all \( x \in X. \) So, \( A = B. \) This proves the uniqueness of \( A. \)

**Theorem 2.2.** Let \( X \) be a linear space and \( Y \) an \( n \)-Banach space. Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there is a function \( \varphi : X^{n+1} \to [0, \infty) \) such that

\[ \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, u_2, \ldots, u_n \right) < \infty, \]

\[ \|Df(x, y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n) \]

for all \( x, y, u_2, \ldots, u_n \in X. \) Then, there is a unique additive mapping \( A : X \to Y \) such that

\[ \|f(2x) - 8f(x) - A(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=1}^{\infty} 2^j \tilde{p} \left( \frac{x}{2^j}, u_2, \ldots, u_n \right) \]

(2.41)

for all \( x, u_2, \ldots, u_n \in X, \) where \( \tilde{p}(x, u_2, \ldots, u_n) \) is defined as in Theorem 2.1.

**Proof.** The proof is similar to the proof of Theorem 2.1.

**Corollary 2.3.** Let \( X \) be a normed space and \( Y \) an \( n \)-Banach space. Let \( \theta \in [0, \infty), p, r_2, \ldots, r_n \in (0, \infty) \) such that \( p \neq 1, \) and let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) such that

\[ \|Df(x, y), u_2, \ldots, u_n\|_Y \leq \theta \left( \|x\|_X^p + \|y\|_X^p \right) \|u_2\|_X^r \cdots \|u_n\|_X^r \]

(2.42)
for all \( x, y, u_2, \ldots, u_n \in X \). Then, there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f(2x) - 8f(x) - A(x), u_2, \ldots, u_n \|_Y \leq \frac{\theta \epsilon \| x \|^p_\| y \|_X \| u_2 \|_X \cdots \| u_n \|_X^p}{(2 - 2^p)(k^3 - k)} \tag{2.43}
\]

for all \( x, u_2, \ldots, u_n \in X \), where

\[
\epsilon = \left( 1 + |k| + 2^{3-p}|k| \right) \left[ (2k + 1)^p + (2k - 1)^p \right] + 2|k| + 13 + 3p + 3|k|^p + 16k^2 + 3p|k|^p + 2^{p+1}k^3
\]
\[
+ 2^p(5 + |k|^p) + 2|k + 1|^p + 2|k - 1|^p + 2^3k^2 \left( 2 + |k| + |k|^p \right) + \frac{(|k| + 1)(|k + 1|)}{|k - 1|}
\]
\[
+ \frac{2^3p|k|}{|k - 1|} \left[ |k + 1|^p + \left( 1 + 8k^2 + 2^p k^2 \right) |k - 1| - |k - 1|^p \right] + \frac{2^p|k|^p(k^2 + |k| - 1)}{|k - 1|}
\]
\[
+ \frac{2}{|k - 1|} \left( |k| + 1 \right) |k - 1|^p + \frac{8k^2 + 2|k - 8, |k|^p}{|k - 1|}. \tag{2.44}
\]

**Proof.** Define \( \varphi(x, y) = \theta \epsilon \| x \|^p_\| y \|_X \| u_2 \|_X \cdots \| u_n \|_X^p \) for all \( x, y, u_2, \ldots, u_n \in X \), and apply Theorems 2.1 and 2.2. \( \square \)

The following example shows that the assumption \( p \neq 1 \) cannot be omitted in Corollary 2.3.

**Example 2.4.** Let \( X = \mathbb{C} \) be a linear space over \( \mathbb{R} \). Define \( \| \cdot, \cdot \| : X \times X \to \mathbb{R} \) by \( \| x_1, x_2 \| = |a_1b_2 - a_2b_1| \), where \( x_i = a_i + b_ij \in \mathbb{C}, \ a_j, b_j \in \mathbb{R}, \ j = 1, 2 \ (i = \sqrt{-1} \) is the imaginary unit). Then, \( (X, \| \cdot, \cdot \|) \) is a 2-normed linear space.

