Research Article

On the Modified q -Bernoulli Numbers of Higher Order with Weight

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The purpose of this paper is to give some properties of the modified q -Bernoulli numbers and polynomials of higher order with weight. In particular, by using the bosonic p -adic q -integral on $\mathbb{Z}_p$, we derive new identities of q -Bernoulli numbers and polynomials with weight.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p -adic norm of $\mathbb{C}_p$ is defined by $|p|_p = 1/p$. When one talks of a q -extension, q can be considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. Throughout this paper we assume that $\alpha \in \mathbb{Q}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p -adic q -integral on $\mathbb{Z}_p$ is defined by Kim (see [1–3]) as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (1.1)$$

where $[x]_q$ is the q -number of x which is defined by $[x]_q = (1 - q^x)/(1 - q)$.

From (1.1), we have

$$q^n I_q(f_n) - I_q(f) = (q-1) \sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l), \quad (1.2)$$

where $f_n(x) = f(x+n)$ (see [2–4]).

As is well known, Bernoulli numbers are inductively defined by

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (1.3)$$

with the usual convention about replacing B^n by B_n (see [3, 5]).

In [2, 5, 6], the q -Bernoulli numbers are defined by

$$B_{0,q} = \frac{q-1}{\log q}, \quad (qB_q + 1)^n - B_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (1.4)$$

with the usual convention about replacing B_q^n by $B_{n,q}$. Note that $\lim_{q \rightarrow 1} B_{n,q} = B_n$. In the viewpoint of (1.4), we consider the modified q -Bernoulli numbers with weight.

In this paper we study families of the modified q -Bernoulli numbers and polynomials of higher order with weight. In particular, by using the multivariate p -adic q -integral on $\mathbb{Z}_p$, we give new identities of the higher-order q -Bernoulli numbers and polynomials with weight.

2. Modified q -Bernoulli Numbers with Weight of Higher Order

For $n \in \mathbb{Z}_+$, let us consider the following modified q -Bernoulli numbers with weight α (see [1, 3]):

$$\begin{aligned} \tilde{B}_{n,q}^{(\alpha)} &= \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n q^{-x} d\mu_q(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{\alpha l}{[\alpha l]_q}, \\ \tilde{B}_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n q^{-y} d\mu_q(y) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l}{[\alpha l]_q}. \end{aligned} \quad (2.1)$$

From (2.1), we note that

$$\tilde{B}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{B}_{l,q}^{(\alpha)} \quad (2.2)$$

(see [1, 3]).

For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, by making use of the multivariate p -adic q -integral on $\mathbb{Z}_p$, we consider the following modified q -Bernoulli numbers with weight α of order k , $\tilde{B}_{n,q}^{(k,\alpha)}$:

$$\tilde{B}_{n,q}^{(k,\alpha)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^n q^{-x_1 - \cdots - x_k} d\mu_q(x_1) \cdots d\mu_q(x_k). \quad (2.3)$$

Note that $\tilde{B}_{n,q}^{(1,\alpha)} = \tilde{B}_{n,q}^{(\alpha)}$ and $\lim_{q \rightarrow 1} \tilde{B}_{n,q}^{(k,\alpha)} = B_n^{(k)}$, where $B_n^{(k)}$ are the n th ordinary Bernoulli numbers of order k .

For $k, N \in \mathbb{N}$, we have

$$\begin{aligned} & \left(\frac{1-q}{1-q^{p^N}} \right)^k \sum_{i_1=0}^{p^N-1} \cdots \sum_{i_k=0}^{p^N-1} [i_1 + \cdots + i_k]_{q^\alpha}^n \\ &= \left(\frac{1-q}{1-q^{p^N}} \right)^k \left(\frac{1}{1-q^\alpha} \right)^n \sum_{i_1, \dots, i_k=0}^{p^N-1} \sum_{j=0}^n \binom{n}{j} (-1)^j q^{\alpha(i_1 + \cdots + i_k)j} \\ &= \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(1-q)^k}{(1-q^{p^N})^k} \underbrace{\left(\frac{1-q^{\alpha p^N j}}{1-q^{\alpha j}} \cdots \frac{1-q^{\alpha p^N j}}{1-q^{\alpha j}} \right)}_{k\text{-times}}. \end{aligned} \quad (2.4)$$

By (1.1), (2.3), and (2.4), we get

$$\tilde{B}_{n,q}^{(k,\alpha)} = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(\alpha j)^k}{[\alpha j]_q^k}. \quad (2.5)$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, one has

$$\tilde{B}_{n,q}^{(k,\alpha)} = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(\alpha j)^k}{[\alpha j]_q^k}. \quad (2.6)$$

Let us consider the modified q -Bernoulli and polynomials with weight α of order k as follows:

$$\tilde{B}_{n,q}^{(k,\alpha)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{-x_1 - \cdots - x_k} d\mu_q(x_1) \cdots d\mu_q(x_k). \quad (2.7)$$

By the same method of (2.5), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{Z}_+$, one has

$$\tilde{B}_{n,q}^{(k,\alpha)}(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^n \binom{n}{j} (-1)^j q^{\alpha x j} \frac{(\alpha j)^k}{[\alpha j]_q^k}. \quad (2.8)$$