Let \( \phi : \mathbb{C} \to \mathbb{C} \) defined by

\[
\phi(x) = \begin{cases} 
  x, & \text{for } |x| < 1, \\
  1, & \text{for } |x| \geq 1. 
\end{cases} \tag{2.45}
\]

Consider the function \( f : \mathbb{C} \to \mathbb{C} \) defined by

\[
f(x) = \sum_{m=0}^{\infty} a^{-m} \phi(a^m x) \tag{2.46}
\]

for all \( x \in \mathbb{C} \), where \( a > |k| \). Then, \( f \) satisfies the functional inequality

\[
\| Df(x, y), u \| \leq \frac{4\alpha^2(\| x \| + 1)}{\alpha - 1} (\| x \| + \| y \|) |u| \tag{2.47}
\]

for all \( x, y, u \in \mathbb{C} \), but there do not exist an additive mapping \( A : \mathbb{C} \to \mathbb{C} \) and a constant \( d > 0 \) such that \( \| f(x) - A(x), u \| \leq d \| x \| u \) for all \( x, u \in \mathbb{C} \).
It is clear that \(|f(x)| \leq \alpha/(\alpha - 1)\) for all \(x \in \mathbb{C}\). If \(|x| + |y| = 0\) or \(|x| + |y| \geq 1/\alpha\) for all \(x, y \in \mathbb{C}\), then the inequality (2.47) holds. Now suppose that \(0 < |x| + |y| < 1/\alpha\). Then, there exists an integer \(n \geq 1\) such that
\[
\frac{1}{\alpha^{n+1}} \leq |x| + |y| < \frac{1}{\alpha^n}.
\] (2.48)

Hence, \(\alpha^m|kx \pm y| < 1, \alpha^m|x \pm y| < 1, \alpha^m|x| < 1\) for all \(m = 0, 1, \ldots, n - 1\). From the definition of \(f\) and (2.48), we obtain that
\[
\left\| Df(x, y), u \right\|
= \left\| \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(kx + y)) + \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(kx - y)) - k \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(x + y))
- k \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m(x - y)) - 2 \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m kx) + 2k \sum_{m=n}^{\infty} \alpha^{-m}\phi(\alpha^m x), u \right\| (2.49)
\leq \frac{4\alpha^2(k + 1)}{\alpha - 1} \left( |x| + |y| \right) |u|.
\]

Therefore, \(f\) satisfies (2.47). Now, we claim that the functional equation (1.1) is not stable for \(p = 1\) in Corollary 2.3. Suppose on the contrary that there exist an additive mapping \(A : \mathbb{C} \rightarrow \mathbb{C}\) and a constant \(d > 0\) such that \(\|f(x) - A(x), u\| \leq d |x||u|\) for all \(x, u \in \mathbb{C}\). Then, there exists a constant \(c \in \mathbb{C}\) such that \(A(x) = cx\) for all rational numbers \(x\). So, we obtain that
\[
\left\| f(x), u \right\| \leq (d + |c|) |x||u| (2.50)
\]
for all rational numbers \(x\) and all \(u \in \mathbb{C}\). Let \(s \in \mathbb{N}\) with \(s + 1 > d + |c|\). If \(x\) is a rational number in \((0, \alpha^{-s})\) and \(u = bi\) \((b \in \mathbb{R})\), then \(\alpha^m x \in (0, 1)\) for all \(m = 0, 1, \ldots, s\), and we get
\[
\left\| f(x), u \right\| = \left\| \sum_{m=0}^{\infty} \frac{\phi(\alpha^m x)}{\alpha^m}, u \right\| \geq \sum_{m=0}^{s} \frac{\phi(\alpha^m x)}{\alpha^m} |b| = (s + 1)x|b| > (d + |c|)x|b| = (d + |c|)|x||u|, (2.51)
\]
which contradicts (2.50).

**Theorem 2.5.** Let \(X\) be a linear space and \(Y\) an \(n\)-Banach space. Let \(f : X \rightarrow Y\) be a mapping with \(f(0) = 0\) for which there is a function \(\varphi : X^{n+1} \rightarrow [0, \infty)\) such that
\[
\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, u_2, \ldots, u_n) < \infty, (2.52)
\]
\[
\left\| Df(x, y, u_2, \ldots, u_n) \right\|_Y \leq \varphi(x, y, u_2, \ldots, u_n) (2.53)
\]
for all \( x, y, u_2, \ldots, u_n \in X \). Then, there is a unique cubic mapping \( C : X \to Y \) such that

\[
\| f(2x) - 2f(x) - C(x), u_2, \ldots, u_n \|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \tilde{\varphi}(2^j x, u_2, \ldots, u_n)
\]  

for all \( x, u_2, \ldots, u_n \in X \), where \( \tilde{\varphi}(x, u_2, \ldots, u_n) \) is defined as in Theorem 2.1.

**Proof.** As in the proof of Theorem 2.1, we have

\[
\| f(4x) - 10f(2x) + 16f(x), u_2, \ldots, u_n \|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n)
\]  

for all \( x \in X \), where \( \tilde{\varphi}(x, u_2, \ldots, u_n) \) is defined as in Theorem 2.1.

Now, let \( h : X \to Y \) be the mapping defined by \( h(x) := f(2x) - 2f(x) \). By (2.55), we have

\[
\| h(2x) - 8h(x), u_2, \ldots, u_n \|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n)
\]  

for all \( x \in X \). Replacing \( x \) by \( 2^j x \) in (2.56) and dividing both sides of (2.56) by \( 8^{j+1} \), we get

\[
\left\| \frac{1}{8^j} h(2^j x) - \frac{1}{8^{j+1}} h(2^{j+1} x), u_2, \ldots, u_n \right\|_Y \leq \frac{1}{8^j} \tilde{\varphi}(2^j x, u_2, \ldots, u_n)
\]  

for all \( x, u_2, \ldots, u_n \in X \) and all integers \( j \geq 0 \). For all integers \( l, m \) with \( 0 \leq l < m \), we have

\[
\left\| \frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n \right\|_Y \leq \sum_{j=l}^{m-1} \left\| \frac{1}{8^j} h(2^j x) - \frac{1}{8^{j+1}} h(2^{j+1} x), u_2, \ldots, u_n \right\|_Y
\]

\[
\leq \sum_{j=l}^{m-1} \frac{1}{8^{j+1}} \tilde{\varphi}(2^j x, u_2, \ldots, u_n)
\]  

for all \( x, u_2, \ldots, u_n \in X \). So, we get

\[
\lim_{l,m \to \infty} \left\| \frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n \right\|_Y = 0
\]  

for all \( x, u_2, \ldots, u_n \in X \). This shows that the sequence \( \{ (1/8^l) h(2^l x) \} \) is a Cauchy sequence in \( Y \). Since \( Y \) is an \( n \)-Banach space, the sequence \( \{ (1/8^j) h(2^j x) \} \) converges. So, we can define a mapping \( C : X \to Y \) by

\[
C(x) := \lim_{j \to \infty} \frac{1}{8^j} h\left(2^j x\right)
\]  

(2.60)
for all \( x \in X \). Putting \( l = 0 \), then passing the limit \( m \to \infty \) in (2.58), and using Lemma 1.6(4), we get

\[
\|h(x) - C(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \varphi\left(2^j x, u_2, \ldots, u_n\right) \tag{2.61}
\]

for all \( x, u_2, \ldots, u_n \in X \).