By Theorem 2.2, we get

$$\begin{aligned}
 \tilde{B}_{n,q^{-1}}^{(k,\alpha)}(k-x) &= \frac{1}{(1-q^{-\alpha})^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(\alpha j)^k}{[\alpha j]_{q^{-1}}^k} q^{-\alpha j(k-x)} \\
 &= \frac{(-1)^n q^{\alpha n}}{(1-q^{\alpha})^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \left(\frac{q^{-1}(q-1)\alpha j}{(q^{\alpha j}-1)q^{-\alpha j}} \right)^k q^{-\alpha j(k-x)} \\
 &= \frac{(-1)^n q^{\alpha n}}{(1-q^{\alpha})^n} \sum_{j=0}^n \binom{n}{j} (-1)^j q^{\alpha j x} q^{-k} \frac{(\alpha j)^k}{[\alpha j]_q^k} \\
 &= (-1)^n q^{\alpha n-k} \tilde{B}_{n,q}^{(k,\alpha)}(x).
 \end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.3. For $n \in \mathbb{Z}_+$, one has

$$\tilde{B}_{n,q^{-1}}^{(k,\alpha)}(k-x) = (-1)^n q^{\alpha n-k} \tilde{B}_{n,q}^{(k,\alpha)}(x), \quad \tilde{B}_{n,q^{-1}}^{(k,\alpha)}(k) = (-1)^n q^{\alpha n-k} \tilde{B}_{n,q}^{(k,\alpha)}. \tag{2.10}$$

From Theorem 2.3, we note that

$$\lim_{q \rightarrow 1} \tilde{B}_{n,q^{-1}}^{(k,\alpha)}(k-x) = B_n^{(k)}(k-x), \quad \lim_{q \rightarrow 1} \tilde{B}_{n,q^{-1}}^{(k,\alpha)}(k) = (-1)^n B_n^{(k)}. \tag{2.11}$$

Thus, we have $B_n^{(k)}(k) = (-1)^n B_n^{(k)}$, where $B_n^{(k)}$ are the n th Bernoulli numbers of order k .

From (2.3) and (2.7), we can derive the following equations:

$$\begin{aligned}
 \tilde{B}_{k,q}^{(l,\alpha)}(x) &= \lim_{N \rightarrow \infty} \frac{1}{[m]_q^l [p^N]_{q^m}^l} \sum_{i_1, \dots, i_l=0}^{m-1} \sum_{n_1, \dots, n_l=0}^{p^N-1} [x + i_1 + \dots + i_l + m(n_1 + \dots + n_l)]_{q^{\alpha}}^k \\
 &= \frac{[m]_{q^{\alpha}}^k}{[m]_q^l} \sum_{i_1, \dots, i_l=0}^{m-1} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left[\frac{x + i_1 + \dots + i_l}{m} + x_1 + \dots + x_l \right]_{q^{\alpha m}}^k \\
 &\quad \times q^{-mx_1 - \dots - mx_l} d\mu_{q^m}(x_1) \dots d\mu_{q^m}(x_l) \\
 &= \frac{[m]_{q^{\alpha}}^k}{[m]_q^l} \sum_{i_1, \dots, i_l=0}^{m-1} \tilde{B}_{k,q^m}^{(l,\alpha)} \left(\frac{x + i_1 + \dots + i_l}{m} \right).
 \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.4. For $k \in \mathbb{Z}_+$ and $l, m \in \mathbb{N}$, one has

$$\tilde{B}_{k,q}^{(l,\alpha)}(x) = \frac{[m]_{q^{\alpha}}^k}{[m]_q^l} \sum_{i_1, \dots, i_l=0}^{m-1} \tilde{B}_{k,q^m}^{(l,\alpha)} \left(\frac{x + i_1 + \dots + i_l}{m} \right). \tag{2.13}$$

In particular,

$$\tilde{B}_{k,q}^{(l,\alpha)}(mx) = \frac{[m]_{q^\alpha}^k}{[m]_q^l} \sum_{i_1, \dots, i_l=0}^{m-1} \tilde{B}_{k,q^m}^{(l,\alpha)}\left(x + \frac{i_1 + \dots + i_l}{m}\right). \quad (2.14)$$

From (1.2), we can derive the following integral:

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x+1)q^{-x}d\mu_q(x) &= \int_{\mathbb{Z}_p} f(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}f'(0), \\ \int_{\mathbb{Z}_p} f(x+2)q^{-x}d\mu_q(x) &= \int_{\mathbb{Z}_p} f_1(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}f'(1) \\ &= \int_{\mathbb{Z}_p} f(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}(f'(0) + f'(1)). \end{aligned} \quad (2.15)$$

Continuing this process, we obtain

$$\int_{\mathbb{Z}_p} f(x+n)q^{-x}d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l). \quad (2.16)$$

By (2.16), we get

$$\int_{\mathbb{Z}_p} [x+n]_{q^\alpha}^m q^{-x}d\mu_q(x) = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^m q^{-x}d\mu_q(x) + \frac{m\alpha}{[\alpha]_q} \sum_{l=0}^{n-1} [l]_{q^\alpha}^{m-1} q^{\alpha l}. \quad (2.17)$$

Therefore, by (2.1) and (2.17), we obtain the following theorem.

Theorem 2.5. For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, one has

$$\tilde{B}_{m,q}^{(\alpha)}(n) - \tilde{B}_{m,q}^{(\alpha)} = m \frac{\alpha}{[\alpha]_q} \sum_{l=0}^{n-1} [l]_{q^\alpha}^m q^{\alpha l}. \quad (2.18)$$

In an analogues manner as the previous investigation [7–10], we can define a further generalization of modified q -Bernoulli numbers with weight. Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$. Then the generalized q -Bernoulli numbers with weight attached to χ can be defined as follows:

$$\begin{aligned} \tilde{B}_{n,\chi,q}^{(\alpha)} &= \int_X \chi(x) [x]_{q^\alpha}^n q^{-x} d\mu_q(x) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \tilde{B}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right). \end{aligned} \quad (2.19)$$

We expect to investigate these objects in future papers. This definition $\tilde{B}_{n,q}^{(\alpha)}$ was also given in a previous paper (see [9]).

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