Now we show that \( C \) is cubic. By Lemma 1.6, (2.52), (2.58), and (2.60), we have

\[
\|C(2x) - 8C(x), u_2, \ldots, u_n\|_Y = \lim_{j \to \infty} \frac{1}{8^j} \left| h(2^{j+1} x) - \frac{1}{8^j} h(2^j x), u_2, \ldots, u_n \right|_Y
\]

\[
= 8 \lim_{j \to \infty} \frac{1}{8^j} \left| h(2^{j+1} x) - \frac{1}{8^j} h(2^j x), u_2, \ldots, u_n \right|_Y \tag{2.62}
\]

\[
\leq \lim_{j \to \infty} \frac{1}{8^j} \varphi\left(2^j x, u_2, \ldots, u_n\right) = 0
\]

for all \( x, u_2, \ldots, u_n \in X \). By Lemma 1.6(3), \( C(2x) = 8C(x) \) for all \( x \in X \). Also, by Lemma 1.6(4), (2.52), (2.53), and (2.60), we get

\[
\|DC(x, y), u_2, \ldots, u_n\|_Y
\]

\[
= \lim_{j \to \infty} \frac{1}{8^j} \left| Dh(2^j x, 2^j y), u_2, \ldots, u_n \right|_Y
\]

\[
= \lim_{j \to \infty} \frac{1}{8^j} \left| Df(2^{j+1} x, 2^{j+1} y) - 2Df(2^j x, 2^j y), u_2, \ldots, u_n \right|_Y \tag{2.63}
\]

\[
\leq \lim_{j \to \infty} \frac{1}{8^j} \left[ \left| Df(2^{j+1} x, 2^{j+1} y), u_2, \ldots, u_n \right|_Y + 2 \left| Df(2^j x, 2^j y), u_2, \ldots, u_n \right|_Y \right]
\]

\[
\leq \lim_{j \to \infty} \frac{1}{8^j} \left[ \varphi\left(2^{j+1} x, 2^{j+1} y, u_2, \ldots, u_n\right) + 2\varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) \right] = 0
\]

for all \( x, y, u_2, \ldots, u_n \in X \). By Lemma 1.6(3), \( DC(x, y) = 0 \) for all \( x, y \in X \). Hence the mapping \( C \) satisfies (1.1). By [11, Lemma 2.3], the mapping \( x \to C(2x) - 2C(x) \) is cubic. Therefore, \( C(2x) = 8C(x) \) implies that the mapping \( C \) is cubic.
To prove the uniqueness of $C$, let $S : X \to Y$ be another cubic mapping satisfying (2.54). Fix $x \in X$. Clearly, $C(2^l x) = 8^l A(x)$ and $S(2^l x) = 8^l S(x)$ for all $l \in \mathbb{N}$. It follows from (2.54) that

$$
\| C(x) - S(x), u_2, \ldots, u_n \|_Y = \left\| \frac{C(2^l x)}{8^l} - \frac{S(2^l x)}{8^l}, u_2, \ldots, u_n \right\|_Y
$$

$$
\leq \frac{1}{8^l} \left[ \left\| f(2^{l+1} x) - 2 f(2^l x) - C(2^l x), u_2, \ldots, u_n \right\|_Y 
+ \left\| S(2^l x) - f(2^{l+1} x) + 2 f(2^l x), u_2, \ldots, u_n \right\|_Y \right]
$$

$$
\leq \frac{1}{8^l} \sum_{j=0}^{\infty} \frac{1}{8^j} \tilde{\varphi}(2^j x, u_2, \ldots, u_n)
$$

$$
\leq \sum_{j=0}^{\infty} \frac{1}{8^j} \tilde{\varphi}(2^j x, u_2, \ldots, u_n) = \sum_{j=0}^{\infty} \frac{1}{8^j} \tilde{\varphi}(2^j x, u_2, \ldots, u_n)
$$

for all $x, u_2, \ldots, u_n \in X$, and $l \in \mathbb{N}$. By (2.52), we see that the right-hand side of the above inequality tends to 0 as $l \to \infty$. Therefore, $\| C(x) - S(x), u_2, \ldots, u_n \|_Y = 0$ for all $u_2, \ldots, u_n \in X$. By Lemma 1.6, we can conclude that $C(x) = S(x)$ for all $x \in X$. So $C = S$. This proves the uniqueness of $C$. \hfill \Box

**Theorem 2.6.** Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that

$$
\sum_{j=1}^{\infty} 8^j \varphi\left( \frac{x}{2^j}, u_2, \ldots, u_n \right) < \infty,
$$

$$
\| D f(x, y), u_2, \ldots, u_n \|_Y \leq \varphi(x, y, u_2, \ldots, u_n)
$$

(2.65)

for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique cubic mapping $C : X \to Y$ such that

$$
\| f(2x) - 2 f(x) - C(x), u_2, \ldots, u_n \|_Y \leq \sum_{j=1}^{\infty} 8^j \tilde{\varphi}\left( \frac{x}{2^j}, u_2, \ldots, u_n \right)
$$

(2.66)

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

**Proof.** The proof is similar to the proof of Theorem 2.5. \hfill \Box

**Corollary 2.7.** Let $X$ be a normed space and $Y$ an $n$-Banach space. Let $\theta \in [0, \infty), p_2, \ldots, p_n \in (0, \infty)$ such that $p \neq 3$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that

$$
\| D f(x, y), u_2, \ldots, u_n \|_Y \leq \theta\left( \| x \|^p_X + \| y \|^p_X \right) \| u_2 \|^p_X \cdots \| u_n \|^p_X
$$

(2.67)
for all $x, y, u_2, \ldots, u_n \in X$. Then, there exists a unique cubic mapping $C : X \to Y$ such that

$$
\|f(2x) - 2f(x) - C(x), u_2, \ldots, u_n\|_Y \leq \frac{\theta \epsilon \|x\|_X^p \|u_2\|_X^p \cdots \|u_n\|_X^p}{|(8 - 2p)(k^3 - k)|} \tag{2.68}
$$

for all $x, u_2, \ldots, u_n \in X$, where $\epsilon$ is defined as in Corollary 2.3.

Proof. Define $\varphi(x, y) = \theta(\|x\|_X^p + \|y\|_X^p)\|u_2\|_X^p \cdots \|u_n\|_X^p$ for all $x, y, u_2, \ldots, u_n \in X$, and apply Theorems 2.5 and 2.6.

The following example shows that the the generalized Hyers-Ulam stability problem for the case of $p = 3$ was excluded in Corollary 2.7.

Example 2.8. Let $X = \mathbb{C}$ be a linear space over $\mathbb{R}$, and let $\|\cdot, \cdot\| : X \times X \to \mathbb{R}$ be defined as in Example 2.4. Then, $(X, \|\cdot, \cdot\|)$ is a 2-normed linear space.

Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$
\phi(x) = \begin{cases} 
x^3, & \text{for } |x| < 1, \\
1, & \text{for } |x| \geq 1.
\end{cases} \tag{2.69}
$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$
f(x) = \sum_{m=0}^{\infty} a^{-3m} \phi(a^m x) \tag{2.70}
$$

for all $x \in \mathbb{C}$, where $a > |k|$. Then, $f$ satisfies the functional inequality

$$
\|Df(x, y), u\| \leq \frac{4a^6 (|k| + 1)}{a^3 - 1} \left(|x|^3 + |y|^3\right)|u| \tag{2.71}
$$

for all $x, y, u \in \mathbb{C}$, but there do not exist a cubic mapping $C : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\|f(x) - C(x), u\| \leq d \|x|^3\|u|$ for all $x, u \in \mathbb{C}$.

It is clear that $|f(x)| \leq a^3/(a^3 - 1)$ for all $x \in \mathbb{C}$. If $|x|^3 + |y|^3 = 0$ or $|x|^3 + |y|^3 \geq 1/a^3$ for all $x, y \in \mathbb{C}$, then inequality (2.71) holds. Now suppose that $0 < |x|^3 + |y|^3 < 1/a^3$. Then, there exists an integer $n \geq 1$ such that

$$
\frac{1}{a^{3(n+1)}} \leq |x|^3 + |y|^3 < \frac{1}{a^{3n}}. \tag{2.72}
$$
Hence, $a^m|kx + y| < 1, a^m|x + y| < 1, a^m|x| < 1$ for all $m = 0, 1, \ldots, n - 1$. From the definition of $f$ and (2.72), we obtain that

$$
\left\| Df(x, y), u \right\| = \left\| \sum_{m=0}^{\infty} \alpha^{3m} \phi(a^m(kx + y)) + \sum_{m=0}^{\infty} \alpha^{3m} \phi(a^m(kx - y)) - k \sum_{m=0}^{\infty} \alpha^{3m} \phi(a^m(x + y)) \right\|
$$

$$
- k \sum_{m=0}^{\infty} \alpha^{3m} \phi(a^m(x - y)) - 2 \sum_{m=0}^{\infty} \alpha^{3m} \phi(a^m kx) + 2k \sum_{m=0}^{\infty} \alpha^{3m} \phi(a^m x), u \right\|
$$

$$
\leq \frac{4a^6(|k| + 1)}{\alpha^3 - 1} (|x^3| + |y^3|)|u|.
$$

(2.73)

Therefore, $f$ satisfies (2.71). Now, we claim that the functional equation (1.1) is not stable for $p = 3$ in Corollary 2.7. Suppose on the contrary that there exist a cubic mapping $C : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $\| f(x) - C(x), u \| \leq d |x|^3|u|$ for all $x, u \in \mathbb{C}$. Then, there exists a constant $\beta \in \mathbb{C}$ such that $C(x) = \beta x^3$ for all rational numbers $x$. So, we obtain that

$$
\| f(x), u \| \leq (d + |\beta|)|x|^3|u|
$$

(2.74)

for all rational numbers $x$ and all $u \in \mathbb{C}$. Let $s \in \mathbb{N}$ with $s + 1 > d + |\beta|$. If $x$ is a rational number in $(0, a^{-s})$ and $u = bi$ ($b \in \mathbb{R}$), then $a^m x \in (0, 1)$ for all $m = 0, 1, \ldots, s$, and we get

$$
\| f(x), u \| = \left\| \sum_{m=0}^{\infty} \frac{\phi(a^m x)}{\alpha^{3m}}, u \right\| \geq \sum_{m=0}^{s} \frac{\phi(a^m x)}{\alpha^{3m}} |b|
$$

$$
= (s + 1)x^3|b| > (d + |\beta|)x^3|b| = (d + |\beta|)|x|^3|u|,
$$

which contradicts (2.74).

**Theorem 2.9.** Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that

$$
\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, u_2, \ldots, u_n) < \infty,
$$

(2.76)

$$
\| Df(x, y), u_2, \ldots, u_n \|_Y \leq \varphi(x, y, u_2, \ldots, u_n)
$$

(2.77)

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$
\| f(x) - A(x) - C(x), u_2, \ldots, u_n \|_Y \leq \frac{1}{6} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \bar{\varphi}(2^j x, u_2, \ldots, u_n)
$$

(2.78)

for all $x, u_2, \ldots, u_n \in X$, where $\bar{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.
Proof. By Theorems 2.1 and 2.5, there exist an additive mapping \( A' : X \to Y \) and a cubic mapping \( C' : X \to Y \) such that

\[
\|f(2x) - 8f(x) - A'(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \varphi(2^j x, u_2, \ldots, u_n),
\]

\[
\|f(2x) - 2f(x) - C'(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \varphi(2^j x, u_2, \ldots, u_n)
\]

for all \( x, u_2, \ldots, u_n \in X \). Hence,

\[
\|f(x) + \frac{1}{6} A'(x) - \frac{1}{6} C'(x), u_2, \ldots, u_n\|_Y \leq \frac{1}{6} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \varphi(2^j x, u_2, \ldots, u_n)
\]

for all \( x \in X \). So, we obtain (2.78) by letting \( A(x) = -(1/6)A'(x) \) and \( C(x) = (1/6)C'(x) \) for all \( x \in X \).

To prove the uniqueness of \( A \) and \( C \), let \( A'', C'' : X \to Y \) be another additive and cubic mapping satisfying (2.78). Fix \( x \in X \). Let \( A_1 = A - A'' \) and \( C_1 = C - C'' \). So,

\[
\|A_1(x) + C_1(x), u_2, \ldots, u_n\|_Y
\]

\[
\leq \|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y + \|f(x) - A''(x) - C''(x), u_2, \ldots, u_n\|_Y
\]

\[
\leq \frac{1}{3} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \varphi(2^j x, u_2, \ldots, u_n)
\]

for all \( x, u_2, \ldots, u_n \in X \). Then (2.76) implies that

\[
\lim_{n \to \infty} \frac{1}{8^n} \|A_1(2^n x) + C_1(2^n x), u_2, \ldots, u_n\|_Y = 0
\]

for all \( x, u_2, \ldots, u_n \in X \). Thus, \( C_1 = 0 \). So, it follows from (2.81) that

\[
\|A_1(x), u_2, \ldots, u_n\|_Y \leq \frac{1}{3} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \varphi(2^j x, u_2, \ldots, u_n)
\]

for all \( u_2, \ldots, u_n \in X \). Therefore, \( A_1 = 0 \). \qed

Similarly to Theorem 2.9, one can prove the following result.
Theorem 2.10. Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \rightarrow [0, \infty)$ such that

$$
\sum_{j=0}^{\infty} 8^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n \right) < \infty,
$$

(2.84)

$$
\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n)
$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ such that

$$
\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \leq \frac{1}{6} \sum_{j=1}^{\infty} \left( 2^{j-1} + 8^{j-1} \right) \varphi \left( \frac{x}{2^j}, u_2, \ldots, u_n \right)
$$

(2.85)

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.6.

Theorem 2.11. Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \rightarrow [0, \infty)$ such that

$$
\sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n \right) < \infty, \ \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi \left( 2^j x, 2^j y, u_2, \ldots, u_n \right) < \infty,
$$

(2.86)

$$
\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n)
$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ such that

$$
\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \\
\leq \frac{1}{6} \left[ \sum_{j=1}^{\infty} 2^{j-1} \varphi \left( \frac{x}{2^j}, u_2, \ldots, u_n \right) + \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi \left( 2^j x, u_2, \ldots, u_n \right) \right]
$$

(2.87)

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.5.

Corollary 2.12. Let $X$ be a normed space and $Y$ an $n$-Banach space. Let $\theta \in [0, \infty), r_2, \ldots, r_n \in (0, \infty), p \in (0, 1) \cup (1, 3) \cup (3, \infty)$, and let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ such that

$$
\|Df(x,y), u_2, \ldots, u_n\|_Y \leq \theta \left( \|x\|_X^p + \|y\|_X^p \right) \|u_2\|_X^{q_2} \cdots \|u_n\|_X^{q_n}
$$

(2.88)
for all \( x, y, u_2, \ldots, u_n \in X \). Then, there exist a unique additive mapping \( A : X \to Y \) and a unique cubic mapping \( C : X \to Y \) such that

\[
\| f(x) - A(x) - C(x), u_2, \ldots, u_n \|_Y \leq \frac{1}{6|k^3 - k|} \left( \frac{1}{|2 - 2^p|} + \frac{1}{|8 - 2^p|} \right) \theta \epsilon \| x \|_X^n \| u_2 \|_X^3 \cdots \| u_n \|_X^3
\]  

(2.89)

for all \( x, u_2, \ldots, u_n \in X \), where \( \epsilon \) is defined as in Corollary 2.3.

Proof. Define \( \varphi(x, y) = \theta(\| x \|_X^n + \| y \|_X^n) \| u_2 \|_X^3 \cdots \| u_n \|_X^3 \) for all \( x, y, u_2, \ldots, u_n \in X \), and apply Theorems 2.9–2.11.

Remark 2.13. The generalized Hyers-Ulam stability problem for the cases of \( p = 1 \) and \( p = 3 \) was excluded in Corollary 2.12 (see Examples 2.4 and 2.8).

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